APPLIED MECHANICS.

APPLIED MECHANICS

AN ELEMENTARY GENERAL INTRODUCTION TO

THE THEORY OF
STRUCTURES AND MACHINES

WITH DIAGRAMS, ILLUSTRATIONS, AND EXAMPLES

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PREFACE.

ON the author's appointment to lecture on Mechanics in the Royal Naval College, a course of elementary lessons was commenced, based on RANKINE'S well-known treatise, with such assistance as could be obtained from other sources. After some years this course assumed a tolerably permanent form, and it was thought desirable to print it, partly from the inconvenience to students of being exclusively dependent on oral instruction, and partly from an idea that it might be useful to others besides those who were immediately addressed. The place which these lectures occupy in the programme of the College will be found explained in an Appendix.

The preparation of the work for the press has extended over a considerable period, and has been subject to many interruptions. There is therefore not always the unity desirable in a scientific treatise; nor is it by any means complete, even when due account is taken of the stringent limitations explained in the Introduction. It is, however, hoped that these deficiencies may be partly compensated for by the fact that the book is the product of a great deal of experience in teaching the subject, and a great deal of consideration as to the matter which ought to find a place in a general elementary treatise. Nearly the whole has been delivered in the form of lectures, and some part has actually been printed from notes taken throughout one session by a member of the junior class (Mr. H. J. Oram, R.N.) at that time, which were afterwards transcribed for the press by the author's assistant. Everything, however, of any importance has been re-written, with alterations and additions, to make it better fit for publication. Throughout, the object has been to give reasons, not rules, and details of application are consequently subordinated to the principles on which the theory is based. Especially has the author endeavoured to distinguish as clearly as possible between those parts of the subject which are universally and necessarily true, and those parts which rest on hypotheses more or less questionable. The book is intended to give that general knowledge of the mechanics of structures and machines which should accompany the detailed study either of naval architecture or of any special branch of engineering to which a student proposes to devote himself. Much, therefore, is excluded which might naturally be expected to form part of the work, simply because, however important, it is required only by a special class of students.

The introduction of descriptive details is not necessary to the plan of this work, except in certain parts of the theory of mechanism, nor, indeed, in a general treatise would it be possible to include them systematically within any reasonable compass. In the chapters on mechanism, however, they are required, and elsewhere it has been thought advisable to introduce them occasionally. Care has been taken to select working examples almost exclusively, the plates representing which have mostly been drawn by Mr. T. A. Hearson, to whom the author is indebted for many suggestions and portions of the descriptive matter, together with some assistance in revising proof sheets and transcribing lecture notes for the press. The proofs have been read by Professor W. C. Unwin, M.I.C.E., to whose great technical knowledge some corrections are due. In a general elementary work there is not room for much that is new: in the references at the end of each chapter and in the Appendix the various sources of information have been stated fully.

GREENWICH, May, 1884.

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WHAT are the conditions of a science? and when may any subject be said to enter the scientific stage? When the facts of it begin to resolve themselves into groups; when phenomena are no longer isolated experiences, but appear in connection and order; when, after certain antecedents, certain consequences are uniformly seen to follow; when facts enough have been collected to furnish a basis for conjectural explanation; and when conjectures have so far ceased to be utterly vague that it is possible in some degree to foresee the future by the help of them.—FROUDE.

A competent view of the world can never be had as a gift; we must acquire it by hard work.—MACH.

INTRODUCTION.

THE province of the Engineer and Architect is to control the forces of nature and apply them to useful purposes, an object which is effected by means of pieces of material suitably connected and arranged. The protection of life and property from destructive forces is accomplished by pieces rigidly connected with one another which transmit their action to bodies to which they are not injurious; while the utilization of such forces in moving weights, changing the form of bodies, and other similar operations, is effected by a set of moving pieces which transmit their action to the required place and modify it in some given way. In the first case the pieces are called, collectively, a STRUCTURE, in the second, a MACHINE. The object of the present work is to give an outline of the principles on which structures and machines are designed.

The actual form of such a construction is almost always the final result of a process of evolution by which it has been gradually perfected by adaptation from some previously existing construction. To meet new wants the engineer selects some arrangement, suggested by experience of some nearly similar case, which appears likely to answer the purpose by its simplicity, facility of construction, and adaptation to the forces which it is proposed to control and utilize. If the new arrangement is merely a copy of the old, this may be sufficient and the construction may be at once proceeded with, but if there be any important difference it is necessary, before incurring the expense and risk of actual construction, to ascertain that the design is in conformity with those general laws governing the action of natural forces upon matter which reason and experience alike show to be necessarily true in all cases. To a certain extent this has already been considered by the designer, whose knowledge and experience enable him to avoid at once arrangements which are obviously inadmissible, but complete conformity can only be secured by comparison with results deduced by reasoning and verified by experiment.

In any branch of knowledge the explanation of a set of facts by a general principle, from which new results can be obtained, is properly described as a Theory of the phenomena to which they relate. When its principles are well established it enables us to predict the results of experiment; when they are not, it is even more necessary, to direct the course which experiment should take for more perfect knowledge. The systematic study of structures and machines with a view to discover the theoretical principles on which their construction is based, and the deduction from those principles of results which may be useful to the designer, forms a branch of science which, following RANKINE, we may describe as Applied Mechanics. In some cases the subject may have been so exhaustively studied, and may be in its nature so limited, that all the arrangements which can be employed for a given purpose may be foreseen and the best determined by *à priori* considerations. The process of invention itself then becomes a problem in science. This, however, is the rare exception; in general, the use of theory is limited to the answering of certain questions relating to an arrangement which has already been proposed. *Among the most important of these are*—

(1.) What should be the dimensions of the parts of the construction that they may be strong enough to resist the action of the forces to which it is exposed?

(2.) Will the construction be sufficiently stable and rigid?

(3.) Are the natural forces, which it is proposed to utilize, sufficient for the proposed purpose and are they under proper control?

It is only in the very simplest cases that these and similar questions can be answered completely, without reference to the direct results of experience in order to interpret theoretical reasoning and render it applicable. Even, therefore, after the general plan of a construction is decided on, the work of the practical designer includes much which cannot be reduced to a mere process of deduction from given data. Nevertheless the part of theory in controlling and directing inventive power is of great and constantly-increasing importance, by furnishing principles of universal application, in conformity with which every mechanical construction must be designed, and by which the researches of the experimentalist must be guided.

The mechanics of structures and machines is based on the properties of materials, and on those general laws connecting matter and motion, the study of which is the object of Abstract Mechanics, but the special nature of the subject-matter occasions a certain difference in the methods employed. In the elementary branches of purely abstract mechanics the number of bodies considered seldom exceeds two; if more are introduced the questions to be considered become impracticably complex considered as abstract mathematical problems. In applied mechanics a number of pieces are connected with comparatively little freedom so as to form *an organic whole, and the results of experience* or of mathematical investigations too complex for ordinary use are admitted freely for the purpose of simplification. Hence the calculations employed are of a coarser type, and, in particular, graphical methods are everywhere employed when possible, not only to exhibit, but also to obtain, numerical results. On the other hand, no investigation is considered as complete until it has been checked by reference to experience, and unless its errors are approximately known. The elementary principles of abstract statics, dynamics, and hydrostatics must be supposed already known, and some practical knowledge of machines and structures is presupposed.

The classification of mechanical constructions depends in great measure on the number of pieces connected and on the mode of connection. We have first the broad distinction between structures, in which the pieces have no movements except such as may be due to changes in their form and dimensions consequent on the forces to which they are exposed, and machines in which the object is attained by means of such movements. This distinction is so fundamental that there is no word in common use which includes both.

Structures may be ranged in order of simplicity according to the degree of constraint with which their parts are connected as follows:—

(1.) Structures with pin joints without redundant parts.
(2.) Structures with pin joints which include redundant parts.
(3.) Blockwork and earthwork structures.
(4.) Structures with riveted or other forms of fastened joints.

A pin joint, such as is shown in a simple form in Figs. 1 and 2, Plate VIII., page 463, is one in which the pieces connected are united by a single pin fitting into holes in the pieces, and, in consequence, neglecting friction, the mutual action between the pieces connected necessarily passes through the axis of the pin. A redundant part is one which may be removed without destroying the structure if the remaining parts be sufficiently strong. The first class of structures therefore possess a peculiar characteristic which renders their theory much more simple than that of any other, namely, that the forces acting on each piece depend only on the external forces acting on the whole and not on the material or the dimensions of the pieces. In the theory of structures, then, this class is first considered, and the answer to the first of the general questions propounded above consists in the solution of two general problems.

(1.) Being given the load on the structure, it is required to find the forces acting on each part.

(2.) Being given the forces acting on a piece of material, it is required to find its dimensions that it may be sufficiently strong and stiff.

The first forms a part of the subject which may be properly described as the "Statics of Structures"; while the second, which depends on the properties of the materials of construction, is known as the "Strength and Stiffness of Materials." The results obtained are in continual requisition in the theory of the more complex structures, but require to be supplemented by further investigations and by results derived from direct experience, peculiar to each class. The present treatise, being simply introductory, refers to the more complex structures only incidentally.

A Machine is a structure the parts of which are in motion. The motion introduces new forces, often of great magnitude and importance, which must be taken into account in its design; but we have, in addition, to consider the third general question mentioned above, namely, the adaptation of the natural forces available, to the work which the machine has to do. The simplest machines consist chiefly of a number of rigid pieces, and their theory is divided into two parts—one concerned with the motion of the machine, the other with the work it does. In many of the most important machines fluids are used, and their theory forms a distinct branch of the subject not less important than the rest, some account of which is indispensable. Thus the whole subject is divided into five parts.

Since the parts of structures as well as machines possess, though to a very limited extent, freedom to move, and since such movements often have to be supposed for the purposes of an investigation, the most natural arrangement perhaps would be to commence with the first part of the theory of machines, and then pass on to the statics of structures. In the present treatise it has, however, been found convenient to invert this order, and we now, therefore, commence with structures.

PART I.—STATICS OF STRUCTURES.

CHAPTER I.

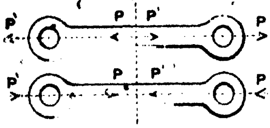
FRAMEWORK LOADED AT THE JOINTS.

1. *Preliminary Explanations and Definitions.*—A frame is a structure composed of bars, united at their extremities by joints, which offer no resistance to rotation. In the first instance we may suppose the centre lines of the bars all in one plane, and in that case the joints may consist simply of smooth pins passing through holes at the ends of the bars, which are to be imagined forked, if necessary, so as to allow the centre lines to meet in a point. A large and important class of structures, known to engineers as "trusses," approach so closely to frames that calculations respecting them may be conducted by treating them as if they were frames. The difference between a truss and a frame will appear as we proceed.

The frame may be acted on by forces applied at points in one or more of its bars, or at the joints which unite the bars together. An important simplification, however, is effected by supposing, in the first instance, that the joints only are loaded, an assumption which will be made throughout this chapter, except in a few simple examples. It will be shown hereafter (p. 75) that all other cases may be derived from this by means of a preliminary reduction.

Assuming, then, that the frame is acted on by forces at the joints, due either to weights or other external causes, or to the reaction of supports on which the frame rests, the problem to be solved is to find the forces called into play on each of the bars of which it is constructed. These forces are caused by the pressure of the pins on the sides of the holes through which they pass, and it at once follows, since no other forces act on the bar, that for each bar these pressures must be equal and opposite, their common line of action being the line joining the centres of the holes. There are two possible cases shown in Figs. 1a, 1b; in the first the bar is acted on by a pair of equal and opposite forces tending to lengthen it, and in the second to shorten it. The pairs of forces are called a Pull and a Thrust respectively, while the bars subjected to their action are called Ties and Struts respectively. Between a pull and a thrust there is no statical difference but that of sign; the constructive difference, however, between a tie and a strut is great. The first may theoretically be a rope or chain, and the second may be made up of pieces simply butting against one another without fastening, while a rigid bar will serve either purpose, though its powers of resistance are generally entirely different in the two cases.

Figs. 1a, 1b.



It often happens that it is unknown whether a bar be a strut or a tie, and the pair of forces are then called a STRESS on the bar. This word "stress" was introduced by Rankine to denote the mutual action between any two bodies, or parts of a body, and here means, in the first instance, the mutual action between the parts of the frame united by the bar we are considering. If, however, we imagine the bar cut into two parts, *A* and *B*, by any transverse section, as shown in Figs. 1a, 1b, those parts are held together in the case of a pull, or thrust away from each other in the case of a thrust, by internal molecular forces called into play at each point of the transverse section, and acting one way on *A* and the other way on *B*. As *A* and *B* must both be in equilibrium, it is obvious that these internal forces must be exactly equal to the original forces, and thus it appears that the stress on the bar may also be regarded as the internal molecular action between any two parts into which it may be imagined to be divided. Stress regarded in this way, will be fully considered in a subsequent division of this work; it will be here sufficient to say that its intensity is measured by dividing the total amount by the sectional area of the bar, and is limited to a certain amount, depending upon the nature of the material of which the bar is constructed.

It is further manifest from what has been said, that the stress on a bar may likewise be regarded as a mutual action between the bar and either of the pins at its ends which are pulled towards the middle of the bar in the case of a pull, or thrust away from it in the case of a thrust; each pin is therefore acted on, in addition to any load which may be suspended from it, by forces, the directions of which are the lines joining the centres of the pins, from which it follows at once that *every joint may be regarded as a point kept in equilibrium by the load at that joint and by forces of which the bars of the frame are the lines of application.* This principle enables us to find the stress on each bar of a frame loaded at the joints whenever such stress can be determined by statical considerations alone, without reference to the material or mode of construction, that is to say, in all cases which properly belong to the present division of our work.

Forces are measured in pounds, or, when large, in tons of 2240 lbs. They are often distributed over an area or along a line, and are then reckoned per square foot or per "running" foot, the last expression being commonly abbreviated to "foot-run."

The bars need not be connected by simple pin joints as has been supposed for clearness, provided that their centre lines if prolonged meet in a point through which passes the line of action of the load on the joint. This point may be called the centre of the joint, and we may replace the actual joint by a simple pin, or, if the bars are not in one plane, by a ball and socket which has the same centre. We shall return to this hereafter, but now pass on to consider various kinds of frames, commencing with the simplest.

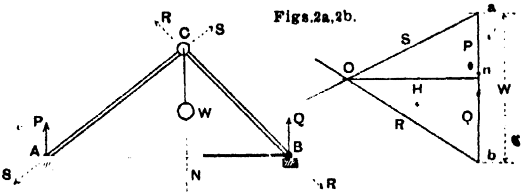
SECTION I.—TRIANGULAR FRAMES.

2. *Diagram of Forces for a Simple Triangular Frame.*—The simplest kind of frame is a triangle.

In Fig. 2a, ACB is such a triangle; it is supported at AB so that AB is horizontal, and loaded at C with a weight W. Then evidently the effect of the weight is to compress AC, BC, and to stretch AB, which is conveniently indicated by drawing AC, BC in double lines, and AB in a single line. Also the weight produces certain vertical pressures on the supports A, B, which will be balanced by corresponding reactions P and Q.

To find the magnitude of the thrust on AC, BC, the pull on AB, and the reactions, the diagram of forces Fig. 2b is drawn; ab is a vertical line representing W on any convenient scale, while aO, bO are lines drawn through a, b respectively, parallel to AC, BC to meet in O, and finally On is drawn parallel to AB, or, what is the same thing, perpendicular to ab. Now, applying the fundamental principle laid down above, we observe that C is a point kept in equilibrium by three forces, the load at C, namely W, the thrust of AC which we will call S, and the thrust of BC which we will call R. In the second figure the triangle

Oab has its sides parallel to these forces, and hence it follows that Oa, Ob represents S, R on the same scale that ab represents W. Again, A is a point kept in equilibrium by three forces, the thrust of AC, the pull of the tie AB, which we will call H, and the upward reaction P of the support A. But referring to the figure 2b, On, an are respectively parallel to the two last forces, so that, by the triangle of forces, they represent H, P on the same scale that Oa represents S. The same reasoning applies to the point B, and therefore bn represents the other supporting force Q, as is also obvious from the consideration that $P+Q=W$. We thus see that all the forces acting upon and within the triangular frame ACB are represented by corresponding lines in Fig. 2b,


Figs. 2a, 2b.

which is thence called the "diagram of forces" for the triangular frame. Such a diagram can be drawn for any frame, however complicated, and its construction to scale is the best method of actually determining the stresses on the several parts of the frame.

The force H requires special notice: it is called the "thrust" of the frame. In the present case the thrust is taken by the tension of the third side of the triangle, but this may be omitted, and the supports A and B must then be solid and stable abutments capable of resisting a horizontal force H. In many structures such a horizontal thrust exists; and its amount and the mode of providing against it are among the first things to be considered in designing the structure. Besides the graphical representation just given, which enables us to obtain the thrust of a triangular frame by constructing a simple diagram, it may also be calculated by a formula which is often convenient. Let AC be denoted by b and BC by a, as is usual in works on trigonometry, and let AN, BN their projections on AB be called b', a', and let the height of the triangle be h and its span l, then by similar triangles,

$$\frac{P}{H}=\frac{an}{On}=\frac{CN}{AN}=\frac{h}{b'},$$

$$\frac{Q}{H}=\frac{bn}{On}=\frac{CN}{BN}=\frac{h}{a'}.$$

Therefore, by addition,

$$\frac{W}{H}=\frac{ab}{On}=h\left(\frac{1}{b'}+\frac{1}{a'}\right)$$

$$\text{or}\quad H=W\frac{a'b'}{lh}.$$

In practical questions it often happens that a', b', h are known by the nature of the question, whence H is readily determined. The case when the load bisects the span may be specially noticed: then $a'=b'=\frac{1}{2}l$ and

$$H=\frac{Wl}{4h}.$$

When the height of the frame is small compared with the span, this calculation is to be preferred to the diagram, which cannot then be constructed with sufficient accuracy.

The simple frame here considered may be inverted, in which case the diagram of forces and the numerical results are unaltered, the only change being that the two struts have become ties and the tie a strut.

3. *Triangular Trusses.*—Triangular frames are common in practice, and the rest of this section will be devoted to some of the commonest forms in which they appear.

Fig. 3a shows a simple triangular truss consisting of a beam, AB, supported by a strut at the centre, the lower extremity of which is carried by tie roads, AC, BC, attached to the ends of the beam. If now a weight, W, be placed at the centre, immediately over the strut, it does not bend the beam (sensibly) as it would do if there were no strut, but is transmitted by the strut to the joint C, so that the truss is equivalent to the simple triangular frame of the last article. This, however, supposes that the strut has exactly the proper length to prevent any bending of the beam; if it be too short or too long the load on the frame will be less or greater than W, a point which will be further considered presently. It should be noticed that D is not necessarily at the centre.

Fig.3a.

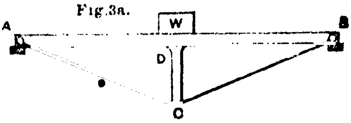


Fig.3b.

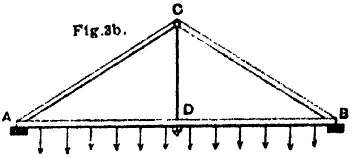


Fig. 3b shows the same construction inverted. CD is a tie by which D is suspended from C; we will suppose this rod to pass through AB and a nut applied below, by means of which D may be raised or lowered.

Let AB now be uniformly loaded with a given weight, then the bending of AB is resisted by CD, which supports it and carries a part of the load, which may be made greater or less by turning the nut. If, however, we imagine AB, instead of being continuous through D, to be jointed at D, then the tie CD necessarily carries half the weight of AD and half the weight of BD, that is to say, half the whole load, whatever be its exact length. This simple example illustrates very well the most important difference between a truss and a mathematical frame; namely, that in the truss one or more of the bars is very often continuous through a joint. Such cases can only be dealt with on the principles of the present division of our work, by making the supposition that the bar in question, instead of being continuous, is jointed like the rest. The error of such a supposition will be considered hereafter; it is sufficient now to say that in order that it may be exact in the particular case we are considering, the nut must be somewhat slackened out so that D may be below the straight line AB, and that being dependent on accuracy of construction, temperature, and other varying circumstances, such errors cannot be precisely stated, but must be allowed for in designing the structure by the use of a factor of safety. The supposition is one which is usual in practical calculations, and will be made throughout this division of our work.

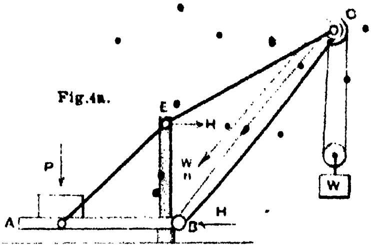
The foregoing is one of the simplest cases where, as is very common in practice, the bars of the frame are loaded and not the joints alone. When such bars are horizontal and uniformly loaded, the effect is evidently the same as if half the load on each division of the loaded bar were carried at each of the joints through which it passes. This is also true if the loaded bars be not horizontal, but the question then requires a much more full discussion, which is reserved for a later chapter (see Ch. IV.).

When one of the joints of the loaded bar is a point of support, like A in Fig. 3, the supporting force is due partly to the half weight of one or more divisions of the loaded bar, and partly to the downward pull or thrust of other bars meeting there: the first of these causes does not affect the stress on the different parts of the truss, and the calculations are therefore made without any regard to it. The explanations given in this article should be carefully considered, as they apply to many of the examples subsequently given.

The triangular truss in both the forms given in this article is frequently employed in roofs and bridges of small span, as well as for other purposes.

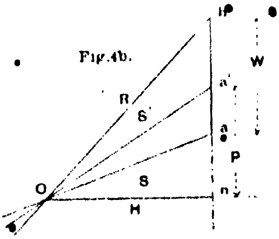
4. *Cranes.*—The arrangements adopted for raising and moving weights furnish many interesting examples of triangular frames. Fig. 4a shows one of the forms of the common crane, a machine the essential members of which are the jib, BC, supported by a stay, CE, attached to the crane-post, BE, which is vertical. In cranes proper this third member rotates, carrying BC and CE with it, but in the sailors' derrick a fixed mast plays the part of a crane-post and the stay, CE, is a lashing of rope frequently capable of being lengthened and shortened by suitable tackle, so as to raise and lower the jib, a motion very common in cranes and hence called a derrick motion. The weight is generally also capable of being raised and lowered directly by blocks and tackle, but for the present will be supposed directly suspended from C.

Fig. 4a.



The diagram of forces now assumes the form shown in Fig. 4b, in which the lettering is the same as in Fig. 2b, page 4, the only difference in the diagrams being that in the present case AC, which is now a tie, is divided into two parts, AE and EC, inclined at an angle. The stress on AE is therefore not the same as on EC, but is got by drawing a third line, Oa', parallel to AE. The perpendicular Oa gives us in this instance not only the stress on AB and the horizontal thrust of CB at B, but also the horizontal pull of CE at E: we may call this H as before. There is an upsetting moment on the structure as a whole which is equal to the product of the weight W by its horizontal distance from B (often called the radius of the crane) and also to the force H, multiplied by the length of the crane-post, BE. One principal difference between different types of cranes lies in the way in which this upsetting moment is provided against.

Fig. 4b.

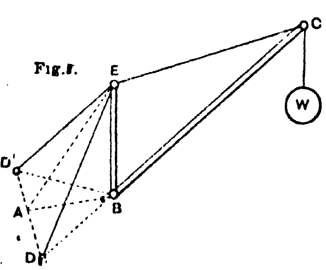


(α.) In portable cranes, such as shown in Fig. 4a, there is a horizontal platform, AB, supported by a stay, AE, and carrying a counterbalance weight, P, sometimes capable of being moved in and out so as to provide for different loads. The right magnitude of counterbalance weight and the pull on the stay, AE, are shown by the diagram, P corresponding to the supporting force at A in the previous case.

(β.) In the pit crane, the post is prolonged below into a well and the lower end revolves in a footstep, the upper bearing being immediately below *B*. In this instance the post has to be made strong enough to resist a bending action at *B*, equal to the upsetting moment, and the bearings have to resist a horizontal force equal to *H* multiplied by the ratio of the length of the crane-post, *BE*, to that of its prolongation below the ground.

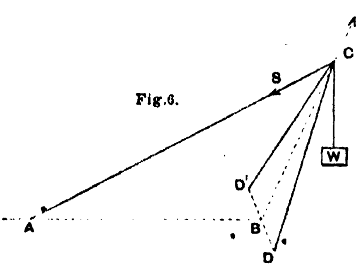
(γ.) The upper end of the crane-post may revolve in a headpiece, which is supported by a pair of stays anchored to fixed points in the ground. The upright mast of a derrick frequently requiring support in the same way, this arrangement is known as a derrick crane. It is shown in Fig. 5, *ED*, *ED'* being the stays. To find the stress on the stays it is necessary to prolong the vertical plane through *EC*, to intersect the line *DD'*, joining the feet of the stays in the point *A*, and imagine the two stays, *ED*, *ED'* replaced by a single stay, *EA*: then a diagram of forces, drawn as in the previous case, determine *S'*, the pull on this stay. But it is clear that *S'* must be the resultant pull on the two original stays, and may be considered as a force applied at *E* in the direction of *AE* to the simple triangular frame *DED'*. A second diagram of forces therefore will determine the pull on each stay, just as in the next following case.

Fig. 5.



5. *Sheer Legs and Tripods.*—Instead of employing an upright post to give the necessary lateral stability to the triangle, one of its members may be separated into two. Thus in moving very heavy weights sheer legs are used, the name being said to be derived from their resemblance to a gigantic pair of scissors (shears) partly opened and standing on their points. In Fig. 6, *CD*, *CD'* are spars, or tubular struts, often of great length, resting on the ground at *DD'* and united at *C*, so as to be capable of turning together about *DD'* as an axis. The load is carried at *C* and the legs are supported by a stay, *CA*, which is sometimes replaced by a rope and tackle, capable of being lengthened or shortened

Fig. 6.



so as to raise or lower the sheers. Drawing *AB* to the middle point of *DD'*, the pair of legs are to be imagined replaced by a single one, *CB*, then the diagram of forces may be constructed just as in Fig. 4b, and we shall obtain the tension of the rope *S* and the resultant thrust on the pair of legs *R*. Now draw the triangle *CDD'*, as in Fig. 7a, and imagine it loaded at *C* with a weight, *R*, then drawing the diagram of forces, Fig. 7b, we get *R'* the thrust on each leg. The horizontal force, *H'*, in this second diagram represents the tendency of the feet of the legs to spread outwards laterally, while the force *H* of the original diagram represents their tendency to move inwards perpendicular to *DD'*. In some cases the guy rope and tackle, *CA*, are replaced by a third leg called the back leg, and the sheers are then raised and lowered by moving *A* by a large screw; the force *H* is then also the force to be overcome in turning the screw.

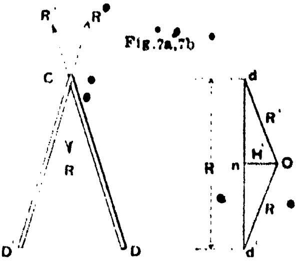


Fig. 7a, 7b

Instead of having only two legs, as in sheers, we may have three forming a tripod. This arrangement is frequently used to obtain a fixed point of attachment for the tackle required to raise a weight, and is sometimes called a "gin," or as military engineers prefer to spell the word, a "gyn." The thrust on each leg and the tendency of the legs to move outwards can be obtained by a process so similar to that in the preceding examples that we need not further consider it.

6. *Effect of the Tension of the Chain in Cranes.*—In most cases the load is not simply suspended from *C* as has been hitherto supposed, but is carried by a chain passing over pulleys and led to a chain barrel, generally placed somewhere on the crane-post. The tension of the chain in this case is W/n, where n is a number depending on the nature of the tackle, and this tension is to be considered as an additional force applied at *C* to be compounded with the load *W*, the effect of which has been previously considered. Fig. 8 shows the form the diagram of forces assumes in this case. Drawing *ba* as before to represent *W*, and *aa'* parallel to the direction in which the chain is led off from the pulley at *C* and equal to the tension W/n, the third side of the triangle *ba'*, must be the resultant force at *C* due to both forces, whence drawing *a'O* parallel to the stay and *bO* parallel to the jib, and reasoning as before as to the equilibrium of the forces at *C*, we see that these lines must be the tension of the stay and the thrust on the jib.

The effect of the tension of the chain is generally to diminish the pull on the stay and increase the thrust on the jib, sometimes very considerably, as for example in certain older types of crane still used for light loads under the name of "whip" cranes. In these cranes the chain passes over a single fixed pulley at the end of the jib, and is attached directly to the weight, so that the tension of the chain is equal to the weight. The other end of the chain is led off along a horizontal stay to a wheel and axle at the top of the crane-post, a chain from the wheel of which passes to a windlass below. This arrangement, the double windlass of which facilitates changes in the lifting power corresponding to the load to be raised, is a development of the primitive machine in which the wheel was a tread wheel worked by men or animal power. In this case the pull on the stay is diminished by the whole weight lifted, and is thus reduced very much. Where a crane has to be constructed of timber only, this is a considerable advantage, from the difficulty of making a strong tension joint in this material.

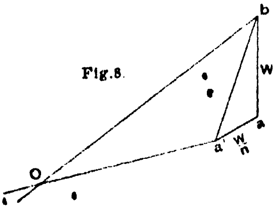


Fig. 8.

EXAMPLES.

1. The slopes of a simple triangular roof truss are each 30°. Find the thrust of the roof and the stress on each rafter when loaded with 250 lbs. at the apex.

Thrust of roof = 216·5 lbs.
Stress on rafters = 250 ,,

2. A beam 15 feet long is trussed with iron tension rods, forming a simple triangular truss 2 feet deep. Find the stress on each part of the frame when loaded with 2 tons in the middle.

Thrust on strut = 2 tons.
Pull on tension rods = 3·88 ,,
Thrust on beam = 3·75 ,,

3. The platform of a foot bridge is 20 feet span, and 6 feet broad, and carries a load of 100 lbs. per sq. ft. of platform. It is supported by a pair of triangular trusses each 3 feet deep, one on each side of the bridge. Find the stress on each part of one of the trusses.

The whole load of 12,000 lbs. rests equally on the two trusses, there is therefore 6000 lbs. distributed uniformly along the horizontal beam of each truss

Thrust on strut = 3,000 lbs.
Tension of tie rods = 5,220 ,,
Thrust on horizontal beam = 5,000 ,,

4. The slopes of a simple triangular roof truss are 30° and 45° and span 10 ft. The rafters are spaced 2½ feet apart along the length of the wall, and the weight of the roofing material is 20 lbs. per sq. ft. Find by graphical construction the thrust of the roof.

Each rafter carries a strip of roof 2½ feet wide, the load on rafter = 50 lbs. per foot length of rafter. Find the lengths by construction or otherwise. The virtual load at apex = ½ weight on the two rafters = 311 lbs.

Thrust of roof = 198 lbs.

5. The jib AC of a ten-ton crane is inclined at 45° to the vertical, and the tension rod BC at an angle of 60°. Find the thrust of the jib, and the pull of the tie rod when fully loaded, the tension of the chain being neglected. If a back stay BD be added inclined at 45°, and attached to the end of a horizontal strut AD, find the counterbalance weight required at D to balance the load on the crane, and find also the tension of the back stay.

Thrust on jib AC	33·5 tons.
Tension of tie rod	= 27·5 ,,
Counterbalance weight	= 23·5 ,,
Tension of back stay	= 33·5 ,,

6. A pair of sheer legs are 40 feet high when standing upright, the lower extremities rest on the ground 20 feet apart, the legs stand 12 feet out of the perpendicular, and are supported by a guy rope attached to a point 60 feet distant from the middle point of the feet. Find the thrust on each leg, and the tension of the guy rope under a load of 30 tons.

Thrust on each leg	19·5 tons.
Tension of guy rope	= 12·8 ,,

7. In example 5 the tension of the chain is half the load, and the chain barrel is so placed that the chain bisects the crane post AB. Find the stress on the jib and tie rod.

Thrust of jib	= 36 tons.
Pull of tie rod	25 ,,

8. In a derrick crane the projections of the stays on the ground form a right-angled triangle each of the equal sides of which is equal to the crane-post. The jib is inclined at 45° and the stay at 60° to the vertical. Find the stress on all the parts (1) when the plane of the jib bisects the angle between the stays; (2) when it is moved through 90° from its first position. Load 3 tons.

Answer.—	Case 1. Pull on each stay	7·1 tons.
	Case 2. Pull on one = thrust on other	7·1 ,,

9. A load of 7 tons is suspended from a tripod, the legs of which are of equal length and inclined at 60° to the horizontal. Find the thrust on each leg. If the load be removed and a horizontal force of 5 tons be applied at the summit of the tripod in such a way as to produce the greatest possible thrust on one leg, find that thrust and determine the stress on the other two legs.

Answer.—	Case 1. Thrust on each leg	2·7 tons.
	Case 2. Thrust on one leg	6·7 ,,
	Pull on each of the others	3½ ,,

SECTION II. INCOMPLETE FRAMES.

7. *Preliminary Remarks.*—A frame may have just enough bars and no more to enable it to preserve its shape under all circumstances, or the number of bars may be insufficient or there may be redundant bars. The distinction between these three classes of frames is very important: in the first the structure will support any load consistent with strength, and the stress on each bar bears a certain definite relation to the load, so that it can be calculated without any reference to the material or mode of construction; in the second, the frame assumes different forms according to the distribution of the load, but the stress on each bar can still be calculated by reference to statical considerations alone; in the third, where the frame has redundant bars, the stress on some or all of the bars depends on the relative yielding of the several bars of the

frame. It is to the second class, which may be called "incomplete" frames, that the present section will be devoted.

In incomplete frames the structure changes its form for every distribution of the load, and, strictly speaking, therefore, such constructions cannot be employed in practice, because the distribution of the load is always variable to a greater or less extent. But when the greater part of the load is distributed in some definite way the principal part of the structure may consist of an incomplete frame, designed for the particular distribution in question, and subsequent moderate variations of distribution may be provided for either by stiffening the joints or by subsidiary bracing. Such cases are common in practice, and investigations relating to incomplete frames are therefore of much importance.

8. *Simple Trapezoidal or Queen Truss.*—We will first consider a frame which is composed of four bars. The most common case is that in

Fig. 9a.

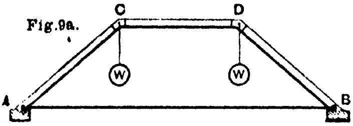
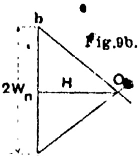


Fig. 9b.



which of the bars are horizontal and the other two equal to one another, thus forming a trapezoid. The structure is called a *trapezoidal frame*.

It is suitable for carrying weights applied at the joints CD, either directly or by transmission through vertical suspending rods from the beam AB. From the symmetry of the figure it is evidently necessary for stability that the loads at C and D should be equal. This fact will also appear from the investigation. Consider first the joint C, and draw the triangle of forces, Oan, for that point; an being taken to represent W, aO will represent the thrust on AC and On that along CD. The triangle Obn will represent the forces at the joint D, Ob representing the thrust of BD; bn will represent the load at D, and from the symmetry of the figure must equal an, and hence weight at D must for equilibrium equal that at C. Now let us proceed to joint A, where there are also three forces acting, one along AC is now known and represented by aO, thus On will represent the tension of AB, and an will be the necessary supporting force at A equal to W, as might be expected. The tension of AB is equal to the thrust on CD. We observe that the diagram of forces is the same as that of a triangular frame, carrying $2W$ at the vertex and of span equal to the difference between AB and CD.

Trapezoidal frames are employed in practice for various purposes.

(α.) A beam, AB (Fig. 10a), loaded throughout its length may be strengthened by suspending pieces, CN, DM, transmitting a part of the weight to the arch of bars, AC, CD, BD, an arrangement common in small bridges.

Fig. 10a.

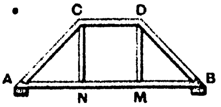
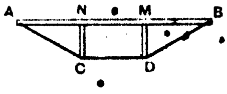


Fig. 10b.



(β.) As a truss for roofs, in which case there will be a direct load at C and D due to the weight of the roofing material, while vertical members serve partly as suspending rods by which part of the weight of tie beam and ceiling (if any) is transmitted to CD, and partly to enable the structure to resist distortion under an unequal load. When made of wood, this is the old form of roof called by carpenters a "Queen Truss," CN, DM, being the "queen posts" (see Section III. of this chapter). This name is constantly used for all forms of trapezoidal truss erect or inverted which include the vertical "queens."

(γ.) Not less common is the inverted form, Fig. 10b, applied to the beams carrying a traversing crane, the cross girders which rest on the main girders of a railway bridge and carry the roadway, and many other purposes. The bars AC, CD, BD are now iron tie rods. In this case also if the two halves of the beam are unequally loaded there will be a tendency to distortion, to resist which completely, diagonal braces, CM, DN, must be provided, as shown in the figure by dotted lines. Such diagonal bars occur continually in framework, and their function will be fully considered in the next chapter. But in the present case they are quite as often omitted, the heavy half of the beam then bends downwards and the light half bends upwards (see Ex. 4, p. 87), but the resistance of the beam to bending is found to give sufficient stiffness.

9. *General case of a Funicular Polygon under a Vertical Load. Example of Mansard Roof.*—We next take a general case. In Fig. 11a, 0 1 2 3 ... 6 is a rope or chain attached to fixed points at its ends and loaded with weights, W_1 W_2..., suspended from the points 1, 2, etc. The figure shows 5 weights, but there may be any number. The rope hangs in a polygon, the form of which depends on the proportions between the weights. It is often called a "funicular polygon" and possesses very important properties. We shall find it convenient to distinguish the sides of this polygon by letters a, b, c, etc. We are about to determine the proportions between the weights when the rope hangs in a given form, and, conversely, the form of the rope when the weights are given.

In Fig. 11b draw ab vertical to represent W_1, the load suspended at the angle of the polygon where the sides a and b meet, then draw aO, bO parallel to a, b respectively to meet in O, thus forming a triangle Oab, which we distinguish by the number 1, which represents the forces acting on the point 1, so that the tensions of the sides a, b are thus determined. Now draw Oc parallel to the side c to meet the vertical in c; we thus obtain a triangle distinguished by the number 2, which represents the forces acting at that point, and as Ob is already known to be the tension of b it follows that bc must be the weight W_2, and Oc the tension of the side c. Proceeding in this way we get as many triangles as there are weights, and the sides of these triangles must represent the weights and the tensions of the parts of the rope to which they are respectively parallel. Thus, if the form of the rope is known and one of the weights, all the rest can be determined. Conversely, to find the form of the funicular polygon when the weights are given in magnitude and line of action, we have only to set downwards on a vertical line the weights in succession and join the points a b..., which will now be known, to any given point O, then the funicular polygon must have its angles on the lines of action of the weights and its sides parallel to the radiating lines Oa, Ob, Oc, etc., so that the sides can be drawn in succession, starting from any point we please.

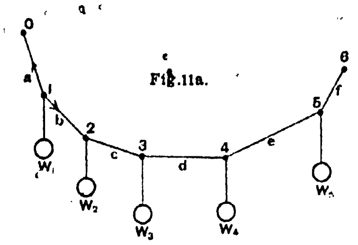


Fig. 11a.

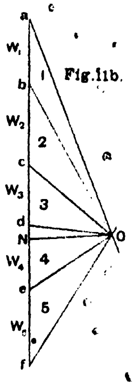


Fig. 11b.

In the diagram of forces, Fig. 11b, if ON be drawn horizontal to meet the vertical a, b, c... in N, this line must represent the horizontal tension of the rope.

The rope may be replaced by a chain of bars which may be inverted, thus forming an arch resting on fixed points of support, the diagram of forces will be unaltered, and ON will represent the thrust of the arch. As an elementary example of an arch of bars we will consider a truss used for supporting a roof of double slope called a Mansard roof. We will take the usual case in which the truss is symmetrical about the centre. Suppose it is loaded at the joints. There is one proportion of load which the truss is able to carry without any bracing bars being added.

From symmetry the weights at 2 and 2′ (see Fig. 12a) must be equal. To find the portion between the weights at 1, and at 2 2′, together with the stresses on the bars of the frame, in Fig. 12b set down aa' to represent W at 1, and draw aO and $a'O$ parallel to a and a', the thrusts along these bars will be determined. Then, considering the equilibrium of either 2 or 2′, say 2, one of the three forces acting at the joint, namely aO, along the bar a being known, the other two forces may be

Fig. 12a.

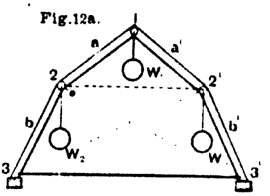
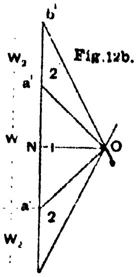


Fig. 12b.



determined by drawing ab and Ob parallel to them, ba parallel to W_2, and Ob to the bar b. If ON be drawn horizontally it will give the amount of the horizontal thrust of the roof or the tension of a tie bar 3 3′, if there is such a bar. If the proportion of W_2 to W_1 is greater than ab to aa' the structure will give way by collapsing, 2 and 2′ coming together; and if the proportion is less, the structure will give way by 2 and 2′ moving outwards and 1 falling down between. In practice it is impossible to secure the necessary proportion of loads, on account of variation of wind pressure and other forces, and therefore stiffening of some kind is always needed. If bracing bars be placed as shown by the dotted lines 2 3′, 2′ 3, 2 2′, the structure will stand whatever be the proportion between the loads. The truss may be partially braced by the horizontal bar 2 2′ only. Then the proportion between the loads W_1 and W_2 may be anything we please, but the loads at 2 and 2′ must be equal, at least theoretically, but in practice the stiffness of the joints will generally be sufficient for stability, especially if vertical pieces be added connecting these points to the tie beam as in a queen truss.

10. *Suspension Chains. Arches. Bowstring Girders.*—We now go on to consider another important example, in which the number of bars composing the frame is very much increased, as found in the common suspension bridge.

Let *AB* (Fig. 13a) be the platform of a bridge of some considerable span, which has little strength to resist bending. Suppose it divided into a number of equal parts, an odd number for convenience, say nine, and each point suspended by a vertical rod from a chain of bars secured at the end to fixed points, *D* and *E*, in a horizontal line. In the figure only half the structure is shown. Suppose the platform loaded with a uniformly distributed weight; we require to know the stress on

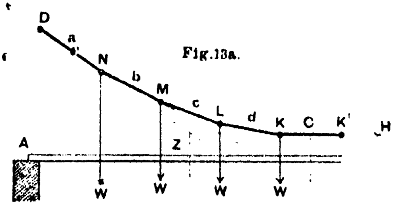


Fig.13a.

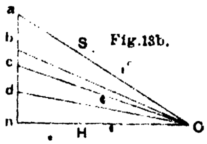


Fig.13b.

each bar and the form on which the chain will hang. Equal weights on each division of the platform will produce equal tensions in the vertical suspending rods, and if we neglect the differences of weight of the rods and bars themselves, the load at each joint of the chain of bars will be the same. (Comp. Art. 11.) Let W = load at each joint. Now the centre link *KK'*, since there is an odd number and the chain is symmetrical, will be horizontal. Let us consider the equilibrium of the half chain between *C* and *D*. The four weights, *W*, hanging at *K*, *L*, *M*, *N*, are sustained in equilibrium by the tension of the bars *KK'* and *ND*.

The resultant of the four *W*'s will act at the middle of the third division from the left end, and since this resultant load together with the tensions of the middle and extreme links maintain the half chain in equilibrium, the three forces must meet in a point, the point *Z* shown in the figure. Thus the direction of the extreme link *DN* may be drawn. The direction and position of the other links may be found also. Considering the portion of the chain *NC* carrying three weights, the resultant of which is in the line through *L*, the link *NM* must be in such a direction as to pass through the point where this resultant cuts *KK'* produced. Having drawn *NM*, *ML* may be drawn in a similar way, and then *LK*. Returning to the consideration of the half chain, the three forces which keep it in equilibrium may be represented by the three sides of a triangle. Set down *an* (Fig. 13b) to represent

$4W$, and draw aO and nO parallel to DZ and ZC; aO will be the tension of DN and nO of KK'. If an be divided into 4 equal parts, and the points b, c, d joined to O, these lines will represent the tensions of links NM, ML, and LK. It may be easily shown that they will be parallel to those links. We see that the tension increases as we pass from link to link, from the centre to the ends.

In many cases in practice, the number of vertical suspending rods and links in the chain is very great. We may then, in what follows, without sensible error, regard the chain as forming a continuous curve.

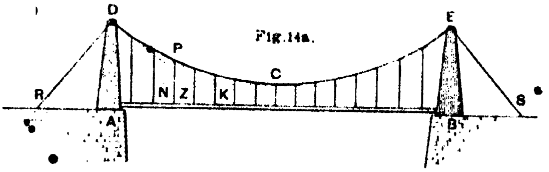


Fig. 14a.

In such a case, C, the lowest point of the chain (Fig. 14a), is over the middle of the platform. The tangent at C, which is horizontal, will meet the tangent to the chain at D, in a point Z, which will be over the middle of the half platform, for that will be a point in the line of action of the resultant load on the half chain. We can now draw a triangle of forces anO, for the half chain as before; On will represent the tension of the chain at the lowest point, or the horizontal component of the tension of the chain at any point. We can easily obtain a convenient expression for this horizontal tension thus:—Let l = span of the bridge, and w = load per foot-run. Then $\frac{1}{2}wl$ = weight on the half chain represented by an. Let H = horizontal tension, then

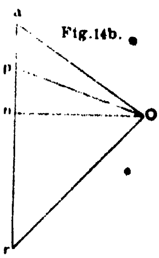


Fig. 14b.

$$\frac{H}{\frac{1}{2}wl} = \frac{On}{an}.$$

But if we drop a perpendicular from D to cut the horizontal tangent in a point V (not shown in the figure), DV will be the dip of the chain d, and comparing the triangles DVZ, aOn,

$$\frac{On}{an} = \frac{VZ}{DV} = \frac{\frac{1}{4}l}{d} = \frac{H}{\frac{1}{2}wl};$$

$$\therefore H = \frac{1}{8}wl\frac{l}{d},$$

which, since wl = total load on chain, may be written

$$H = \frac{1}{8}\text{ load on chain }\frac{\text{span}}{\text{dip}}.$$

This is the same as the horizontal thrust of a triangular frame of the same height which carries a uniformly distributed load of the same intensity.

Having found the magnitude of the horizontal tension of the chain we can calculate the tension at D, the highest point of the chain. Let S be this greatest tension, represented in the diagram of forces by aO, then since $\overline{aO}^2 = \overline{an}^2 + \overline{nO}^2$

$$S^2 = \left(\frac{W}{2}\right)^2 + H^2.$$

The tension at any point P of the chain may be found by drawing from O a line op parallel to the tangent to the chain at P. It will cut an in a point p such that $np : na$: : length of platform below PC : ½ span.

$$\text{Since } \overline{Op}^2 = \overline{np}^2 + \overline{On}^2,$$

$$\text{Tension at } P = \sqrt{\left(\frac{np}{na}\frac{W}{2}\right)^2 + H^2}.$$

The loaded platform, instead of being suspended from the chain of bars, may rest by means of struts on an arch of bars as in the figure.

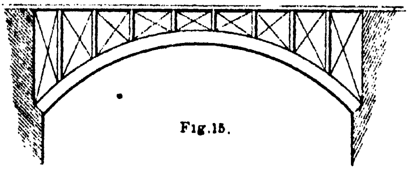


Fig. 15.

In this case all the bars will be in compression instead of tension, as in the previous case. If the form of the arch is similar to that in which the chain hung, it will have no tendency to change its form under the load. There will be simple thrust of varying amount at different parts of the arch. The horizontal thrust at the top of the arch is given by the same expression as for the horizontal tension of the chain, and the thrust of any bar of the arch may be determined in a manner similar to that for finding the tension of any link of a chain. We shall show presently that the proper form of the arch and chain under a uniform load is a parabola. Hence, the structure just described is called a Parabolic Arch. In iron bridges the platform is not unfrequently carried by a number of ribs placed side by side. Each rib is approximately parabolic in form, usually of I section, of depths from $\frac{1}{40}$th to $\frac{1}{80}$th the span at the crown, increasing somewhat towards the abutments. The roadway is supported sometimes by simple vertical struts, as in the ideal case just considered, sometimes by spandrils of more complex form, chiefly for the sake of appearance. When uniformly loaded, the stress on the ribs is nearly as found above: for resistance to variations in the load reliance is placed on the resistance to bending of the ribs and platform. The case of a stone or brick arch is far more complex, and *is not considered here.*

There is yet another very common structure designed on the same principles. In this the platform, instead of resting on an arch below, it, is suspended from an arch above it. In this case the thrust of the arch is taken by the platform, which serves as a tie, just as the string ties together the ends of a bow. Hence it is called a *Bowstring Girder.* In this case, as in the others, the loading proper to the parabolic form is uniformly distributed, and any variation of the loading will tend to distort the bow. The structure may, however, be enabled to sustain a varying load by the addition of bracing bars as shown by the diagonal lines. When the bridge is heavily loaded it will almost always happen that the greater part of the weight is uniformly distributed, and is sustained by simple thrust of the arch, so that the bracing is only a subsidiary part of the structure.

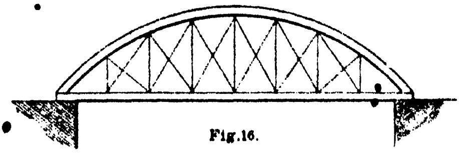


Fig. 16.

11. *Suspension Chains (continued). Bowstring Suspension Girder.*—In describing the supension bridge we spoke of the chain as being secured at the ends to fixed points. In practice the securing of the ends is effected thus. The chain is led to the top of a pier of cast-iron or masonry, and instead of being simply attached to the top of the pier, and thus producing an enormous tendency to overturn the pier, the chain is secured to a saddle which rests on rollers on the top of the pier, and on the other side the chain is prolonged to the ground, passes through a tunnel for some little distance, and is finally secured by means of anchors to a heavy block of masonry. By this arrangement the only force acting on the pier is a purely vertical one, and a comparatively slender pier will be sufficient to sustain it. It is not necessary that the tension of the chain should be the same on each side of the pier, or that it should be inclined at the same angle. What is necessary is that the horizontal component of the tension on each side should be the same. If an (Fig. 14b, page 17) = half weight on chain as before, and $On = H$, the horizontal tension (which may either be calculated from the formula just obtained, or found by construction), then aO will be the pull of the chain S at the top of the pier. Then

considering the equilibrium of the saddle, the pull of the chain Q on the short side and the upward reaction of the pier may be found by completing the triangle of forces aOr; Or will be the pull on the anchor, and ar the total vertical pressure on the pier.

In connection with this description of the method of securing the ends of the suspension chain, we may mention a form of structure in which the arch and chain are combined, a good example of which occurs in the railway bridge at Saltash. The horizontal pull of the chain is here balanced by the thrust of an arch, so that the combined effect is to produce simply a vertical pressure on the piers. The suspending rods are secured to the chains and prolonged to the arch above, so that a portion of the load is carried by the arch, producing a thrust, and a portion by the chain, causing a pull. To prevent any tendency to overturn the piers (this is ensured by means of saddles resting on rollers) the horizontal component of the thrust of the arch must equal the horizontal component of the pull of the chain. The proportion between the loads on arch and chain will depend on the proportion between the rise of the arch and dip of the chain.

If W_1 = load on arch, and W_2 = load on chain,

d_1 = rise of arch, and d_2 = dip of chain,

then
$$H = \frac{W_1 l}{8d_1} = \frac{W_2 l}{8d_2}; \quad \therefore \frac{W_1}{W_2} = \frac{d_1}{d_2};$$

also
$$W_1 + W_2 = \text{total load on bridge}:$$

from which the stresses on the structure may be determined. It is known as a Bowstring Suspension Girder (pp. 42, 71).

We shall next show that the form of the curve of a chain carrying a uniformly loaded platform is a parabola. Referring to Fig. 14a, let P be any point in the chain, drop a perpendicular PN to meet the tangent at C, and bisect CN in K. Then KP must be the direction of the pull of the chain at P in order that the portion PC may be kept in equilibrium. The triangle PNK has its sides parallel to the three forces, which act on PC, and the sides are therefore proportional to the forces. Let $CN = x$ so that the load hanging on $PC = w.x$, also let $PN = y$.

Then
$$\frac{H}{wx} = \frac{NK}{PN} = \frac{\frac{1}{2}x}{y};$$

$$\therefore x^2 = \frac{2H}{w} y; \text{ or, since } H = \frac{wl^2}{8d},$$

$$x^2 = \frac{l^2}{4d} y;$$

therefore x^2 is proportional to y.

Now the curve whose co-ordinates have this relation one to another is called a *parabola*.

If the load, instead of being uniformly distributed on a horizontal platform, were simply due to the weight of the chain itself, then the curve in which the chain would hang would deviate somewhat from the parabola; for in that case, since the slope increases as we approach the piers, the load also, per horizontal foot, would increase as we approach the piers, causing the chain near the piers to sink and become more rounded, and at the centre to rise and become more flattened. The curve in which the chain hangs by its own weight is called the *catenary*. In the catenary, as in the parabola, the tension increases as we approach the piers. This may be taken account of by proportioning the section of the chain to the tension at the various points; this would tend still more to make the weight of chain, per horizontal foot, increase as we approach the piers, and cause the chain to deviate still further from the parabolic form. Such a curve is called a catenary of uniform strength.

In an actual suspension bridge, where there is a uniformly loaded platform as well as a heavy chain, the true curve in which it hangs will lie somewhere between the parabola and the catenary; but since in most cases the deviation from uniformity of the weight of chain is small compared with the load it carries, the deviation from the parabola is not great. The error involved in assuming the curve to be parabolic is generally greatest in bridges of large span: in such cases a preliminary calculation of approximate weights may be necessary so as to be able to apply the general process of article 9.

EXAMPLES.

1. A trapezoidal truss is 16 feet span and 4 feet deep, the length of the upper bar is 6 feet. Find the stress on each part when loaded with 2 tons at each joint.

Stress on sloping bars = 3·2 tons.
,, horizontal bars = 2·5 tons

2. The platform of a bridge, 8 feet broad and 27 feet span, is loaded with 150 pounds per square foot. It is supported on each side by an inverted queen truss placed below, the queen posts, each 3 feet deep, dividing the span into three equal portions. Find the stress on each part.

Load on each truss = half the whole load on platform = 16,200.
$\frac{1}{3}$ 16,200 = 5,400 is the load at each of the two joints of one of the queen trusses.
Tension of sloping bars = 17,074 lbs.
Tension and thrust of horizontal bars = 16,200

3. The height of a Mansard roof without bracing is 10 feet and span 14 feet. The height of the triangular upper portion is 4 feet and span 8 feet. The load being 1 ton at the ridge, find the necessary load at each intermediate joint and the thrust of the roof.

By the construction described in the text, load at each intermediate joint = $\frac{1}{2}$ ton, and the thrust of the roof = $\frac{1}{2}$ ton.

4. If the roof in the last question be partly braced by a bar joining the intermediate joints, find the stress on the bar when the load at each intermediate joint is 1 ton.

Thrust on bar $= \frac{1}{2}$ ton.

5. The load on the platform of a suspension bridge, 600 feet span, is $\frac{1}{2}$ ton per foot-run, inclusive of chains and suspending rods. The dip is $\frac{1}{13}$th the span. Find the greatest and least tensions of one of the chains.

Least tension $=$ horizontal tension $= 243\frac{3}{4}$ tons.
Greatest tension $= 255$ tons.

6. The load on a simple parabolic arch, 200 feet span and 20 feet rise, is 360 tons, determine the thrust and greatest stress on the arch.

Thrust $= 450$ tons; greatest stress $= 484$ tons.

7. The rise of a bowstring bridge is 15 feet and span 120 feet, find the thrust when the load on each girder is 2,000 lbs. per foot-run.

Thrust 240,000 lbs. $= 107\frac{1}{7}$ tons.

8. In example 5 the ends of the chain are attached to saddles resting on rollers on the tops of piers 50 feet high, and prolonged to reach the ground at points 50 feet distant from the bottom of the piers, where they are anchored. Find the load on the piers and the pull on the anchors.

Load on the pier $= 637\frac{1}{2}$ tons;
Pull on each anchor $= 344{\cdot}6$ tons.

9. A light suspension bridge is to be constructed to carry a path 8 feet broad over a channel 63 feet wide by means of 6 equidistant suspending rods, the dip to be 7 feet. Find the lengths of the successive links of the chain. Supposing a load of 1 cwt. per square foot of platform, find the sectional areas of the links of the chain, allowing a stress of 4 tons per square inch.

$\frac{6}{7}$ of the whole load is carried by the chains and the remaining portion by the piers directly. Tension of each suspending rod $= 36$ cwt.

Links.	Tensions.	Areas.	Lengths.
Centre,	277·7	3·47	9·
2nd,	280·	3·5	9·08
3rd,	287·	3·6	9·3
4th,	298·	3·72	9·66

10. Construct a parabolic arch, the thrust of which is half the total load.

Span $=$ four times the rise.

11. If a weight of a uniformly loaded platform be suspended from a chain by vertical rods, show that the corners of the funicular polygon lie on a parabola.

SECTION III.—COMPOUND FRAMES.

12. *Compound Triangular Frames for Bridge Trusses.*—By a compound frame is meant a frame formed from two or more simple frames by uniting two or more bars. Many frames of common occurrence in practice may conveniently be considered as combinations of the simpler

examples already described. They are generally dealt with by use of what we may call the principle of superposition, which may be thus stated:—*The stress on any bar due to any total load is the algebraical sum of the stresses due to the several parts of the load.*

We will now consider some examples of compound frames which are used in bridge trusses. In these structures the object is to carry a distributed load by means of a comparatively slender beam. A prop in the centre may still leave the halves too weak to carry the weight on them, and the beam may be strengthened by supporting it in more than one point.

(1) Suppose the beam supported by a number of equidistant struts, the lower ends of which are carried by tension rods attached to the ends of the beam, we then have a structure called a *Bollman truss.* There may be any number of struts—2, 3, 4, or more; the structure has been used for bridges of comparatively large span. If the actual load is distributed in some manner over the beam, we must first reduce the case to that of a structure loaded at the joints only. The loads on the struts are due to the weights resting on the adjacent divisions of the beam, and may be determined by supposing the beam broken or jointed at the points where the struts are applied.

Let us suppose the beam has three divisions, and that the load on the two struts are W_1 and W_2. These loads will be transmitted down the struts to the apices (Fig. 17) E and F, and will be independently supported, each by its own pair of tension rods.

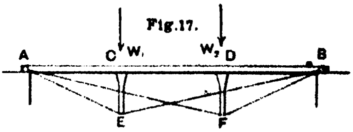


Fig. 17.

We may thus separately determine the stress on each part of either of the elementary triangular frames, AEB or AFB. AB will be in compression on account both of the load at E and also at F. On account of W_1, using the formula previously obtained, the horizontal thrust $H_E = W\frac{a'b'}{lh}$, and on account of W_2 *at* F, $H_F = W_2\frac{a'b'}{lh}$.

Tension of AE, $T_{AE} = H_E \sec EAB$, $\quad T_{FB} = H_F \sec FBA$;
„ EB, $T_{EB} = H_E \sec EBC$, $\quad T_{AF} = H_F \sec FAD$.

The actual tensions of the sloping rods are simply as written, but since AB is a part of both triangular frames, the total thrust along it is found by summing the thrusts due to each: so

$$H = H_E + H_F.$$

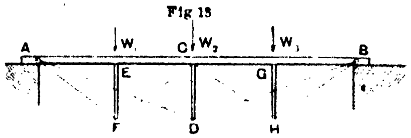
This is an example of the *principle of superposition* stated above.

(2) Suppose the beam which carries the distributed load to be

supported by a central strut forming a simple triangular truss, and further let the halves of the beam, not being strong enough to carry the load on them, each be subdivided and trussed by a simple triangular truss, the tension rods from the bottom of the subdividing struts proceeding only to the ends of each half beam. If the quarter spans are still too great, they may each of them be trussed in a similar way, and so on. Such a structure is called a *Finck truss.*

Suppose, for example, we have three struts. (Fig. 18.) We must first determine the load at the joints—that is, in this case, the load on the struts due to the distributed load on the beam. Suppose that on account of the *adjacent* subdivisions those loads are W_1, W_2, W_3. If the load is uniformly distributed over the beam the W's are each of them equal to $\frac{1}{4}$ total weight on beam.

Fig 18



We may now separately consider the triangular frame, *AFC*, carrying the load, W_1. On account of it there will be a thrust on *AC*.

$$H_F = W_1 \frac{AC}{4h} = W_1 \frac{l}{8h}.$$

The tensions of *AF* and *FC* are each $= H_F \sec FAE$. We get similar results from the triangle *CHB*. Just in the same way we may consider the principal triangle frame, *ADB*, but in this case the thrust down the strut, *CD*, which is the load at *D*, is not simply W_2, but greater by the amount of the downward pull of the two tension rods, *CF* and *CH*. The vertical components of these tensions are $\frac{1}{2}W_1$ and $\frac{1}{2}W_3$, so that the total thrust down the strut $= W_2 + \frac{1}{2}(W_1 + W_3)$. This is the load which must be taken to act at *D* in determining the stresses on the members of *ADB*. Hence it appears that

$$H_D = (W_2 + \tfrac{1}{2}W_1 + \tfrac{1}{2}W_3)\frac{l}{4h},$$

and the tensions of *AD* and *DB* are each equal to $H_D \sec DAB$.

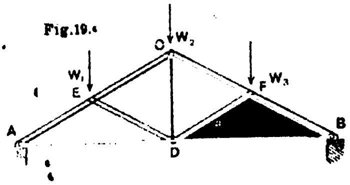
It will be seen that the thrust on the central strut and tensions of the longer rods are the same as if the secondary trusses had not been introduced. For example, if the W's each $= \frac{1}{4}$ whole load on beam, then the virtual load at $D = \frac{1}{2}$ weight on beam. The mere strengthening of each half the beam by trussing it can no more relieve the central strut of the load it has to carry, than the fact of strengthening a structure of any kind can relieve the two points of support from the duty each must have of bearing its own proper share of the weight. In stating the thrust on the beam we must divide it into two portions, AC and CB. The portion AC is subjected to the thrust of the triangles AFC and ADB; $\therefore H_{ac} = H_f + H_d$; and CB being a portion of the triangles CHB and ADB, $H_{cb} = H_h + H_d$. When W_3 is not equal to W_1, the thrusts on the two portions will be different. This is quite possible although the beam AB may be a continuous one.

Both these simple forms of truss have been used for bridges of considerable span. As an example of the first may be mentioned the bridge at Harper's Ferry, U.S., destroyed during the war. It was 124 feet span in 7 divisions. The great length of the tension rods and their inequality appears objectionable. The second in 8 or 16 divisions has been much used in America; but in England other forms mentioned in a later chapter are much more common.

13. *Roof Trusses in Timber.* In roofs of small span, 10 or 12 feet only, the roofing material, slates or tiles, rests on a number of laths set lengthways to the roof, and these laths rest on sloping rafters spaced 1 or 2 feet apart, with their feet resting on the walls of the building; the stability of the walls being depended on for taking the thrust.

When we come to larger and more important roofs we find additional members added for strength and security. The closely spaced rafters just mentioned are called common rafters. These being too long and slender to carry the weight of the roofing material and transmit it to the walls, are supported, not only at the ends by the walls and ridge piece, but also at the middle by a longitudinal beam of wood called a *purlin*, and the purlin is supported at intervals of its length by principal rafters. The principal rafters again are supported by struts at their central points, immediately below the purlins. To carry the lower ends of the struts, a vertical tension piece is introduced, by which they are suspended from the apex of the principals, while the thrust is taken by a tie beam connecting the feet of the rafters. In such a roof, a ceiling or floor may frequently be required to be supported by the tie beam, and to prevent it from sagging under the weight an additional tension will come on the vertical suspending rod. This rod is then a very important member of the structure, and is called the king post, and the whole structure, consisting of the principal rafters, king post, etc., is called a *King post truss.* This truss is often constructed entirely of wood. The sloping struts then for constructive reasons (Ch. XV.) butt on an enlarged part at the bottom of the king post above the point where the horizontal tie beam is attached, but for calculation purposes may be regarded as meeting at that point as shown in Fig. 19.

Fig. 19.



By means of the purlins and the ridge piece the weight of the roofing material will produce loads at the joints $ECF = W_1 W_2 W_3$ suppose.

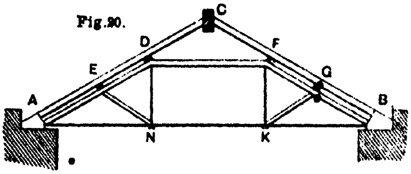
Now treat the structure as made up of three simple triangular frames, AED, DFB, and ACB. First consider AED with the load W_1 at vertex E. The horizontal thrust of this frame $H_E = W_1 \frac{AD}{4h}$ where h is the height of point E above AD. Also the thrust along AE and ED due to the load $E = H_E \sec EAD$. In an exactly similar manner we may consider the triangle DFB; the results for this will be to those for AED in the proportion of W_3 to W_1. Next as to the primary triangle ACB. There is at C a direct load of W_2 due to the weight between E and C, and F and C. But beside this, the king post pulls the point C downwards, so that the total load at $C = W_2$ + tension of king post. In addition to a portion of the weight of the ceiling (if any) the post has to support D against the downward thrust of the two struts ED and FD. The vertical components of these thrusts are $\frac{1}{2} W_1$ and $\frac{1}{2} W_3$, therefore, neglecting the weight of ceiling, the virtual load at $C = W_2 + \frac{1}{2}(W_1 + W_3)$. Let us call this total load W, then H_C the horizontal thrust of $ACB = W \frac{AB}{4CD}$, and the thrusts along AC and CB due to load $C = H_C \sec A$.

Now in the complete structure, since AD is a member both of the triangular frame AED and ACB, the total tension of $AD = H_E + H_C$.
For the same reason tension of $DB = H_F + H_C$,
and thrust of $AE = (H_E + H_C) \sec A$,
" " $FB = (H_F + H_C) \sec A$.

The other members of the structure are portions of one elementary frame only, and the stress is due only to the load at the apex of that frame.

The king post truss serves for roofs of spans under 30 feet, but for spans greater than this trusses of more complicated construction are required. If the span is from 30 to 50 feet, then instead of supporting the common rafters by a purlin at the centre of its length only, as in the king post truss, two supporting purlins may be used, dividing the length of the rafter into three equal portions. These purlins may be carried by a queen truss, the sloping members of which are supported in the middle by struts, as shown in the figure (Fig. 20).

Fig. 20.



The vertical queen posts, DN and FK, serve to sustain the downward thrust of the struts, EN and GK, and also to support the weight of a ceiling, if there is one. Supposing the weight of the ceiling omitted, let W be the weight of roofing material on one side for a length of roof equal to the spacing of the trusses, then $\frac{1}{3}W$ will, through the common rafters and purlins, act at E, and $\frac{1}{3}$ at D; and similarly for the other side. At the ridge C there will also be $\frac{1}{3}W$ acting; but this will be distributed equally amongst the common rafters which are carried by the truss, and will produce compression in those rafters without directly affecting the truss. The part of the thrust of the roof arising from this will, however, generally, like the rest, ultimately come on the principal tie beams.

To find the stresses on the different members of the truss. Consider first the small triangles AEN and BGK, each carrying $\frac{1}{3}W$ at the vertex. We then consider the trapezoidal truss $ADFB$. The loads at D and F will be $\frac{1}{3}W +$ tension of queen post. Since the tension of the queen post $DN =$ the vertical component of the thrust along EN it will equal $\frac{1}{2} \cdot \frac{1}{3}W = \frac{1}{6}W$, and the total load at each joint of the trapezoidal truss will be $\frac{1}{3}W + \frac{1}{6}W = \frac{1}{2}W$, the same as would have acted if there had been no purlin at E and no strut EN. After having determined the respective stresses due to the triangles and trapezoid separately, we must add the results for any bar which is a part of both. Were it not for the friction at the joints and the power of resistance of the continuous rafters AC, CB to bending, this structure would be stable only under a symmetrical load. In practice, however, it is able to sustain an unsymmetrical load, such as roofs are frequently subjected to.

14. *Queen Truss for large Iron Roofs.*—As the span of the roof is still further increased we find other kinds of trusses employed to support them. A common form in iron roofs is constructed, as shown in

Fig. 21. It is in reality a further development of the wooden queen truss, and is known by the same name. AC and CB are divided into

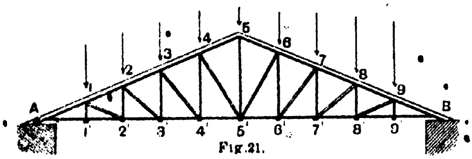


Fig. 21.

a number of equal parts, and sloping struts and vertical suspending rods are applied as shown. Suppose the load the same at each joint on one side of the roof, the load on the right, however, not being necessarily equal to that on the left. Let the upward supporting force at $A = P$. P will be half the total weight if the loading is symmetrical, but in any other case it may be found by taking moments of the loads about B. We might solve the problem of finding the stress on each member of the structure by treating separately each elementary triangle into which the structure may be divided, and summing the stresses for any bar which may form a part of two or more triangular frames. But we will describe another method.

First, to find the tension of the vertical suspending rods consider $A12'$ as an independent triangle, carrying a load W at its vertex. The slope of $12'$ being the same as that of $A1$, the tension rod $22'$ must supply a supporting force to the joint $2' = \frac{1}{2}W$. Considering next the triangle $A23'$ and its equilibrium about the point A. The forces along 23 and $3'4'$ have no moment about A, so that the moment of the two weights W at 1 and 2 about A must be balanced by the upward pull of the tension rod $33'$. $\therefore$ tension of $33' = W$.

In a similar way we can see that the tension of $44' = \frac{3}{2}W$. However many divisions of the roof there may be, the tensions of the vertical suspending rods will increase in arithmetical progression, with the same difference between each. The road $11'$, except so far as may be due to the weight of the rod $A2'$, will have no tension on it. Calling this the 1st tension rod, the tension of the $n^{\text{th}} = \frac{n-1}{2}W$. We must notice that the rod $55'$ is common to both sides of the roof, and we must add the two tensions to get the total. Now consider any joint, say 4 in the tie bar AB, and resolve vertically and horizontally. If $R =$ thrust of $34'$, θ its inclination to the horizontal, and T the pull on that division of AB which is indicated by the numerical suffix placed below it,

$$R \sin\theta = \tfrac{3}{2}W,$$

$$R\cos\theta = T_{3'4'} - T_{4'5'};$$

$$\therefore\ T_{3'4'} - T_{4'5'} = \tfrac{3}{2}W\cot\theta.$$

But from figure $\cot\theta = \frac{1}{3}\cot A$;

$$\therefore\ T_{34} - T_{45} = \tfrac{1}{2}W\cot A.$$

Whichever joint we select we should find the same result—namely, that the difference between the tensions of two consecutive portions of the tie rod is a constant quantity $= \frac{1}{2}W\cot A$. So that these tensions are in arithmetical progression diminishing towards the centre.

If we call $A2'$ the 1st division of the rod, then for the joint between the $n-1$th and nth we have

$$R\sin\theta = \frac{n-1}{2}W,$$

$$R\cos\theta = T_{n-1} - T_n \text{ and } \cot\theta = \frac{1}{n-1}\cot A;$$

$$\therefore\ T_{n-1} - T_n = \tfrac{1}{2}W\cot A.$$

If $A1$ is the 1st division of the rafter, then the thrust on the nth division $= T_n \sec A$.

Now, the tension of the tie rod in the

1st division $= P\cot A$,
2nd " $= (P - \frac{1}{2}W)\cot A$,
nth " $= \left(P - \frac{n-1}{2}W\right)\cot A$.

The thrust on the nth division of rafter $= \left(P - \frac{n-1}{2}W\right)\operatorname{cosec} A$.

The thrust on any strut may best be found by squaring and adding the two equations of equilibrium of the lower joint of it. We get

$$\text{Thrust of } n\text{th strut} = \frac{W}{2}\sqrt{n^2 + \cot^2 A}.$$

15. *Concluding Remarks. General Method of Constructing Diagrams of Forces.*—Cases of framework often occur which are much more complicated than those which we have hitherto considered, but if there are no redundant bars the stress on each part depends on statical principles only, without reference to the relative yielding of the several parts of the structure. Such cases may always be treated by use of the general principle stated in Art. 1, and we shall conclude this chapter by explaining briefly a graphical method of applying that principle invented by the late Professor Clerk Maxwell. The forces will be supposed all in one plane, and each of them will be supposed known, that is to say, if there be any unknown reactions at points of support they will be supposed previously found by a graphical or other process, from the consideration that the whole must form a set of forces in equilibrium. In Fig. 22a a frame is shown acted on by known forces $PQR\ldots$, an ideal example is chosen which is better suited for the purpose of explaining the method than any case of common occurrence in practice. First seek out a joint where only two bars meet: there will usually be two such joints if there be no redundant bars in the frame, and in the present instance we will choose the joint where *P* acts. Distinguish all the triangles, making up the frame by letters *A*, *B*, *C*, etc., and place numbers or letters outside the frame, one for each bar. In Fig. 22b draw 18 parallel to the force *P* and representing it in magnitude, 8*a* parallel to 8, 1*a* parallel to 1, to intersect in the point *a*; then, as in previous examples, 8*a*, 1*a* represent the stress on the two bars to which they are parallel. Pass now to the joint where *Q* acts: this joint is chosen because only three bars meet there, on one of which we have just determined the stress; draw 12 parallel to *Q* and representing it, then *ab* parallel to the bar lying between the triangles *A* and *B*, and 2*b* parallel to the bar 2; we thus get a polygon 12*ba*, the sides of which are parallel to the four forces acting at the joint where *Q* acts, while two of them represent two forces already known, the other two, therefore, will represent the remaining two forces. Proceed now to the joint where *W* acts and complete in the same way the polygon 8*abc*7, then to the joint where *R* acts, and so on. We at length arrive at the triangle 4*f*5, the third side of which, if we have performed the construction accurately, and if the forces be really in equilibrium, must be parallel to the last force *T*. On examination of the diagram of forces (Fig. 22b) it will be seen that to every joint of the frame corresponds a polygon representing the forces at that joint, while each line, such as *ab* or 7*c*, gives the stress on the bars separating those letters or numbers in the frame-diagram. The polygon 12...8 is the polygon of external forces, each side representing the force to which it is parallel.

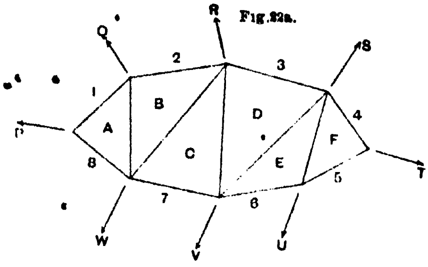


Fig. 22a.

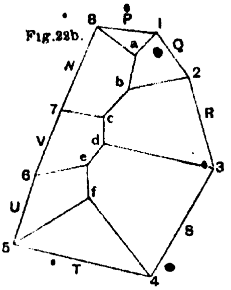


Fig. 22b.

The method here described is easy to understand in the general case we have considered, and with a little practice the transformations the diagram of forces undergoes will offer no difficulty. Some joints are usually unloaded, and the corresponding lines in the polygon of external forces vanish; the forces may be parallel, in which case the polygon becomes a straight line, while not unfrequently the sides of two of the polygons representing the forces at the joints coincide. The figure, however, always possesses the same properties.

In Mr. Bow's excellent work referred to at the end of this chapter over 200 examples will be found of the application of this method, including almost all known forms of bridge and roof trusses.

EXAMPLES.

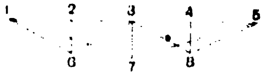
1. A Bollman truss of three divisions is 21 feet span, and is loaded uniformly with 1 ton per foot. The depth of the truss is 3½ feet. Find the stress on each part.

Load on each strut = 7 tons.
Tension of short rods = 10·4 ,,
,, longer rods = 9·6 ,,
Total thrust on beam = 18⅔ ,,

being 9⅓ due to each triangle.

2. A Finck truss of 4 divisions, 20 feet span and 3 feet deep, is loaded with 1 ton per foot, find the stress on each part.

Thrust on 26 and 48 = 5 tons.
,, 37 = 10 ,,
Tensions of 16, 63, 38, and 85 = 4·86 ,,
,, 17 and 75 = 17·4 ,,
Thrust on 13 and 35 = 4⅙ + 16⅔ = 20⅚ tons.



3. In the last question suppose one half the truss loaded with an additional 1 ton per foot. Find the stress on each part.

Suppose the additional load on the right-hand side.

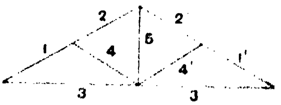
Thrusts.
On 26 = 5 tons.
,, 37 = 15 ,,
,, 48 = 10 ,,
,, 13 = 4⅙ + 25 = 29⅙.
,, 35 = 8⅓ + 25 = 33⅓.

Tensions.
On 16 and 63 = 4·86 tons
,, 38 ,, 85 = 9·72 ,,
,, 17 ,, 75 = 26·1 ,,

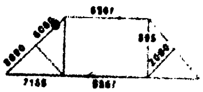
4. A roof 28 feet span, height 7 feet, rests on king post trusses spaced 10 ft. apart. The weight of roof is 20 lbs. per square foot. Find the stress on each part. Also obtain results when an additional load of 40 lbs. per square foot rests on one side.

Load at each joint. 1st case = 1566·6 lbs.

Bars.	Stress in lbs.		Bars	Stress.	
	Equal Load.	Additional Load		Equal Load	Additional Load
1	5254	8756	1′	5254	12261
2	3503	7006	2′	3503	7006
3	4700	7833	3′	4700	10966
4	1752	1752	4′	1752	5255
5	1566·6	3113			



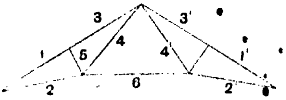
5. A roof 48 feet span, 12 feet high, rests on queen trusses 8 feet high, spaced 10 feet apart. Find the stresses for a load of 20 lbs. per square foot.



6. An A roof, braced as in the figure, is 40 feet span, and 10 feet high; the horizontal

tie bar is 8 feet below the vertex. Find the stresses on each part when loaded with 2 tons at each joint by constructing a diagram of forces or otherwise.

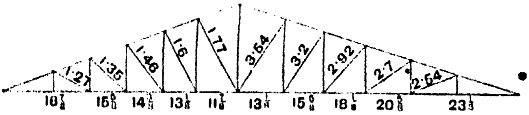
Bars.	Stress.
1	10·4
2	9·4
3	9·4
4	4·7
5	1·8
6	5·



7. In the last question suppose an accumulation of snow on one side equivalent to an additional load of 2 tons at the middle of the rafter, and 1 ton at the ridge. Find the stress on each part.

Bars.	Stress.	Bars.	Stress.
1	13·9	1'	17·3
2	12·5	2'	15·7
3	12·8	3'	15·4
4	5·5	4'	8·6
5	1·8	5'	3·6
6	7·5		

8. Suppose there are 11 suspending rods in iron roof shown in the figure, the height of which is $\frac{1}{10}$th the span. Find the stress on each part 1st, when loaded with $\frac{1}{2}$ ton at each joint on both sides, and, 2nd, when loaded with an additional $\frac{1}{3}$ ton at each joint on one side, not including the ridge.



Additional load is on right-hand side, and the figures on the diagram refer to case 2.

9. The roadway of a bridge, 80 feet span, is carried by a pair of compound trapezoidal trusses, each consisting of three simple trapezoids of the same height, the six "queens" of which are equidistant, forming seven divisions of length four thirds the height of the truss. Find the stress on all the bars due to $\frac{1}{4}$ ton per foot run on the bridge.

10. Find the stress on each part of a "straight-link suspension" bridge formed by inverting the truss of the last question, assuming the pull at the centre of the platform zero.

REFERENCES.

For further information on the subjects treated of in the present chapter the reader may refer amongst other works to

GLYNN *Construction of Cranes.* Weale's series.
HURST—*Carpentry.* Spon, 1871.
BOW—*Economics of Construction.* Spon, 1873.

CHAPTER II.

STRAINING ACTIONS ON A LOADED STRUCTURE.

16. *Preliminary Explanations.*—In the preceding chapter we have considered only those structures in which the parts are subject to compression and tension alone, except by way of anticipation in a few special cases. But the parts of a structure are generally subject to much more complex forces, and besides, although the forces acting on each bar have been determined, we should, if we stopped here, have a most imperfect idea of the way in which the load affects the structure as a whole.

If we imagine a structure to be made up of any two parts, A and B, united by joints, or distinguished by an ideal surface cutting through the structure in any direction, the whole of the forces acting on the structure may be separated into two sets, one of which acts on A, the other on B. Since the structure is in equilibrium as a whole the two sets of forces must balance one another, and must therefore produce equal and opposite effects on A and B, effects which are counteracted by the union existing between the parts. The two sets of forces taken together constitute a STRAINING ACTION of which each set is an element, and the object of this and the next two chapters is to consider the straining actions to which loaded structures and parts of structures are subject.

Straining actions differ in kind, according to the nature of the effects which they tend to produce. Four simple cases may be distinguished:—

(1) The parts A and B may tend to move towards each other or away from each other perpendicular to a given plane. This effect is called Compression or Extension, and the corresponding straining action is a thrust or a pull.

(2) A and B may tend to slide past each other parallel to a given plane. This effect is called Shearing.

(3) A and B may tend to rotate relatively to each other about an axis lying in a given plane. This is called Bending.

(4) *A* and *B* may tend to rotate relatively to each other about an axis perpendicular to a given plane. This is called Twisting.

In the first two cases the straining action reduces to two equal and opposite forces, and in the second two to two equal and opposite couples. In general, straining actions are compound, consisting of two or more simple straining actions combined. The given plane with reference to which the straining actions are reckoned may always be considered as an ideal section separating *A* and *B* even when the actual dividing surface is different. We shall commence by considering the straining actions on a beam of small transverse section.

SECTION I.—BEAMS.

17. *Straining Actions on a Beam.*—The action of a simple thrust or pull on a bar has already been sufficiently considered in Chapter I. They are usually considered as separate cases, and the simple straining actions on a bar are therefore reckoned as five in number. The other three are (1) shearing, (2) bending, and (3) twisting, of which the last rarely occurs, except in machines, and will, therefore, be considered in a later division of this work, under that head.

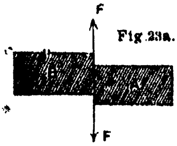


Fig. 23a.

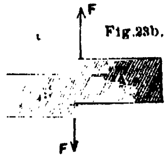
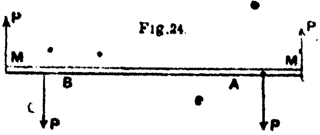


Fig. 23b.

Shearing and bending are due to the action of forces, the directions of which are at right angles to the bar: in structures, the forces usually lie in one plane passing through the axis of the bar. A bar loaded in this way is called a beam.

Simple shearing is due to a pair of equal and opposite forces, *F* (Fig. 23), applied to points very near together, tending to cause the two parts *A* and *B* to slide past one another, as shown in the figure (Figs. 23a, 23b). Either element is called the shearing force, and is a measure of the magnitude of the shearing action, but in considering the sign we must consider both together. In this work, if the right hand portion, *A*, tends to move upwards, and *B* downwards, as in Fig. 23b, the shearing action will usually be reckoned negative, while in the converse case (Fig. 23a) it will be reckoned positive.

Fig. 24.



Simple bending is due to a pair of equal and opposite couples applied to the bar, one acting on *A*, the other on *B*, as in Fig. 24, tending to make A and B rotate in opposite directions. The magnitude of the bending is measured by the moment of either couple, which is called the bending moment. In this work bending moments will usually be reckoned positive when the left-hand half, B, rotates with the hands of a watch, and the right-hand half in the opposite direction; that is to say, when the beam tends to become convex downwards, as in the ordinary case of a loaded beam supported at the ends. In loaded beams shearing and bending generally exist together, and vary from point to point of the beam. We shall now consider various special cases.

18. *Example of a Balanced Lever.—General Rules for calculating S.F. and B.M.*—First take the case of a beam, AB, supported at C (Fig. 25), and loaded with weights, PQ, at its ends.

If the weights are such that $P.AC = Q.BC$ the beam will be in equilibrium, but the two parts, AC, BC, tend to turn about C in opposite directions; there is therefore a bending action at C, of which the equal and opposite moments $P.AC$, $Q.BC$ are the elements. Either of these is the bending moment usually denoted by M, so that we write

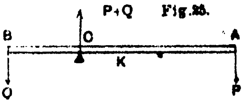


Fig. 25.

$$M_c = P.AC = Q.BC.$$

Not only is there a bending action at C, but if we take any point, K, and consider the forces acting on AK, BK separately, we see that AK tends to turn about K under the action of the force P, while BK tends to turn about K under the action of the forces $P+Q$ at C and Q at B. The first tendency is immediately seen to be simply the moment $P.AK$, while the second is $Q.BK-(P+Q)CK$. The last quantity reduces to $Q.BC-P.CK$, or, remembering that $Q.BC = P.AC$, to $P.AK$. The two moments, then, as before, are equal and opposite, and constitute a bending action at K, measured by the bending moment

$$M_K = P.AK.$$

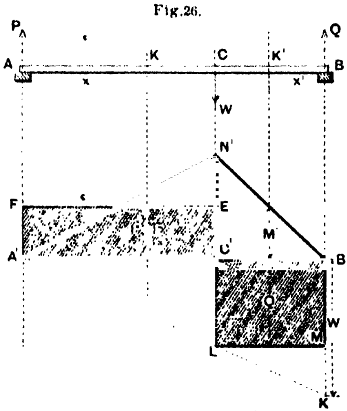
This example will sufficiently explain the general rule for calculating the bending moment at any point, K, of a beam. *Divide the forces into two sets, one acting to the right and the other to the left of K, and estimate the moment of either set about K, then the result will be the bending moment at K.* The example shows that the calculation of one of the two moments will generally be more simple than that of the other, and cases constantly occur, as where a beam is fixed at one end in a wall, where nothing is known about one set of forces except that they balance the other set. In each case the simplest calculation is of course to be preferred.

Moments are measured numerically by unit weight acting at unit leverage, as, for example, 1 ton acting at a leverage of 1 foot, for which the expression "foot-ton" is commonly employed. This phrase, however, is used also for a wholly different quantity, namely, the unit of mechanical work, and for this reason it would be preferable to call the unit of moment a "ton-foot" for the sake of distinction.

The peculiar action called shearing will be better understood when we come to consider the action of forces on a framework girder in the next section; it will here be sufficient to say that if the sum of the forces acting on AK, BK are not separately equal to zero, they must tend to cause AK, BK to move past each other in the vertical direction, thus constituting a shearing action measured by the magnitude of the shearing force, which may be thus calculated for any point K. *Divide the forces into two sets, one acting to the right of K, and the other to the left of K, the algebraical sum of either set is the shearing force at K.* As before, either set may be chosen, whichever gives the result most simply. In the example just given the shearing force at any point of AC is P; and at any point of BC, Q.

19. *Beam Supported at the Ends and Loaded at an Intermediate Point.*—We will next consider the case of a beam supported at the ends and loaded at some intermediate point. Before we can apply the rules previously enunciated, to find the shearing force and bending moment at any point, we must first determine the supporting forces at the two ends. We find the force P acting at A (Fig. 26), by taking moments about B, thus,

Fig. 26.



$$P(a+b) = Wb\,;\ \therefore\ P = \frac{Wb}{a+b},$$

and similarly $$Q = \frac{Wa}{a+b}.$$

First as to the shearing force. Taking any point K in AC, and considering the forces acting on AK, of which there is only one,

$$F_K = P = \frac{Wb}{a+b}.$$

At any point K' between C and B we have

$$F_{K'} = Q = \frac{Wa}{a+b}.$$

It will be noticed that at K the tendency is for the left hand portion to slide upwards relatively to the right, whereas at K' the tendency is for the right-hand portion to slide upwards relatively to the left. It is advantageous to distinguish between these two tendencies, as previously stated, by calling the one positive and the other negative.

We may draw a diagram to represent the shearing force at any point thus. Let $A'B'$ be drawn parallel to and below AB to represent the length of the beam, and let $CC'L$ be the line of action of the weight. If we set up an ordinate $A'F = P$, and downwards an ordinate $B'M = Q$, and draw FE and ML parallel to $A'B'$ to meet the vertical $EC'L$; the shearing force at any point will be represented by the ordinates of the shaded figure $A'FELMB'$, measured from the base line $A'B'$. Not only should the magnitude of the shearing force be represented, but also the direction of the sliding tendency. This is why the ordinate was set downwards on the right-hand side of C'.

In this example the supporting forces may be found by construction, and thus the whole operation of determining and representing the shearing force performed graphically. For, set down $B'K = W$, join $A'K$, and where the vertical through C' cuts $A'K$, draw LM horizontal, then $B'M = Q$ and $MK = P$. Then set up $A'F = MK$, and draw FE horizontal.

Next as to the bending moment at any point. Take any point K in AC distant x from A, then

$$M_K = Px = \frac{Wb}{a+b}x,$$

and similarly at K' in CB distant x' from B,

$$M_{K'} = Qx' = \frac{Wa}{a+b}x';$$

so for either side of C, the bending moment is greater the greater the distance of the point from the end of the beam. Thus the greatest bending moment is at C.

If in the value of M_K we put $x = a$,
or „ $M_{K'}$ „ $x' = b$,

we get the same result, viz., that

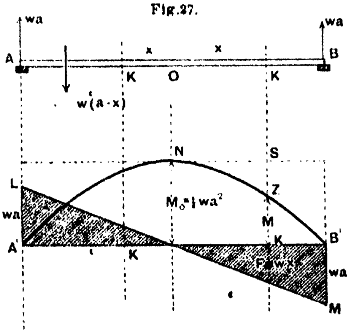
$$M_c = \frac{Wab}{a+b} = \text{greatest bending moment.}$$

The graphical representation of the bending moment at any point is very useful and instructive. We may construct the diagram thus:—$A'B'$ representing the length of the beam, set up from C', $C'N'$ the bending moment at $C' = \frac{Wab}{a+b}$ on some convenient scale, on such a scale for instance as 1 inch = 20 ft.-lbs. Then joining $A'N'$ and $B'N'$, the ordinate of the figure $A'N'B'$, measured from the base line $A'B'$, will express on the scale chosen the bending moment at any point of the beam. If $a = b = \frac{1}{2}$ span, so that the load is applied at the centre of the beam, then

$$M_c = \tfrac{1}{4} W \times \text{span} = \text{greatest bending moment.}$$

20. *Beam Supported at the Ends and Loaded Uniformly.*—The next example for consideration is that of a beam supported at the ends and loaded uniformly throughout its length with w lbs. per foot (Fig. 27).

Fig. 27.



Let the span $= 2a$. Take any point, K, distant x from the centre O The load on AK is $w.AK$, and therefore the shearing force at K. reckoning the forces on the left-hand side, must be

$$F_x = wa - w.AK = wa - w(a - x) = wx.$$

That is, the shearing force is proportional to the distance of the point from the centre of the beam. At the end A where $x = a$,

$$F_A = wa,$$

and at B where $x = -a$,

$$F_B = -wa.$$

If from $A'B'$, below AB in the diagram, we set up and down ordinates at A' and $B' = wa$ on some scale, and join LM, the ordinates of the sloping line will represent the shearing force at any point. The shearing force at the centre of the beam is zero.

In finding the bending moment at K, reckoning still from the left hand side, we must clearly take account not only of the supporting force at A, but also of the effect of the load which rests on the portion of the beam AK. The moment of this load about K is the same as if it were all collected at its centre of gravity, namely, at the centre of AK. Thus

$$M_K = wa\,.\,AK - w.AK\,.\,\frac{AK}{2}$$

$$= \frac{w}{2}AK(2a - AK) = \frac{w}{2}AK\,.\,KB.$$

That is to say, the bending moment at any point is proportional to the product of the segments into which the beam is divided by the point.

Putting $AK = a - x$ and $BK = a + x$,

$$M_K = \tfrac{1}{2}w(a^2 - x^2),$$

which is greater the less x is. At the centre $x = 0$, and we have the maximum bending moment

$$M_o = \tfrac{1}{2}wa^2.$$

If we put $2wa = W$, the total load on the beam

$$M_o = \tfrac{1}{8}W \times \text{span}.$$

This is only one half the bending moment due to the same load when concentrated at the centre of the beam.

If ordinates be set up from $A'B' = \frac{1}{2}w(a^2 - x^2)$, at all points, the extremities of the ordinates will lie on a curve which may easily be seen to be a parabola with its axis vertical and vertex above the middle point of the beam. For

$$SZ = SK - KZ = \tfrac{1}{2}wa^2 - \tfrac{1}{2}w(a^2 - x^2) = \tfrac{1}{2}wx^2.$$

So that SZ is proportional to SN^2, showing that the curve is a parabola.

21. *Beam Loaded at the Ends and Supported at Intermediate Points.*— Next, suppose a beam (Fig. 28) supported at A, B, and loaded with weights P, Q, at the ends C, D, which overhang the supports. If AC, AB, BD are denoted by a, l, b respectively, the supporting force S at A (by taking moments about B) is given by

$$Sl = P(a + l) - Qb.$$

Similarly R, the supporting force at B, is given by

$$Rl = Q(b + l) - Pa.$$

Take now a point K distant x from A; then

$$F_K = S - P = \frac{Pa - Qb}{l} = \frac{M_A - M_B}{l},$$

where M_A, M_B are the bending moments at A, B.

Also for the bending moment at K,

$$M_K = -Sx + P(a+x) = -\frac{M_A - M_B}{l}.x + M_A,$$

or, as we may write it,

$$M_K = M_A\frac{l-x}{l} + M_B\frac{x}{l}.$$

These formulae show that the shearing force is constant while the bending moment varies uniformly. In the diagram this is indicated by setting up ordinates Aa, Bb, to represent the bending moments at A, B, and joining a, b; the ordinate Kk of this line corresponding to an intermediate point K, will represent the bending moment there.

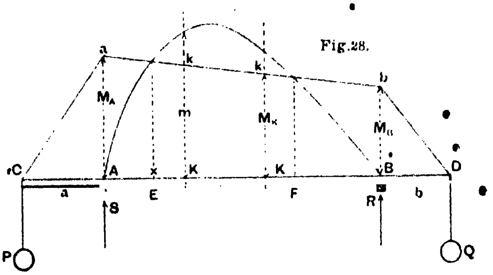


Fig. 28.

The moments are in this example reckoned positive for upward bending if the ordinates are considered as drawn upwards from the base line CD, and it is therefore better to suppose them drawn downwards from the broken line C, a, b, D.

An important special case is when $M_A = M_B$; then the bending moment is constant and the shearing force zero. We have then no shearing but only bending. Simple bending is unusual in practice, but an instance occurs in the axle of a carriage.

The ordinates of the straight lines Ca, Db, represent the bending moment at any point of the overhanging parts of the beam.

22. *Application of the Method of Superposition.*—When a beam is acted on by several loads, the principle of superposition already stated in Chapter I. is often very useful in drawing diagrams and writing down formulae for the straining action at any point. Thus, for example, in the preceding case, if there be many weights on the overhanging end of a beam, the bending moment and shearing force at each point must be the sum of that due to each taken separately; and hence it follows that, whatever be the forces acting on a beam, if there be a part AB under the action of no load, and the bending moments at the ends of that part be M_A, M_B, the straining actions at any intermediate point K will always be given by the formulae just written down. And, further, if there be a load of any kind on AB, and m be the bending moment, on the supposition that the beam simply rests on supports at A, B, then the actual bending moment must always be given by

$$M = M_A \cdot \frac{l - x}{l} + M_B \cdot \frac{x}{l} + m,$$

a general formula of great importance. The result is shown graphically in the diagram, where the curve represents the bending moment m, and the straight line ab the effect of the bending moments at the ends, supposed, as is frequently the case, to be in the opposite direction to m; then the intercept between the curve and the straight line represents the actual bending moment.

If several weights act on a beam, triangles may readily be constructed showing the bending moment due to each weight; then adding the ordinates of all the triangles at the points of application of the weights, and joining the extremities by straight lines, a polygon is obtained which is the polygon of bending moments for the whole load. This process may also be applied to shearing forces. It is simple, but somewhat tedious when there are many weights, and other methods of construction will be explained hereafter. In superposing two loads the artifice just employed in Fig. 28 is very useful. A propped beam (Ex. 11, p. 42) is an important example.

EXAMPLES

1. A beam, AB, 10 feet long, is fixed horizontally at A, and loaded with 10 tons distributed uniformly, and also with 1 ton at B. Find the bending moment in inch-tons at A, and also at the middle of the beam.

$M = 720$ inch tons at A.
$= 210$,, at the centre.

2. In the last question find the shearing force at the two points mentioned.

$F = 11$ tons at A.
$= 6$,, at the centre.

3. A beam, AB, 10 feet long, is supported at A and B, and loaded with 5 tons at a point distant 2 feet from A. Find the shearing force in tons, and the bending moment in inch-tons at the centre of the beam. Find also the greatest bending moment.

F at the centre $= 1$ ton.
M at the centre $= 60$ inch-tons.
Maximum bending moment $= 96$,,

4. In the last question suppose an additional load of 5 tons to be uniformly distributed. Find the shearing force and bending moment at the centre of the beam.

F at centre $= 1$ ton as before.
M at centre $= 11\frac{1}{4}$ foot-tons $= 135$ inch-tons.

5. A beam, AB, 20 feet long, is supported at C and D, two points distant 5 feet from A and 6 feet from B respectively. A load of 5 tons is placed at each extremity. Find the bending moment at the middle of CD in inch-tons.

Moment $= 330$ inch-tons.

6. In the example just given draw the diagrams of shearing force and bending moment at each point of the beam.

7. A foundry crane has a horizontal jib, AC, 21 feet long, attached to the top of a crane-post 14 feet high, which turns on pivots at A and B. The crane carries 15 tons, which may be considered as suspended at the extremity of the jib. The jib is supported by a strut attached to a point in it 7 feet from A, and resting on the crane-post at B. Find the stress on crane-post and strut, and the shearing force and bending moment at any point of the jib.

Tension of crane-post = 30 tons.
Thrust on strut = 50 ,,

8. A rectangular block of wood, 20 feet long, floats in water; it is required to draw the curves of shearing force and bending moment when loaded (1) with 1 cwt. in the middle, (2) with $\frac{1}{2}$ cwt. at each end, and (3) $\frac{1}{2}$ cwt. placed at two points equidistant from the middle and each end.

9. A beam, *AB*, 20 feet long, is supported at the ends, and loaded at two points distant 6 feet and 11 feet respectively from one end with weights of 8 tons and 12 tons; employ the method of superposition to construct the polygons of shearing force and bending moment. Find the maximum bending moment in inch-tons.

Maximum moment = 972 inch-tons.

10. A beam is supported at the ends and loaded uniformly throughout a part of its length; show that the diagram of moments for the part of the beam outside the load is the same as if the load had been concentrated at the centre of the loaded part, and for the remainder is a parabolic arc. Construct this arc. Also, draw a diagram of shearing force.

11. A beam is supported at the ends and uniformly loaded. The beam is also supported in the middle by a prop which carries a given fraction of the total load; employ the method of superposition to draw diagrams of shearing force and bending moment. Find the fraction when the beam is strongest.

Ans. Fraction = ·586.

12. A beam is supported at the ends and uniformly loaded: if the span be divided into any number of equal parts, and half the weight on each division be concentrated at the dividing points, show that the corners of the polygon of moments lie on the parabola due to the uniform load.

SECTION II.—FRAMEWORK GIRDERS WITH BOOMS PARALLEL, AND WEB A SINGLE TRIANGULATION.

23. *Preliminary Explanations.*—Hitherto we have only considered beams of small transverse section, but the part of a beam may be played by a framework or other structure under the action of transverse forces. Such a structure, when employed as a beam, is called a Girder, and consists essentially of an upper and a lower member called the Booms of the girder, connected together by a set of diagonally placed bars, called collectively the Web. The web consists sometimes of several triangulations of bars crossing each other, and may even be continuous. In the present section the booms will be supposed straight and parallel, and the web a single triangulation. The action of a load on such a girder furnishes the simplest and best illustration of the nature of the straining actions we have just been considering.

Suppose, in the first place, we have a rectangular beam of considerable transverse dimensions, which has one end fixed horizontally, and the other end loaded with a weight *W*. Now let a part of the length, *CD* (see Fig. 29), be cut away, and replaced by three bars, *CD*, *EF*, *DE*, jointed at their ends to the two parts of the beam—*CD*, *EF*, forming a rectangle of which *DE* is a diagonal. With this construction the load *W* will be sustained, as well as by the original beam, but the three bars will be subject to stresses which we shall now determine. To do this, suppose each of the three bars (in succession) removed, and examine the effect on the structure—an artifice which often enables us to see very clearly the nature of the stress on a given part of the structure.

In the first place, suppose *CD* removed; then the portion *EB* will turn about the joint *E*, as shown in the lower part of the diagram, so that the function of the bar *CD* must be to prevent this turning, which is exactly what we have previously described as bending. The tendency to turn round *E*—that is, the bending moment at *E*—is in this case simply $= W \times CB$. But if there is a system of loads, the bending moment at *E* may be found by methods previously described.

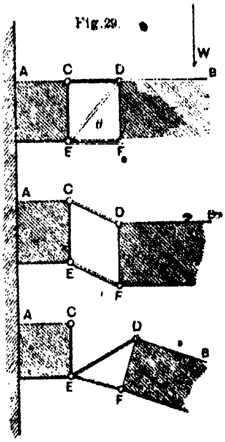


Fig. 29.

Now let H = stress on *CD*. It may readily be seen to be a tensile stress, because, on the removal of the bar, the ends *C* and *D* separate from one another. Also, let $h = CE$ or DF, the depth of the beam. The power of *CD* to prevent *EB* from turning about *E* is measured by the moment about *E* of the force H which acts along it. Therefore

$$Hh = M_E$$

And dividing the bending moment at *E* by the depth of the beam, we obtain the magnitude of the tension of *CD*.

Next, let the bar *EF* be removed. The structure will yield by turning round the joint *D*, the point *F* approaching *E*. Thus the bar *EF* is in compression, and by its thrust, $= H'$ say, towards *F*, it prevents *FB* from turning round *D*.

The tendency to turn round *D*, due to the action of the external forces $= M_D$, will be equal to the resisting moment $H'h$.

$$\therefore H'h = M_D.$$

Therefore, if we divide the bending moment for the joint *D* opposite to the bar, by the depth of the beam h, we obtain the magnitude of the compressive force H'.

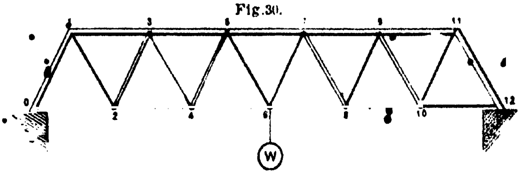
Lastly, let us suppose the diagonal bar *ED* to be removed, the effect is quite different from the two former cases; for instead of the overhanging portion of the beam turning about some point, it now gives way by sliding downwards (as shown in the centre of the diagram), remaining horizontal all the time. CD and EF turn about C and E, remaining parallel to one another. The rectangle $CDFE$ becomes distorted by the shortening of the diagonal ED and the lengthening of CF. In the structure then the function of the diagonal bar ED is to prevent the sliding, by resisting the tendency to shorten. Thus the bar ED must be in compression, and by its thrust upon the point D it maintains FB from sliding downwards. Let S = thrust along ED and θ = angle it makes with the vertical. The force S may be resolved into two components, a horizontal one, $S \sin \theta$, and a vertical one, $S \cos \theta$. It is the vertical component alone which resists the sliding action, and maintains D in its proper position. Now the tendency to slide is no other thing than the shearing force on the structure, which we have previously been investigating. In this example the shearing force is simply W for all sections between A and B. But in other cases of loading the shearing force may be estimated by previously given methods. Since the downward tendency of the shearing force is balanced by the upward thrust of the vertical component of S, we have in all cases

$$S \cos \theta = F.$$

Instead of the points E and D being joined there might have been a bar CF, which, by the resistance to lengthening which it would offer, would have sustained the portion FB from sliding downwards. Such a bar would be in tension just as the bar ED is in compression, and in finding the stress on it we should use exactly the same equation. Now, instead of having three bars only, the whole structure may be built up of horizontal and diagonal bars. The same principles will apply. On removing any one of the horizontal bars, we see that the structure yields by turning round a joint opposite: so we say the function of the horizontal bars is to resist bending. This is expressed by the equation $Hh = M$. On the other hand, the function of the diagonal bars being to resist the shearing tendency, we have always $S \cos \theta = F$.

24. *Warren Girders under various Loads.*—Fig. 30 shows a Warren Girder, so called from the name of the inventor, Captain Warren, a type of girder much used for bridges since its first introduction about the year 1850. It consists of a pair of straight parallel booms connected together by a triangulation of bars inclined to each other, generally at 60°, so that the triangles formed are equilateral. The booms in the actual structure are generally continuous through the junctions with the diagonal bars, but, if well constructed, there is no sensible error regarding the structure as a true frame, in which the several divisions are all united by perfectly smooth joints. Any three bars forming a parallelogram and its diagonal may be considered as playing the same part as regards the rest of the structure as in the case just considered.

Fig. 30.



When a Warren girder is used, it is generally supported at the ends, and the loads are applied at one or more joints in the lower boom. We will examine some examples.

(1) Suppose there is a single load applied at a joint in the centre of the span.

First as to the diagonal bars. It was shown above that the duty of these bars was to prevent the structure yielding under the action of the shearing force; the vertical component of the stress on either of the diagonal bars being equal to the shearing force for the interval of the length of the girder within which the diagonal bar lies. This is expressed by the equation

$$S \cos \theta = F.$$

Now, in the example which we are considering with the load in the centre, the shearing force will be the same at all sections to the right and left, namely, $= \frac{1}{2}W$. Therefore the stress on all the diagonal bars is of the same magnitude.

$$S = \frac{W}{2 \cos 30^\circ} = \frac{W}{\sqrt{3}}.$$

If we consider the effect of removing either of the bars we shall find that commencing from one end they prevent alternately the shortening and lengthening of the diagonals which they join, so that, commencing with one end, the bars are alternately in compression and tension. The compression bars are shown in double lines.

Next, as to the several portions of the length of the top and bottom booms. As was shown above, the stress on any division of the horizontal bars has the effect of preventing a bending round the joint opposite; so that the moment of the stress about the joint is equal to the bending moment at the joint, due to the external forces. This is expressed by the equation

$$Hh = M.$$

Let a = length of a division.

Then, since the supporting force at the joint 0 is $\frac{1}{2}W$, the bending moment at the joints numbered 1, 2, 3, etc., are

$$M_1 = \frac{W}{2}\frac{a}{2} = \frac{Wa}{4},$$

$$M_2 = \frac{W}{2}a = \frac{Wa}{2},$$

$$M_3 = \frac{W}{2}\frac{3}{2}a = \frac{3Wa}{4},$$

and so on, the bending moments increasing in arithmetical progression.

Since the depth of the girder h is the same at all parts of the length, if we divide the M's each of them by h, we obtain the magnitude of the stress on the bars opposite the respective joints. Thus

$$H_{02} = \frac{Wa}{4h},\ H_{13} = \frac{Wa}{2h},\ H_{24} = \frac{3Wa}{4h},\ \text{and so on.}$$

We see, then, that the stress on the several divisions increases in arithmetical progression as we proceed from the ends towards the centre. By observing the effect of removing either of the bars, we see that all the divisions of the upper boom are in compression. This is expressed by drawing them with double lines in the figure. All the divisions of the lower boom are in tension.

(2) Next suppose the load is applied at some other joint not in the centre—the joint 4 for example. We must first calculate the supporting forces. Suppose they are P at 0 and Q at 12. For the portion of the girder to the left of 4 the shearing force will be the same at all sections and be equal to P. So the stress on all the diagonals between 0 and 4 will be equal to $P\sec 30^\circ$.

To the right of joint 4 the shearing force $= Q$, and the stress on all the diagonal bars from 4 to 12 will be $Q\sec 30^\circ$.

Proceeding from either end towards the joint where the load is applied, we observe that the diagonal bars are alternately in compression and tension—so that the bar 56 is now in compression, whilst the bar 54 is in tension. On these bars the nature of the stresses is just opposite to that to which they were exposed when the load was at the centre joint. Thus by varying the position of the load, we not only vary the magnitude of the stress, but we may in some cases change the character of the stress, requiring a diagonal bar to act sometimes as a strut and sometimes as a tie.

For the divisions of the horizontal booms on the left of W the stresses are

$$\frac{Pa}{2h},\ \frac{2Pa}{2h},\ \frac{3Pa}{2h},\ \text{etc.},$$

in arithmetical progression up to the bar opposite the joint to which the load is applied; and to the right of W,

$$\frac{Qa}{2h}, \frac{2Qa}{2h}, \frac{3Qa}{2h}, \text{etc.},$$

in arithmetical progression also up to the bar opposite the load. The upper bars are all in compression and the lower in tension as before.

When there are a number of loads placed arbitrarily at the different joints, the simplest way of determining the stresses is often to find the stress on the bars due to each load taken separately, and then apply the principle of superposition. In applying the principle due regard must be paid to the nature of the stress. A compressive stress must be considered as being of opposite sign to a tensile stress, and, in compounding, the algebraical sum of the stresses for each load will be the total stress on the bars.

(3) There is one particular case, that in which the girder is uniformly loaded, which it is advisable to examine separately.

In general, the load on the platform of the bridge is by means of transverse beams or girders transferred to the joints of the lower boom. The transverse beams may be the same in number as the joints in the lower boom. In that case the girder will be loaded with equal weights at all the bottom joints. If the transverse beams are more numerous their ends will rest on the bottom booms, and tend to produce a local bending action in each division, in addition to the tensile stress which, as the bottom member of the girder, it will have to bear. In some cases, to lessen or get rid of this bending action, vertical suspending rods are introduced, by which means the middle points of the lower divisions are supported, and the loads transmitted to the upper joints of the girder. In such a case we may take all the joints both in the upper and lower booms to be uniformly loaded.

We will, however, suppose equal weights applied to the joints of the lower boom only. First as to the shearing forces. Between the end and the first weight the shearing force = the supporting force = half the total load = P say. In the next division the shearing force is less by the amount of the load at the first lower joint = $P - W$. In the third division of the lower boom from the end the shearing force = $P - 2W$, and so on. The stresses on the diagonals can now be found by multiplying the shearing force in the division within which any one diagonal lies by the secant of the angle which the diagonal makes with the vertical. The stresses will diminish in arithmetical progression as we pass inwards from the ends towards the centre. It will be observed that on the first and second diagonals from the end the stress is of the same magnitude. On the third and fourth t is alike also, and so on. The stresses are alternately compression and tension, commencing with compression on the first bar.

To find the stresses on the booms we must determine the bending moments at all the joints.

$$M_1 = \frac{P}{2}a. \qquad M_2 = \frac{P}{2}a.$$

$$M_3 = \frac{P}{2}3a - W\tfrac{1}{2}a. \qquad M_4 = \frac{P}{2}4a - Wa$$

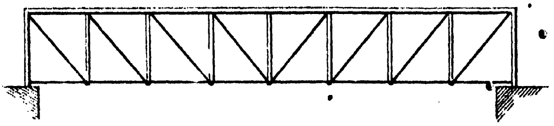
$$= \frac{a}{2}(3P - W). \qquad = \frac{a}{2}(4P - 2W).$$

$$M_5 = \frac{a}{2}(5P - 4W). \qquad M_6 = \frac{a}{2}(6P - 6W).$$

Division of the M's by h, the depth of the girder, will give the several horizontal stresses. They will be found to increase as we pass from the ends towards the centre.

25. *N. Trusses.* The web of the girder, instead of consisting of bars sloping both ways, forming a series of equilateral triangles, may be constructed of bars placed alternately vertical and sloping at an angle, so forming a series of right-angled triangles, looking like a succession of capital letters N. (See Fig. 31.) For this reason it is sometimes called an N girder. The ordinary practice is to divide the girder into a number of squares by means of the vertical bars, so that the diagonals slope at an angle of 45°. It is advantageous to place the diagonals so as to be in tension. For a load in the centre, or a uniformly distributed load, they should slope upwards from the centre towards the ends. The vertical bars will then be in compression. A short bar is better able to resist compression than a long one, whereas a tension bar is of the same strength whether short or long; so it is manifestly economical of material, and a saving of weight, to place the long bars, that is the sloping bars, so as to be in tension. The same methods will apply to find the stresses on the bars, since, as before, the web resists the shearing action, and the booms the bending.

Fig. 31.



The simple queen truss, considered in Chapter I., Section II., is another example of a web consisting of alternate vertical and diagonal bars, but the diagonal is not usually inclined at 45° to the vertical.

EXAMPLES.

1. A trapezoidal truss is 24 feet span and 3 feet deep. The central part is 8 feet long and is braced by a diagonal stay so placed as to be in tension. Find the stress on each part when loaded with 4 tons at one joint and 5 tons at the other.

Stress on diagonal stay ·95 ton.

2. A bridge is constructed of a pair of Warren girders, with the platform resting on the lower booms, each of which is in 6 divisions. The bridge is loaded with 20 tons in the middle. Find the stress on each part.

3. In example 2 obtain the result when the load is supported at either of the other joints.

4. From the results of examples 2 and 3 deduce the stress on each part of the girder when the bridge is loaded with 60 tons, divided equally between the three pairs of joints from one end to the centre.

Results for questions 2, 3, 4. the bars being numbered as in Fig. 30.

Bars.	Stress on Boom				Bars.	Stress on Diagonals.			
	Load at 6.	at 4	at 2	at 6, 4, and 2		Load at 6	at 4	at 2	at 6, 4, and 2.
02	2·88	3·85	4·8	11·53	01	5·76	7·7	9·6	23·06
13	−5·76	7·7	9·6	23·06	12	5·76	7·7	9·6	23·06
24	8·64	11·55	8·64	28·83	23	5·76	7·7	1·92	−11·54
35	−11·52	15·36	7·68	34·56	34	5·76	7·7	1·92	11·54
46	14·4	13·44	6·72	34·56	45	5·76	3·85	1·92	0
57	−17·28	11·52	5·76	34·56	56	5·76	3·85	1·92	0
68	14·4	9·6	4·8	28·8	67	5·76	3·85	1·92	11·54
79	11·52	7·68	3·84	23·04	78	5·76	3·85	1·92	11·54
8, 10	8·64	5·76	2·88	17·28	89	5·76	3·85	1·92	11·54
9, 11	5·76	3·84	1·92	11·52	9, 10	5·76	3·85	1·92	11·54
10, 12	2·88	1·92	·96	5·66	10, 11	5·76	3·85	1·92	11·54
					11, 12	5·76	3·85	1·92	11·54

5. A bridge of 80 feet span is constructed of a pair of N girders in 10 divisions, the platform resting on cross girders supported by the lower booms, and the diagonals so arranged as to be all in tension. A load of 80 tons is uniformly distributed over the platform. Find the stress on each bar. Draw polygons of shearing force and bending moment.

SECTION III.—GIRDERS WITH REDUNDANT BARS.

26. *Preliminary Explanations.*—Again, returning to the (p. 43) beam out of which a portion has been cut and replaced by bars, let us suppose that instead of one diagonal bar only, there are two. We require to find the stresses on the bars. First, on the diagonal bars. In this case also, the stress on these bars will be due to the shearing force. Together they prevent the structure yielding under the shearing action, but the amount each one bears is indeterminate until we know how the diagonals are constructed and attached to the rest of the structure. Suppose, for example, the diagonals are simple struts placed across the corners of the rectangle, but not secured at the ends. The struts will be incapable of taking tension; and the diagonal *ED*, which slopes in the direction, to be subject to compression will have to bear the whole shearing force. The other diagonal is ineffective. Secondly, suppose

the diagonals to be simple ties, such as a chain or slender rod, and so incapable of withstanding compression. Then the bar CF will carry the whole shearing force. We may have any number of intermediate cases between these extremes according to the material of the diagonals and the method of attachment. In all cases one diagonal tends to lengthen, and the other to shorten, and according to their powers of resistance to these tendencies they offer resistance to the shearing. If S_1 and S_2 be stresses on the two bars, then in all cases

$$(S_1 + S_2) \cos \theta = F.$$

If the diagonals are exactly similar rigid pieces similarly secured at the ends, equal changes of length will produce the same stress whether in compression or tension, so that each will bear an equal share of the shearing force. We shall then have

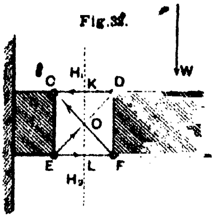
$$S_1 = S_2 = \tfrac{1}{2} F \sec \theta.$$

The foregoing is one of the simplest examples of a frame with redundant bars, and shows clearly why, in such cases, the stress on each bar cannot be determined by statical considerations alone, but depends upon the materials and mode of construction. In structures such as those considered in Chapter I., Section II., in which the principal part is an incomplete frame, stiffened by bracing or other means to provide against variations of the load, the bracing is usually redundant, and the stress on it cannot be calculated with certainty. Allowance has to be made for this in designing the structure by the use of a larger factor of safety. Redundant material is often no addition at all to the strength of the structure, and may even be a source of weakness, as will appear hereafter.

When framework girders were first introduced, it was objected by eminent engineers that failure of a single part would destroy the structure. Experience appears to have shown that risks of this kind are not serious, and the tendency of modern engineering design appears to be rather towards the employment of structures with as few parts as possible.

Next, as to the horizontal bars. These still sustain the bending moment, but not precisely in the same way as when there is only one diagonal. To find the magnitude of the forces we employ a method similar to that used before, but instead of removing a bar we suppose the girder cut through one or more bars at any place convenient to our purpose; then the principle which we make use of is, that the action of each of the two halves on the other must be in equilibrium with the external forces which are applied to either half. In Fig. 32 let us take a vertical section through the point of intersection of the diagonals, four bars are cut by the section, and through the medium of these four bars the structure to the left will act on the portion of the structure to the right of the section, and sustain it against the action of the external loads which rest on it.

Fig. 31.



First, there is the force H_1 pulling at K, and the force H_2 thrusting at L, and at O there are the two forces S_1 and S_2 on the two diagonals. Now, if we consider the tendency for the external forces to bend the right hand portion round O, we see that the diagonal bars offer no resistance to this bending action, and must so far be left out of account. The whole resistance to bending is due to the bars CD and EF along which the forces H_1 and H_2 act, so that if M_o be the bending moment at O due to the external forces,

$$(H_1 + H_2)\frac{h}{2} = M_o.$$

This will be true whatever be the proportion between S_1 and S_2, and H_1 and H_2. Instead, therefore, of taking the bending moment about a joint, as we did previously, we have in this case to take the moment about the point where the two diagonals cross.

But besides the balancing of the bending moment, there are other conditions to which the forces are subject, in order that the right-hand portion may be in equilibrium. One is, that all the forces which act on this portion must balance horizontally. There are no external forces which have any horizontal action, so that it is only the four internal forces which act along the bars cut, of which we have to take any account, and these must, on the whole, have no resultant horizontal action. The two thrusts must equal the two pulls; that is,

$$H_2 + S_2 \sin\theta = H_1 + S_1 \sin\theta.$$
$$\therefore\ H_2 - H_1 = (S_1 - S_2)\sin\theta.$$

This also is true whatever be the distribution of the shearing force between the two diagonals.

If, now, we suppose $S_2 = S_1$, then $H_2 = H_1 = H$, say. And the above formula becomes $Hh = M_o$, the same as we had before; but it must be applied a little differently, the moment now being taken about the point of intersection of the diagonals. If S_1 is not equal S_2, then H will be the mean of H_1 and H_2.

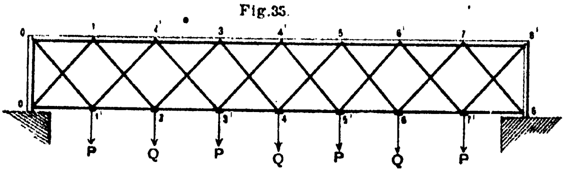
27. *Lattice Girders, Flanged Beams.*—Constructions with a double set of diagonals are common in practice. If, for example, in the N girder

(Fig 31) we place in each division two diagonals instead of one only, the construction is called a lattice or trellis girder. When employed for heavy loads, the diagonals are generally inclined at an angle of 45° to the vertical. In light structures, or when used for giving stiffness, they are often inclined at a much greater angle.

To determine the stresses, it will be necessary to make an assumption for the distribution of the shearing force between the two diagonals for each division of the girder, and it will generally be sufficiently correct to suppose each to carry half, and to write $S = \frac{1}{2} F \sec \theta$, and $Hh = M$ for the points where the diagonals intersect.

In lattice girders we more frequently find the double set of sloping bars introduced, but the vertical bars omitted. In this case it will not be true that the two diagonals in any one division are exposed to the same stress. We can determine the stresses otherwise. The structure may be divided into two elementary girders, each with its own system of diagonal bracing, and each with its own set of loads. Suppose, for simplicity, the number of divisions in the complete girder even, and each half girder loaded with equal weights applied to all the lower joints. Then if we make the simple, and in most cases safe, assumption that the thrusts on the two end vertical bars are equal, the forces on all the bars of the structure will be determinate. In the example shown in Fig. 33 the thrusts on the vertical end bars will be $2P$.

Fig.35.



After we have calculated the stresses on each bar in each elementary girder, then, for any bar which is a portion of both, we must compound to obtain the total stress.

We may further increase the number of diagonal bars and obtain a girder, the web of which is a network of bars. In this case it will not be exactly, but will be very nearly, true that the horizontal bars take the bending, and the sloping bars the shearing action, the shearing force being regarded as equally distributed between all the diagonals cut by any one vertical section.

We may go on adding diagonal bracing bars until the space between the booms is practically filled up, and even then assume that the bending is taken by the horizontal bars and the shearing by the web. The numerous bracing bars may then be replaced by a vertical plate, which will form a continuous web to the girder. Such a construction is a very common one in practice, the horizontal members are called the top and bottom flanges of what is still a girder, and often called so, but more often a flanged or I beam. In the smallest class of these beams, they are rolled or cast in one piece; but for large spans they are built up of plates and angle irons riveted together. For figures showing the transverse sections of such beams see Part IV. In taking the depth of such a girder, to make use of in the equation $Hh = M$, we ought to measure the vertical distance between the centres of gravity of the parts which we consider to be the flanges of the beam or girder. In the simple rolled or cast beam this will be the distance from centre to centre of depth of flanges. In the built-up beam account must be taken of the effect of the angle irons.

It must be remembered that this method of determining the strength of an I beam is only approximate. Its strength will be determined in a more exact way hereafter, when it will be found that the web itself assists in resisting the bending moment, but, area for area, to the extent only of about one-third that borne by the flange. On the other hand, the effective depth is less than the distance from centre to centre of the flanges. In rough preliminary calculations we may often neglect this, and employ the same formula as for lattice girders.

Girders are often of variable depths, so that the booms are not parallel; when this is the case the booms assist in resisting the shearing action of the load, as will be seen hereafter.

EXAMPLES.

1. A beam of I section is 24 feet span and 16 inches deep; the weight of the beam is 1,380 lbs. It is loaded in the centre with 5 tons. Assuming the resistance to bending to be wholly due to the flanges, find the maximum total stress on each flange and the sectional area of each the resistance to compression being taken to be 3 tons and to tension 4 tons per square inch.

Maximum total stress = 53,505 lbs. = 23·88 tons.
Sectional area of upper flange = 8 square in.
" " bottom " = 6 "

2. A trellis girder, 24 feet span and 3 feet deep, in three divisions, separated by vertical bars, with two diagonals in each division, is supported at the ends and loaded (1) with 20 tons symmetrically distributed over the middle division of the top flange, (2) with 20 tons placed over one of the vertical bars. Find the stress on each part of the girder, assuming each diagonal to carry half the corresponding shearing force.

Stress on diagonals—Case 1.	14·2	0	14·2
Case 2.	$18\frac{7}{8}$	$9\frac{1}{2}$	$9\frac{1}{2}$

Remark.—These results show the unsuitability of this construction for carrying a heavy load on account of the great inclination of the diagonals to the vertical.

3. A water tank, 20 feet square and 6 feet deep, is wholly supported on four beams, each carrying an equal share of the load. The beams are ordinary flanged ones, 2 feet

deep. Find approximately the maximum stress on each flange, assuming that the weight of the tank is one-fourth the weight of water it contains.

$$\text{Distributed load on one beam} = \frac{187{,}500}{4} = 46{,}875 \text{ lbs.}$$

$$H_{\max} = 58{,}593 \text{ lbs.} = 26{\cdot}1 \text{ tons.}$$

4. The Conway tubular bridge is 412 feet span. Each tube is 25 feet deep outside and 21 inside. The weight of tube is 1,150 tons, and the rolling load is estimated at $\frac{3}{4}$ ton per foot-run. Find approximately the sectional areas of the upper and lower parts of the tube, the stress per square inch being limited to 4 tons.

$$H_{\max} = 3{,}267 \text{ tons.}$$

$$\text{Area} = 817 \text{ square in.}$$

5. In the girder shown in Fig. 33, p. 52, suppose the weights P and Q are each 1 ton. Find the stress on each member. If the girder be stiffened by the addition of vertical members at each joint of the beams; find the stress on each member, making the usual assumption.

6. A rectangular tank with vertical sides and flat bottom is filled with water to a depth of 15 feet. The sides of the tank are constructed of iron plates riveted together and stiffened by vertical ⊥ irons spaced 4 feet apart. Assuming these stiffening pieces to take the whole bending action due to the water pressure: find the maximum bending moment on one of the stiffening pieces.

REFERENCES.

For details of construction of girders the reader is referred to

Girder Making . . . in Wrought Iron. E. HUTCHINSON. Spon, 1879.

CHAPTER III.

STRAINING ACTIONS DUE TO ANY VERTICAL LOAD.

28. *Preliminary Remarks.*—The preliminary discussion in the preceding chapter of the straining actions to which loaded beams and framework girders are subject will have given some idea of the importance of the effect of shearing and bending on structures, and we shall now go on to consider the question somewhat more generally.

Let us suppose any body or structure possessing, as it usually will, a longitudinal vertical plane of symmetry, to be acted on by a set of parallel forces in equilibrium symmetrically disposed with respect to this plane, as, for example, gravity combined with suitable vertical supporting forces. Then these forces will be equivalent to a set of parallel forces in the plane of symmetry in question. Let the structure now be divided into two parts, A and B, by an ideal plane section, parallel to the forces and perpendicular to their plane. Then the forces acting on A may be reduced to a single force F lying very near the section considered and a couple M, while the forces acting on B may be reduced to an equal and opposite force F lying very near the section and an equal and opposite couple M. The pair of forces are the elements of the shearing action on the section, and the pair of couples are the elements of the bending action on the section. As the nature of the structure is immaterial, we may consider these straining actions for a given vertical section quite independently of any particular structure, and describe them as the Shearing Force and Bending Moment *due* to the given Vertical Load. We shall first consider the connection which exists between the two kinds of straining action and the method of determining them for any possible load.

CONNECTION BETWEEN SHEARING AND BENDING.

29. *Relation between the Shearing Force and the Bending Moment.*—Figure 34 shows the lines of action of weights W_1, W_2, etc., placed at the successive intervals a_1, a_2, etc.

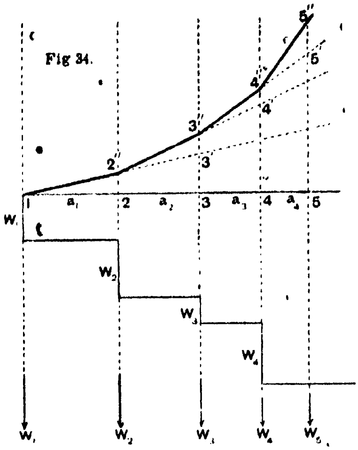


Fig 34.

In the first division the shearing force is

$$F_1 = W_1;$$

in the second $F_2 = W_1 + W_2 = F_1 + W_2$,

$$\therefore F_2 + F_1 = W_2;$$

in the third $F_3 = W_1 + W_2 + W_3 = F_2 + W_3$,

$$\therefore F_3 - F_2 = W_3;$$

and so on for all the divisions, so that in the n^{th} division

$$F_n - F_{n-1} = W_n.$$

We express this in words by saying that *the difference between the shearing forces on two consecutive intervals is equal to the load applied at the point between the two intervals*; or it may be written

$$\Delta F = W.$$

By setting down ordinates to a horizontal base line we obtain the stepped figure as the graphical representation of the shearing force at any point of the beam. It is drawn by first setting downwards at 1 an ordinate for the shearing force on the first interval, and then passing along the beam to the other end, on meeting the lines of action of the successive weights the length of the ordinates is increased by the amount of the weights. In so doing we make use of the proposition which has just been proved.

This is called the *Polygon of Shearing Force*, or more generally, when the loads are continuous, the *Curve of Shearing Force*.

Next as to the bending moment. At the first point where W_1 is applied $M_1 = 0$,

at the second point $M_2 = W_1 a_1 = F_1 a_1$;

at the third point $M_3 = W_1(a_1 + a_2) + W_2a_2 = W_1a_1 + (W_1 + W_2)a_2$

$= M_2 + F_2a_2$,

$\therefore\ M_3 - M_2 = F_2a_2$;

at the fourth point $M_4 = W_1(a_1 + a_2 + a_3) + W_2(a_2 + a_3) + W_3a_3$

$= W_1(a_1 + a_2) + W_2a_2 + (W_1 + W_2 + W_3)a_3$

$= M_3 + F_3a_3$,

$M_4 - M_3 = F_3a_3$,

and generally, $M_n - M_{n-1} = F_{n-1}a_{n-1}$

We may express this in words by saying that the *difference between the bending moments at the two ends of an interval is equal to the shearing force, multiplied by the length of the interval.* Or the result may be written

$$\Delta M = Fa.$$

We will now take a numerical example and see how we may make use of this property to determine a series of bending moments.

Let AB be a beam fixed at one end, and loaded with weights of 2, 3, 5, 11, 13, 7 tons, placed at intervals of 3, 2, 3, 5, 4, 6 feet,

W.	F.	a.	Fa.	M.
2				0
	2	3	6	
3				6
	5	2	10	
5				16
	10	3	30	
11				46
	21	5	105	
13				151
	34	4	136	
7				287
	41	6	246	
				533

commencing from the free end. We adopt a tabular method of carrying out the work of calculation.

First set down a column of weights applied, as shown by the figures in the column headed W. In the next column write the shearing forces. Since the shearing forces are uniform over the intervals between the weights, it will be best to write the F's opposite the spaces between the weights. Any F is found by adding to the F above it the adjacent W. In the third column we set down the lengths of the intervals. Then multiplying the F's and corresponding a's together, set the results in column 4. Lastly, we can write down the column of bending moments by the repeated addition of the Fa's—the bending moment at any point being found by adding to the bending moment at the point above the value of Fa between the points.

If instead of all the forces acting one way some of them act upwards, a minus sign should be set opposite, and all the operations performed algebraically.

The method is equally applicable however the beam is supported.

For example, let a beam 23 feet long be supported at the ends and loaded with 3, 2, 7, 8, 9 tons, placed at intervals of 2, 2, 3, 4, 5, 7 feet, reckoning from one end.

First calculate one supporting force, say at the left-hand end by taking moments about the other end. In the column of *W*'s set this for the first force, and since all the loads act in the contrary direction, put negative signs opposite them, and in writing down the next column of *F*'s add algebraically. We shall at the bottom of the column determine the supporting force at the right-hand end. At the bottom of the column of *M*'s, that is, at the point where the right-hand supporting force acts, we ought to get a zero moment. The obtaining of this will be a test of the accuracy of the work. In this example the small difference between 89·72 and 89·81 is due to our having taken the supporting force only to two places of decimals.

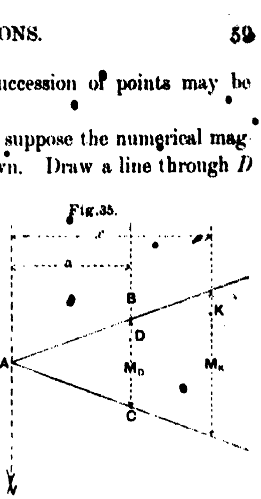
W.	*F*.	*a*.	*Fa*.	*M*.
16·17				0
	16·17	2	32·34	
3				32·34
	13·17	2	26·34	
2				58·68
	11·17	3	33·51	
7				92·19
	4·17	4	16·68	
8				108·87
	3·83	5	19·15	
9				89·72
	12·83	7	89·81	
12·83				0

Observation of the process of calculation leads us to a very important proposition, viz., *where the shearing force changes sign, the bending moment is at that point a maximum.* This will be true for all important practical cases, but exceptional cases may be imagined in which, where the shearing force changes sign, the bending moment is a minimum. Since $\Delta M = Fa$, then, so long as F is positive, M will be an increasing quantity as we pass from point to point. But where F changes to negative there M commences to diminish.

We will now explain the construction of a diagram of bending moment for a system of loads: and first let us consider how the moment of a force about any point or succession of points may be graphically expressed.

Let W be a force and D any point, and suppose the numerical magnitude of the moment of W about D known. Draw a line through D parallel to the force at a distance a (Fig. 35), and anywhere in this line take a length BC to represent on some convenient scale the moment, $M_D = Wa$, of W about D. The scale must be so many inch-tons, foot-lbs., or similar units to the inch. Then choose any point A in the line of action of the force, join AB and AC, and produce these lines indefinitely. The moment of W about any point whatever is represented by the intercept by the radiating lines AB, AC of a line drawn through the point parallel to the force. For example, the moment about $K = M_K = Wx$, where x is the perpendicular distance of K from the line of action of W.

Fig. 35.



$$\therefore \frac{M_K}{M_D} = \frac{Wx}{Wa} = \frac{x}{a}.$$

By similar triangles the intercepts are to one another in the ratio $x : a$ so that they correctly represent the moments.

We will first draw the diagram of bending moments for a beam fixed at one end and loaded at intervals along its length. Returning to Fig. 34, take a line representing the length of the beam as base line. Produce upwards the lines of action of the loads. Commence by setting up at the point where W_2 acts a line to represent the moment of W_1 about that point, that is, take $2' 2$ to represent $W_1 a_1$. If $1\ 2'$ be joined and produced, then the intercept between this line and base line $1\ 5$ will represent on the same scale the moment of W_1 about any point in the beam. Next at the point $3'$, where $1\ 2'$ cuts $W_3\ 3\ 3'$, set up $3'\ 3''$ to represent $W_2 a_2$, join $2''\ 3''$ and produce it. The intercept between $2'\ 3'$ and $2''\ 3''$ will represent the moment of W_2 about any point in the beam. Then at the point $4'$, where $2''\ 3''$ cuts W_4, set up $4'\ 4''$ to represent $W_3 a_3$. Join $3''\ 4''$, produce it, and so on with all the weights. The polygon $1, 2'', 3'', 4'', 5''$ … will be obtained, the ordinates of which measured from the base line AB will represent the bending moment at any point due to all the weights on the beam. This is called the *Polygon of Bending Moment*. In the case of a continuous distribution of load it is called the curve of bending moment.

There is a very important relation between the polygons of shearing force and bending moment which have been drawn in all cases of loading.

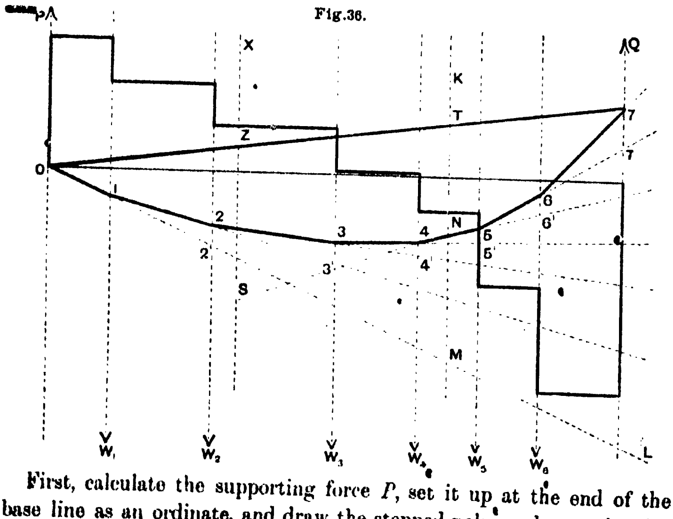
The bending moment at the point $2 = W_1a_1$. Now, referring to the shearing force diagram, we observe standing underneath the interval a_1 a rectangle whose area $= W_1a_1$. Next, for the point 3,

$$M_2 = W_1(a_1 + a_2) + W_2a_2.$$

This is represented on the diagram of bending moment by the ordinate 33″. In the shearing force diagram we notice that the area under the portion of the beam from 1 to 3 consists of two rectangles, $W_1(a_1 + a_2) + W_2a_2$. So that at this point also the bending moment is represented by the area of the polygon of shearing force, reckoned from the end up to the point 3. And so on for every point. This important deduction may be stated generally thus:—*The ordinate of the curve of bending moment at any point is proportional to the area of the curve of shearing force reckoned from one end of the beam up to that point.*

30. *Application to the case of a Loaded Beam.*—We will next take the case of a beam supported at the two ends.

Fig.36.



First, calculate the supporting force P, set it up at the end of the base line as an ordinate, and draw the stepped polygon by continually subtracting the W's. At some point in the beam we shall cross the base line. At that point the shearing force changes sign, and there the bending moment is a maximum. The shearing force on the last interval will give the magnitude of the supporting force Q. The polygon thus drawn will be the polygon of shearing force.

The polygon of bending moment may be drawn without previously determining the supporting force at either end thus:—

Commencing at O (Fig. 36), the point of application of P, draw any sloping line 0 1 2′ cutting W_1 in 1, and W_2 in 2′. Then set up

2′ 2 to represent W_1a_1, join 1 2, produce it to cut W_3 in 3′.
3′ 3 „ W_2a_2, „ 2 3, „ W_4 in 4′.
4′ 4 „ W_3a_3, „ 3 4, „ W_5 in 5′, and so on.
7′ 7 will represent W_6a_6.

Now join 7 with the point 0, where 0 1 2′ cuts the line of action of P. This is called the Closing Line of the polygon of moments. Any vertical intercept of this polygon will represent the bending moment at the corresponding point of the beam. The proof of this may be stated shortly thus:—If we produce 0 1 to meet the line of action of Q in L, then $L7$ will, from what has been said before, represent the sum of the moments of all the weights W about the end of the beam where Q acts. And from the conditions of equilibrium this must equal the moment of P about that end. Accordingly, if we take any point K, the vertical intercept MT below it will represent the moment of P about K. This is an upward moment. The four weights which lie to the left of K will together have a downward moment about K represented by MN. Therefore, the difference NT will represent the actual bending moment at the point K.

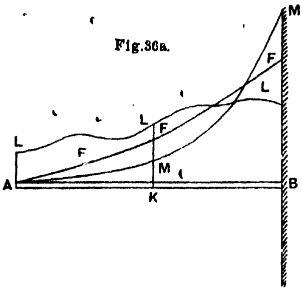
It sometimes happens that we want the moment of the forces, not about K, the section which separates the two parts of the structure, but about some other point, say X, in the figure. We can obtain this moment also with equal facility; for if we prolong the line 4 5 of the polygon to meet the vertical through X in the point S, we find, reasoning in the same way, that SZ, the intercept between the side so prolonged and the closing line, is the moment required.

Polygons of moments and shearing forces may also be constructed by making use of the fundamental relations shown above to exist between them and the load, as will be seen presently, while a third purely graphical method is explained farther on, based on a most important property which they possess.

31. *Application to the case of a Vessel floating in the Water.*—**We sometimes meet with cases in which the beam or structure is loaded not at intervals, but continuously, the distribution of the load not being uniform, but varied in some given way. In such a case the diagrams of shearing force and bending moment become continuous curves. The most convenient way of expressing how the load is distributed is by**

means of a curve, the ordinate of which at any point represents the intensity of the load at that point. Such a curve is called a *curve of loads*. It may be regarded as the profile of the upper surface of a mass of earth or other material resting on the beam:

Fig.36a.



We will consider, first, the case of a beam fixed at one end and loaded continuously throughout, in a manner expressed by a curve of loads *LL* (Fig. 36a). The total area enclosed by the curve of loads will represent the total load on the beam, and between the two ordinates of any two points will be the load on the beam between the two points. Now, the area of the curve of loads, reckoned from the end *A* up to any point, *K* say, since it represents the total load to the left of *K*, will be the shearing force at *K*. If at *K* we erect an ordinate *KF*, to represent on some convenient scale the area *ALK*, and do this for many points of the beam, we shall obtain a second curve *FF*, the curve of shearing force. Having done this, we may repeat the process on the curve *FF*, and obtain the curve of bending moment. For we have previously proved that if the load on the beam is concentrated at given points, then the ordinate of the curve of bending moments is at any point proportional to the area enclosed by the curve of shearing force for the portion of the beam between the end and that point. The truth of this is not affected by supposing the points of application of the load to be indefinitely close to one another, in which case the load becomes continuous. Accordingly, if we set up at *K* an ordinate, *KM*, to represent on some convenient scale the area *AFK* of the shearing force curve, and repeat this for many points, we obtain the curve of bending moment, *MM*. Thus the three curves form a series, each being the graphical integral of the one preceding.

This process has an important application in the determination of the bending moment to which a ship is subjected on account of the unequal distribution of her weight and buoyancy along the length of the ship. On the whole, the upward pressure of the water, called the buoyancy, must be equal to the downward weight of the ship; and the lines of action of these two equal and opposite forces must be in the same vertical. But for any portion of the length, the upward pressure and the downward weight will not, in general, balance one another; so on account of the difference, shearing and bending of the ship will be induced. In the case of a rectangular block of wood floating in water, the upward pressure of the water will, for every portion of its length, equal the downward weight, and there will be no shearing and bending action on it. But in actual ships, the disposition of weight and buoyancy is not so simple. Taking any small portion of the length of the ship, the difference between the weight of that portion of the ship and the weight of the water displaced by that portion of the ship, will be a force which acts on the vessel sometimes upwards and sometimes downwards, according to which is the greater, just in the same way as forces act on a loaded beam producing shearing and bending. In the construction of the vessel, strength must be provided to resist these straining actions, and it is a matter of great practical importance to determine accurately the magnitude of them for all points of the length of the ship. We will select an example of very frequent occurrence, that in which at the ends of the ship the weight exceeds the buoyancy, whilst at the centre the buoyancy exceeds the weight. If the ship were very bluff ended, and carried a cargo of very heavy material in the centre hold, the distribution of weight and buoyancy would probably be the reverse of this.

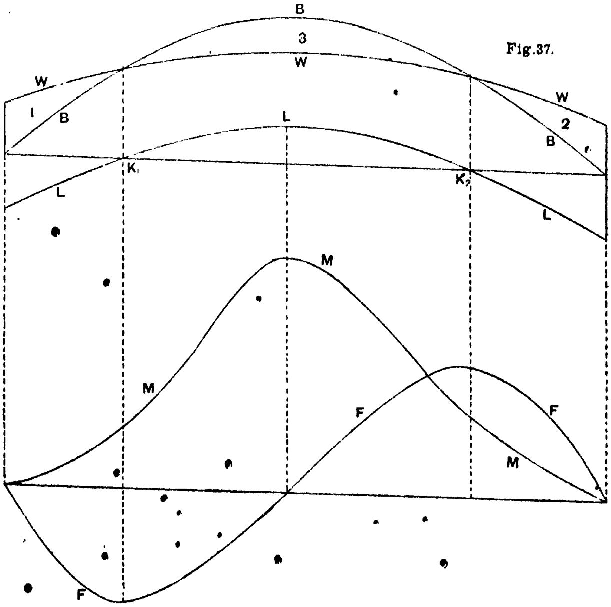


Fig. 37.

In the example the ship is supposed to be divided into any number of equal parts, and the weight of water displaced by each of those parts determined; ordinates are set up to represent those weights, and so, what is called a curve of buoyancy, *BBB* (Fig. 37), is drawn. The whole area enclosed by the curve will represent the total buoyancy or displacement of the vessel, and is the same thing as the total weight of the vessel. Next we suppose that the weights of the different portions of the ship are estimated, and ordinates set up to represent these weights, then what is called a curve of weight, *WWW*, is obtained. In the figure it is set up from the same base line. The total area enclosed by this curve will also be the total weight of the ship, and must therefore equal the area enclosed by the curve of buoyancy. Thus the sum of the two areas marked 1 and 2 must equal the area marked 3. Not only must this be true, but also the centres of gravity must lie on the same ordinate. The difference at any point between the ordinates of the two curves will express by how much at the ends the weight exceeds the buoyancy, and in the middle portion by how much the buoyancy exceeds the weight, representing, in the first case, the intensity of the downward force, and, in the second, the intensity of the upward force. Where the curves cross one another and the ordinates are the same height, as at K_1 and K_2, *the sections are said to be water-borne.* If now we set off from the base line ordinates equal to the difference between the ordinates of the two curves *BBB* and *WWW*, we obtain the curve of loads *LLL*; some portions where the weight is in excess will lie below the base line, and the rest, where the buoyancy exceeds the weight, will lie above the base line. From what has been said before, the area above the base line must equal the area below. Having obtained the curve of loads, the curve of shearing force is to be obtained from it in the manner previously described, by setting up, at any point, an ordinate to represent the area of the curve *LLL* between the end of the ship and that point. In performing the operation, due regard must be paid to the fact that the loads on different parts of the ship act in different directions, and for one direction they must be treated as negative, and the corresponding area of the curve as a negative area.

Having thus determined the curve of shearing force *FFF*, the same operation must be repeated on that curve to determine the curve of bending moment. In drawing the curve of shearing force it will be found that at the further end of the ship we return again to the base line from which we started at first, for the shearing force at the end must be zero. Also the bending moment at the end must be zero. This gives us tests of the accuracy of our work.

In this example the bending is wholly in one direction, tending to make the ends of the ship droop or the ship to "hog" in the technical language of the naval architect, but in some examples the direction of bending changes one or more times. Curves of shearing force and bending moment were first explained in relation to a vessel floating in the water by the late Professor Rankine in his work on shipbuilding. It does not, however, appear that any such curves were ever constructed in any actual example until 1869, when some were drawn for vessels of war by Mr. (now Sir E.) Reed, at that time chief constructor of the Navy. The results obtained by him are described in a paper read before the Royal Society (*Phil. Trans. for* 1871, part 2). They now form part of the ordinary calculations of a vessel.

Since the water exerts on the vessel not only vertical but also horizontal forces, the straining actions upon her do not consist solely of shearing and bending, but include also a thrust. The horizontal pressure also produces bending in a manner which we shall hereafter explain.

32. *Maximum Straining Actions.*—The set of forces we are considering are in equilibrium, and must therefore be partly upwards and partly downwards. The downward force is the total weight W, and is generally more or less distributed, the upward force is of equal magnitude, and is usually concentrated near two or more points. In the case of the vessel, however, the upward force is distributed like the weight, though not according to the same law. In any case the greatest shearing force must be some fraction of the weight, and the greatest bending moment must be some fraction of the weight multiplied by the length l over which the weight is distributed. We may therefore express the maximum straining actions by the formulae

$$F_0 = k \,.\, W\,;\quad M_0 = m \,.\, Wl,$$

where k, m are numerical quantities depending on the distribution of the load and the mode of support. Thus for a uniformly loaded beam supported at the ends $k = \frac{1}{2}$, $m = \frac{1}{8}$. The greatest value m can have in a beam resting on supports without attachment is $\frac{1}{4}$; this occurs when the beam is supported at the ends and the load concentrated in the middle or conversely. In vessels where the supporting force is distributed m is much less; its maximum value is estimated by Mr. (now Sir W. H.) White at $\frac{1}{35}$ in ordinary merchant steamers.

EXAMPLES.

1. The buoyancy of a vessel is 0 at the ends and increases uniformly to the centre, while the weight is 0 at the centre and increases uniformly to the ends. Draw the curves of shearing force and bending moment, and find the maximum values of these quantities in terms of the displacement and length of the vessel.

Answer—$k = \frac{1}{4}$; $m = \frac{1}{12}$.

C.M. E

2. A Warren girder with 12 divisions in the lower boom is supported at the ends and loaded with 250 tons, which may be supposed to be equally distributed among all the 25 joints. Find the stress on each bar by calculating the series of shearing force and bending moments.

RESULTS FOR LEFT-HAND HALF OF GIRDER.

F	115	105	95	85	75	65	55	45	35	25	15	5	5
S	132·2	120·7	109·2	97·7	86·2	74·7	63·2	51·7	40·2	28·7	17·2	5·7	
$\Delta H = F\frac{1}{\sqrt{3}}$	66·1	60·3	54·6	48·8	43·1	37·3	31·6	25·8	20·1	14·3	8·6	2·	
H		126·4		229·8		310·2		367·6		402		413·4	
	66·1		181		272·9		341·8		387·7		410·6		

3. A beam, 48 feet span, is supported at the ends and loaded with weights of 6, 9, 10, 13, 5, and 7 tons, placed at intervals of 4, 5, 9, 7, 13, and 8 feet respectively, commencing at one end. Calculate the shearing force in each interval and the series of bending moments.

4. In the last question construct the polygons of shearing force and bending moment.

5. In the case of a uniformly loaded beam supported at the ends, verify the principle that the area of the curve of shearing force is proportional to the ordinate of the curve of bending moment.

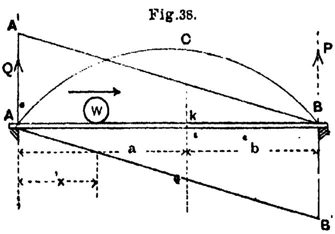
6. When a beam is supported at the ends and loaded in any way, show that an ordinate at the point of maximum moment divides the area of the curve of loads into parts, which are equal to the supporting forces. Further, if a, b are the distances of the centres of gravity of these parts from the ends of the beam, and l the span, show that the maximum moment is mWl where

$$\frac{1}{m} = \frac{l}{a} + \frac{l}{b}.$$

TRAVELLING LOADS.

33. We have hitherto been investigating the effect of a permanent fixed load on a structure in producing straining actions on it. We next examine the effect of a load which is not permanent, but which at different times takes up different positions on the structure, and we require to know what position of the load will produce the greatest straining action at any particular part of the structure, and also the amount of that maximum straining action.

Fig.38.



This question arises principally in the design of bridges across which a *travelling load*, such as a train, may proceed. We will take first the simple case of a beam of span l, supported at the ends, and suppose a single concentrated load W to travel across it in the direction of the arrow. Let us consider any point K (Fig. 38) in the beam, distant a and b from the ends. As the load traverses the beam, each position of the load will produce a certain shearing force and bending moment at the point K. To find their greatest value let x = distance of W from A, then the supporting force at $B = P = W\frac{x}{l}$. So long as the weight lies between A and K the shearing force at K will be simply P.

$$F_K = W\frac{x}{l},$$

consequently the shearing force will increase as x increases, until the load reaches the point K. So long as the weight lies to the left of K, the tendency will be for the portion KB to slide upwards relatively to the portion AK. This we describe in accordance with our definition on p. 34 as a *negative* shearing force. Therefore, putting $x = a$,

$$\text{Max. negative shearing force at } K = W\frac{a}{l}.$$

Now, supposing the weight to move onward, it will in the next instant have passed to the other side of K, and the shearing force will have undergone a sudden change. It will now be equal to the supporting force at the end A,

$$Q = W\frac{b}{l}.$$

But not only is the magnitude of the shearing force suddenly changed, but the tendency to slide is now in the other direction, and the shearing force is positive. As the weight moves further to the right of K the shearing force diminishes, thus

$$\text{Max. positive shearing force at } K =$$

Wherever we take the point K it will always be true that the maximum positive shearing force will occur when the weight lies immediately to the right of K, and the maximum negative when the weight lies immediately to the left. The maximum negative shearing force for every point in the beam may be represented by the ordinates of a sloping line AB' below the beam, the length BB' being taken to represent W. And similarly the maximum positive shearing force at any point by the ordinates of the sloping line $A'B$ about AA' also being taken to represent W.

Next as to the bending moment. When the weight lies to the left of K, and is at a distance from A equal to x, the bending moment at K is given by

$$Pb = W\frac{b}{l}x.$$

This goes on increasing as x increases until the weight reaches the point K. After having passed K the bending moment at K must be differently expressed, being then

$$\frac{W(l-x)}{l}a,$$

which becomes smaller as x increases; so that the greatest bending moment at K occurs when the load is immediately over K, and then the

$$\text{Max. bending Moment at } K = \frac{Wab}{l}$$

If the point K is taken in the centre of the beam,

$$\text{Max. Moment at centre} = \tfrac{1}{4}Wl \text{ as before.}$$

If ordinates be set up at all points to represent the maximum bending moments at these points, a parabola (ACB) will be obtained. For the expression for the maximum bending moment is just twice that previously obtained for the same weight distributed uniformly.

If there are more weights, W_1, W_2, etc., on the beam, and W_1 lie to the right of K, the shearing force at $K = P - W_1$, where P is the right-hand supporting force. Now, suppose we shift W_1 to the left of K, we shall diminish the supporting force to P' say, and this will be the new shearing force at K. The difference between P and P' will be less than W_1, and the shearing force will be increased by passing W_1 to the left of K. If we were to remove W_1 altogether the diminution of P will be less than the whole of W_1, and so the shearing force at K will be increased by so doing. We obtain the greatest positive shearing force at K when all the weights are to the right of K, but as near to K as possible. The greatest negative shearing force will occur when all the weights lie to the left of K, as near to K as possible.

The maximum bending moment at K will occur when the weights are as near K as possible, whether to the right or left. Any addition to the load, on whichever side of K it is placed, will cause an addition to the bending moment.

There is another important case, that in which we have a continuous load of uniform intensity passing over the beam, as when a long train passes on to a bridge. We observe that as the train, coming from A, approaches K, the supporting force at B, and therefore the shearing force at K, increases. When any portion of the weight lies to the right of K, the supporting force will be increased by a part of the weight lying to the right of K; but when we have subtracted the whole of that weight, the difference, which will be the shearing force at K, will be less than before; thus the maximum negative shearing force at K will occur when the portion AK is fully loaded, and no part of the load is on *KB*. To find its value we have only to determine the supporting force at *B*, by taking moments about *A*; then

$$F_K = wa\frac{\frac{1}{2}a}{l} = \frac{\frac{1}{2}wa^2}{l},$$

that is, the magnitude is proportional to the square of the distance of the point from the end *A*. It will be graphically represented by the ordinates of a parabola which has its vertex at *A*, and axis vertical, cutting the vertical through *B* in a point *B'* such that $BB' = \frac{1}{2}wl$, that is, half the weight on the beam when fully loaded. As the load travels onward the shearing force diminishes at last to zero, and then changes sign, becoming positive, the numerical magnitude increasing as the rear of the load approaches *K*. The maximum positive shearing force will occur when the portion *KB* only is loaded. The ordinates of a parabola set below the line of the beam having its vertex at *B* and axis vertical, will represent the maximum negative shearing force.

The question of maximum bending moment is more simple. It will occur at any point when the beam is fully loaded; for at any point the bending moment is the sum of the bending moments due to all the small portions into which the load may be divided, and the removal of any one of them will cause a diminution of bending action throughout the whole length of the beam. A parabola, with its highest ordinate at the centre $= \frac{1}{8}wl^2$, will represent it at any point.

34. *Counter-bracing of Girders.*—In the design of a framework girder it is very important to take account of the maximum positive and negative shearing forces due to a travelling load.

In such a structure the shearing force is resisted by the diagonal bars, and in general these bars are so placed as to be in tension, for the bar may then be made lighter than if subject to a compressive force of the same amount. Suppose the diagonal bars so arranged as to be all in tension when the girder is fully loaded, or when there is only the dead weight of the girder itself to be taken account of. There may be ample provision made for withstanding the tensile forces, and yet it will be important to examine if there may not be some disposition of the travelling load which would cause a thrust on some of the diagonals. If so, the maximum amount of this must be calculated, and the structure made capable of withstanding it. If the shearing force at any section of the girder is what we have called a positive shearing force, that in which the left-hand portion tends to slide upwards relatively to the right, then, in order that it may be withstood by the tension of a diagonal bar, the bar must slope upwards to the left. If the bar so slopes, and by the movement of the travelling load the shearing force becomes negative, then the bar will be subjected to compression. Now, it will frequently happen that in the central divisions of a girder the positive or negative shearing forces due to the dead load are less than the negative or positive shearing forces due to the travelling load, so that if those bars are arranged to be in tension under the dead load, then, on the passage of the travelling load, the stress will be changed to compression. In some cases the bars are slender and not suited to sustain compression; the shearing force is then provided for by the addition of a second diagonal, sloping in the opposite direction, which, by its tension, will perform the duty the first bar would otherwise have to perform by compression. Such a bar is called a counter-brace. We frequently see such additional bars fitted to the middle divisions of framework girders.

Again, the powers of resistance of a piece of material to a given maximum load are greater the smaller the fluctuation in the stress to which it is exposed; and therefore, in determining its dimensions, it is important to know not only the maximum but also the minimum stress to which it is exposed. This can be done on the principles which have just been explained.

EXAMPLES.

1. A single load of 50 tons traverses a bridge of 100 feet span. Draw the curves of maximum shearing force and bending moment, and give the values of these quantities for the quarter and half span.

2. A train weighing one ton per foot-run, and more than 100 feet long, traverses a bridge 100 feet span. Draw the curves of maximum shearing force and bending moment, and give the values of these quantities at the quarter and half span.

3. In the last question, suppose the permanent load $\frac{3}{4}$ths ton per foot-run. Find within what limits counter-bracing will be required. *Ans.*—21 feet at the centre.

4. In Ex. 5, p. 49, the maximum rolling load is estimated at 1 ton per foot-run. Determine which of the diagonals will be in compression, and the amount of that compression, assuming a complete number of divisions to be loaded.

The two centre diagonals are the only ones which can be in compression, the maximum amount of which will be $=(3{\cdot}2-2)\sqrt{2}=1{\cdot}7$. It will occur when the rolling load occupies four divisions only of the bridge.

5. In the last question, suppose a single load of 20 tons to traverse the bridge. Find the maximum stress, both tension and compression, on each part of the girder.

Divisions.	1	2	3	4	5
Max. tension, bottom boom, - -	0	27	48	63	72
Max. compression, upper boom, -	27	48	63	72	75
Max. tension of diagonals, - -	38·1	31·1	24	17	9·8
Max. compression of diagonals, -	—	—	--	0	2·8

6. In the two preceding questions, find the fluctuation of stress on each part of the girder.

METHOD OF SECTIONS.

35. *Method of Sections applied to Incomplete Frames. Culmann's Theorem.*—The straining actions due to a vertical load may either be wholly resisted by internal forces called into play within the structure itself, or also in part by the horizontal reaction of fixed abutments: the supporting forces being in the first case vertical, and in the second having a horizontal component. The distinction is one of the greatest importance in the theory of structures, which are thus divided into two classes; Girders and Arches, including under the last head also Chains. It is the first class alone which we consider in this chapter.

The general consideration of internal forces is outside the limits of this part of our work, and we shall here merely consider some cases of framework structures, commencing with that of an incomplete frame.

Incomplete frames are in general, as in Chapter I., structures of the arch and chain class, but by a slight modification we can readily convert such a frame into a girder and thus obtain very interesting results.

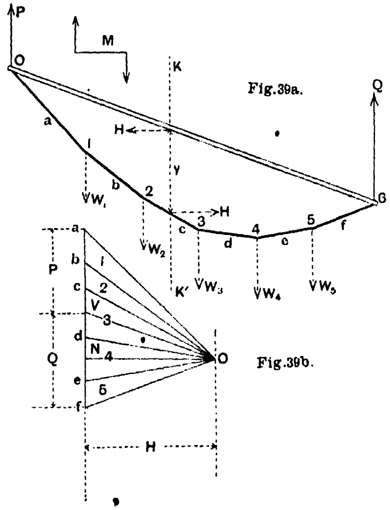


Fig. 39a.

Fig. 39b.

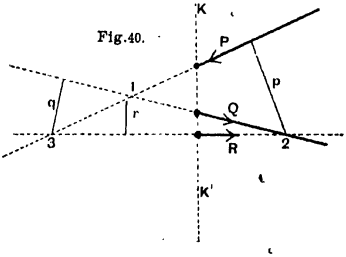
Fig. 39a shows a funicular polygon such as that in Fig. 11, page 14, except that the supports are removed and replaced by a strut *O*6. By this addition the polygon becomes a closed figure, and *O*6 is therefore called its "closing line." The structure is carried by suspending rods at the joints *O*6, and loaded as shown. The construction of the diagram of forces, Fig. 39b, has been sufficiently explained on the page referred to, and it only remains to observe that the supporting forces PQ are immediately derived from the diagram by drawing OV parallel to the closing line, which is not necessarily horizontal. The horizontal thrust of the strut and tension of the rope is found as before by drawing ON horizontal.

The structure may now be regarded as a girder, the load on which, together with the vertical supporting forces, produce definite straining actions M and F on any section. Let the section be KK' in the figure, cutting one of the parts of the rope and the strut as shown in the figure: let the intercept be y. Consider the forces acting at the section on the left-hand half of the girder, the horizontal components of these forces are equal and opposite, acting as shown in the figure, each being H or ON in the diagram of forces. The vertical components are balanced by the shearing force, and the horizontal components by the bending moment, which last fact we express by the equation

$$Hy = M,$$

that is to say, the funicular polygon corresponding to a given load is also a polygon of bending moments, the intercept between the polygon and its closing line multiplied by the horizontal force is equal to the bending moment due to the load. Hence, by a purely graphical process, we can construct a polygon of moments, for we have only to construct a funicular polygon corresponding to the load as shown in the article already cited, and complete it by drawing its closing line. This is one of the fundamental theorems of graphical statics, a subject which of late has been extensively studied. The construction is intimately connected with the process of Art. 29 as the reader should show for himself. In its complete form it is due to Culmann and is generally known by his name, having been given in his work on graphical statics.

Fig. 40.



36. *Method of Sections in general. Ritter's Method.*—In frames which are complete the number of bars cut by the section, instead of being two only, as in the preceding case, is in general three at least.

In Fig. 40 let KK' be the section cutting the three bars in three points, which may be considered as the points of application of three forces PQR due to the reaction of the bars, which balance the shearing and bending actions to which the section is subject. Resolving horizontally and vertically, and taking moments, we should—remembering that the load being wholly vertical the sum of the horizontal components must be zero—obtain three equations which would determine P, Q, R. It is, however, simpler to employ a method introduced by Ritter which enables us to obtain the value of each force at once. Let the lines of action of P, Q intersect in the point 1, Q and R in 2, P and R in 3, and let the perpendicular dropped from each intersection on to the line of action of the third force be r, p, q respectively: by measurement on the drawing of the framework structure we are considering it is always easy to determine these perpendiculars. Then taking moments about the three points we get

$$Rr = L_1\,;\quad Pp = L_2\,;\quad Qq = L_3,$$

where L_1, L_2, L_3 are the moments of the forces acting on the left-hand half of the structure about the points 1, 2, 3 respectively. On page 59 it was shown how to get these moments graphically from the polygon of moments, but they also may be obtained by direct calculation.

We may write down a general formula for this method, thus—

$$Hh = L,$$

where H is the stress on any bar, h its perpendicular distance from the intersection of the two others cut by a section, and L is the moment of the forces about that intersection. The special case in which the intersection lies on the section considered so that the moment L becomes the bending moment (M) on the section has already been considered in Chapter II. When the stress on a single bar is required as a verification of results obtained by graphical methods, or where the maximum stress due to a travelling load has to be determined, this method is often serviceable, but as a general method it is inconvenient from the amount of arithmetical labour involved.

The shearing action on the section is resisted by the components parallel to the section of the stress on the several bars. In the case of the incomplete frame of Fig. 39, p. 71, these components are given at once by the diagram of forces. In general, however, three bars and only three, must be cut by the section if the frame be neither incomplete nor redundant; when two of these are perpendicular to the section the case is that considered in Chap. III. of a framework girder with booms parallel, in which the diagonal bars alone resist the shearing. When one bar only is perpendicular to the section, the other two collectively resist the shearing action: this case is common in bowstring and other girders of variable depth. The upper boom together with the web here resists the shearing.

When more than three bars are cut by the section, the stress in each is generally indeterminate on account of the number of bars being redundant. On this question it will be sufficient for the present to refer to Chapter II., Section II.

EXAMPLES.

1. In example 3, page 66, construct the polygon of bending moments by Culmann's method.

2. In example 6, page 32, find the stress on each part of the roof by Ritter's method.

3. In example 7, page 32, find the stress on each by Ritter's method.

4. If a parabolic bowstring girder be subject to a uniform travelling load, represented by the application of equal weights to some or all of the verticals, show that the horizontal component of the maximum stress on each diagonal is the same for all.

5. In the roof shown in Fig. 21, p. 28, employ Ritter's method to find the stress on the sloping struts and deduce the stress on each division of the tie rod.

6. The curve of shearing force for a vessel consists of two similar parabolas plotted with vertical axes on a base line representing the length of the vessel. The excess of weight over buoyancy of each end of the vessel up to the nearest waterborne section is $\frac{1}{15}$th her displacement; find the maximum bending moment. *Ans.*—$\frac{1}{45}Wl$.

7. A uniform raft of rectangular section, which when floating freely is immersed to $\frac{2}{3}$rds of its depth, has one end stranded so that the lower edge of that end is in the plane of flotation. Draw a diagram of shearing force, giving the value of some ordinates in terms of the whole weight of the raft, and show that the maximum bending moment is $\frac{1}{27}Wl$.

8. A circular ring cut out of a piece of sheet metal is balanced in a horizontal plane upon knife edges placed in a central line. Find the shearing force and bending moment at any radial section.

9. In question 6, p. 54, draw curves of shearing force and bending moment for one of the stiffening pieces.

10. A flat-bottomed vessel of length L and beam B floats horizontally in the water The sides are vertical and the water lines curves of sines given by the equation

$$y=\frac{B}{2}\sin\pi\frac{x}{L},$$

where x is measured from one end. The weight is uniformly distributed.—Find the curves of shearing force and bending moment. Deduce their maximum values.

$$\textit{Ans.}\text{—}F_0=\frac{2}{19}W;\ M_0=\frac{1}{29}WL \text{ nearly.}$$

REFERENCES.

For further information on subjects connected with the present chapter the reader may refer to

Naval Architecture. W. H. WHITE. Murray.

Elements of Graphic Statics. L. M. HOSKINS. Macmillan.

Graphical Statics. CREMONA. Clarendon Press.

The last mentioned work is a translation by Professor Beare, revised by Professor Cremona, of two small treatises on graphical calculation and reciprocal figures. (See Note in Appendix.)

CHAPTER IV.

FRAMEWORK IN GENERAL.

37. *Straining Actions on the Bars of a Frame. General Method of Reduction.*—When the bars of a frame are not straight, or when they carry loads at some intermediate points, the straining action on them is not generally a simple thrust or pull, but includes a shearing and bending action. The present and two following articles will be devoted to some cases of this kind.

First suppose the bars straight, but let one or more be loaded in any way, and in the first instance consider any one bar, AB (Fig. 41), apart from the rest of the frame, and suspended by strings in an inclined position. Let any weights act on it as shown in the figure, then the tensions of the vertical strings will be just the same as in a beam, AB, supported horizontally at the ends and loaded at the same points with the same weights. Resolve the forces into two sets, one along the bar, the other transverse to the bar. The second set produce shearing and bending just as if applied to a beam in a horizontal position, while the first set produce a longitudinal stress, which will be different in each division of the bar. Let θ be the inclination of the bar to the vertical, then the pulls on the successive divisions are

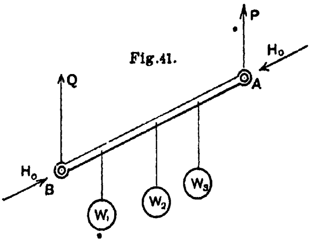


Fig. 41.

$$P.\cos\theta : (P - W_3)\cos\theta : (P - W_3 - W_2)\cos\theta : \ldots,$$

the last being a thrust equal to $Q.\cos\theta$, so that the stress varies from $Q\cos\theta$ to $-P.\cos\theta$. Now observe that we can apply to AB at its ends, in the direction of its length, a thrust, H_0, of any magnitude we please without altering P and Q, but that we cannot apply a force in any other direction, whence it follows that when AB forms one of the bars of a frame, its reaction on the joint A must be a downward force, P, and a force H_0, which must have the direction BA, while the reaction on B in like manner consists of a downward force, Q, and an equal force, H_0, in the direction AB. The downward forces P, Q are described as the part of the load on AB carried at the joints A, B, and it is now clear that if these quantities be estimated for each bar and added to the load directly suspended there, we must be able to determine the forces H_0 by exactly the same process as that by which we find the stress on each bar of a frame loaded at the joints. The actual thrust on AB evidently varies between $H_0 - P . \cos\theta$ at the top, to $H_0 + Q . \cos\theta$ at the bottom, so that H_0 may be described as the mean thrust on the bar, while the shearing and bending depend solely on the load on the bar itself, and not on the nature of the framework structure of which it forms part, or on the load on that structure. In the particular case where the load on the bar is uniformly distributed, the forces P, Q are each half the weight of the bar, and the thrust H_0 is the actual thrust at the *middle point* of the bar.

This question may also be treated by the graphical method of Art. 35 with great advantage. Through A and B draw a funicular polygon corresponding to the load on AB, the line OV in the diagram of forces will be parallel to AB and may be taken to represent H_0. This funicular polygon will be the curve of bending moment for the bar, and the other straining actions at every point are immediately deducible. It will be seen presently that the bar need not be straight.

For simplicity it has been supposed that the forces acting on the bar are parallel: if they be not, the reduction is not quite so simple. It will then be necessary to resolve the forces into components along the bar and transverse to the bar, the second set can be treated as above, while the total amount of the first set must be considered as part of the force supplied to the joints either at A or B. Such cases, however, do not often occur, and it is therefore unnecessary to dwell on them.

The joints have been supposed simple pin joints or their equivalents, but the method used for frames loaded at the joints will apply even if the real or ideal centres of rotation of the bars are not coincident, provided only the centre lines prolonged pass through the point where the load is applied. The method of reduction just explained then requires modification. Such cases are of frequent occurrence, and the next article will be devoted to them.

38. *Hinged Girders. Virtual Joints.*—The case of a loaded beam, the ends of which overhang the supports on which it rests, has already been considered in Art. 21, where it was shown that the straining actions at any point might be expressed in terms of the bending moments at the points of support, which of course will be determined by the load on the overhanging part. If the overhanging parts be supported, as in the case of a beam continuous over several spans, or with the ends fixed in a wall, the same formula will serve to express the straining actions at any point in terms of the bending moments at the points of support, but those bending moments will not be known unless the material of the beam and the mode of support are fully known. Hence the full consideration of such cases forms part of a later division of our work. Certain general conclusions can be drawn, however, which are of practical interest.

The graphic construction for the bending moment at any point of a beam, *CD*, which is not free at the points of support, is given in Fig. 28, p. 40. The figure refers to the case where the bending action at *C* and *D* is in the opposite direction to the bending action near the centre, as it is easily seen must be the case in general. The points of intersection of the moment line with the curve of moments drawn, as explained in the article cited, on the supposition of the ends being free, show where the negative bending at the ends passes into the positive bending of the centre. Here, there is no bending at all, and the central part of the beam (*EF* in figure) is exactly in the position of a beam supported but otherwise free at its ends. We may therefore treat the case as if *E* and *F* were joints, the position of which will be known if the bending moments at the ends are known, and conversely. In some cases there may be actual joints in given positions, while in others there will be "virtual joints," the position of which may be supposed known for the purposes of the investigation.

Fig. 42.

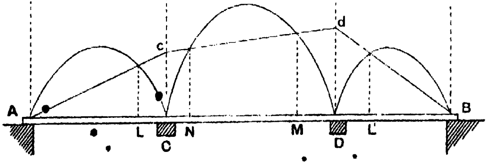


Fig. 42 shows a beam *AB* continuous over three spans, the moment curves for which will be known when the load resting on each span is known. It is evident from what has been said that the moment line must be the broken line *AcdB*, cutting the moment curve of the centre span in two points, and the moment curves of the end spans each in one point, the others being the ends of the beam. Thus there are four virtual joints, of which two must be supposed known in order to find the straining actions at any point. Their position will depend (1) on whether the supports are on the same level or not, (2) on the material and mode of construction of the beam, (3) on the load. Such a beam is in a condition analogous to that of a frame with redundant bars, considered in Chapter II., Section III.; the straining actions are indeterminate by purely statical considerations, for the same reason as before. We can, however, see that the bending action at each point is in general less than if the beam were not continuous.

In one particular case the position of the virtual joints can be foreseen. Suppose a perfectly straight beam, of uniform transverse section, to be continuous over an indefinite number of equal spans: let the weight of the beam be negligible, and let equal weights be placed at the centre of each span. Then since the pressure on each support must be equal to the weight, the beam is acted on by equal forces at equal distances alternately upwards and downwards, and there being perfect symmetry in the action of the upward and downward forces, the virtual joints must be midway between the centre and the points of support of each span.

In the special case where the beam is uniformly loaded we can further see that the load resting on the supports is not one half the weight of the parts of the beam resting there, as it would be if the beam were not continuous, but must in general be greater for the centre supports and less for the end supports. For if the virtual joints be *LNML'*, as in the figure, it is easily seen that *A* carries half the weight of *AL*, not of *AC*, while *C* carries half the weight of *A'L* and *NM*, together with the whole weight of *CL* and *CN*. This observation shows that in trussed beams where, as is usually the case, the loaded beam is continuous through certain joints, the effect of the continuity is generally to transfer a part of the weight from the joints where the ends are free to the joints where the beam is continuous. We shall return to this point hereafter.

The principle of continuity is frequently taken advantage of in the construction of girders of uniform depth by making them continuous over several spans. The virtual joints then vary in position for each position of the travelling load, rendering it a complicated matter to determine the maximum straining actions, while there is always an element of uncertainty about the results, for reasons already referred to and afterwards to be stated more fully.

In some structures, however, the joints have a definite position.

Fig. 43 shows a cantilever bowstring girder, consisting of a central bowstring girder *NM*, the ends of which rest on parts *ACN*, *BDM*, projecting from the piers, technically described as "cantilevers." The joints here are at *N* and *M*. In structures of great span, in which

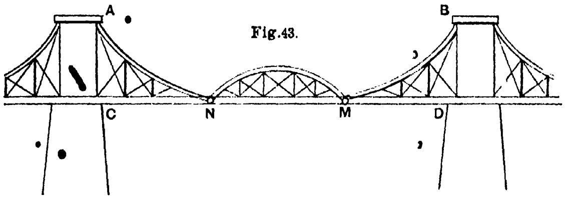


Fig.43.

the weight of the structure is the principal element, so that the variations in distribution are small, this type of girder is economical in weight. In the great bridge over the Forth, the central portion for each of two principal openings consists of a simple girder 350 feet span, while the cantilevers are each no less than 675 feet in length, making a total span of 1700 feet. These cantilevers are of great depth near the piers, and to provide against wind pressure, they are there likewise greatly increased in breadth, and solidly united to them. Full descriptions of this bridge, a structure which, from its gigantic dimensions and other unusual features, deserves attentive study, appeared in the engineering journals for 1890, and several have since been re-published in a separate form.

39. *Hinged Arches.*—In the second section of Chapter I. certain forms of arches were considered which are simply inverted chains, and require for equilibrium a load of a certain definite intensity at each point. We shall now take the case of an arch rib capable of sustaining a load distributed in any way. We shall suppose the load vertical, and, to take the thrust of the arch, we shall imagine a tie rod introduced so as to convert it into a bowstring girder. If the straining actions at each point of the rib are to be determinate without

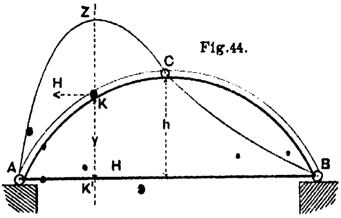


Fig.44.

reference to the relative flexibility of the several parts of the rib, and other circumstances, we must have, as in the case of the continuous beam, joints in some given position. The necessary joints are in this instance three in number, and, we shall suppose, are at the crown C (Fig. 44), and one at each springing A and B.

Taking a vertical section KK' through the rib and tie, let the bending moment due to the vertical load and supporting forces be M. This bending moment is resisted, *first*, by the horizontal forces called into play; that is to say, the pull of the tie rod H at K', and the equal and opposite horizontal thrust of the rib at K; *secondly*, by the resistance to bending of the rib itself, the moment of which we will call μ. Hence if y be the ordinate of the point considered, we must have

$$M = Hy + \mu.$$

To determine H we have only to notice that at the crown where $y = h$ there is a joint, that is, $\mu = 0$,

$$\therefore\ M_0 = Hh,$$

where M_0 is the bending moment due to the load for the central section. Thus, to determine μ we have the equation

$$\mu = M - M_0 \cdot \frac{y}{h}.$$

The graphic representation of μ is very simple. Let us imagine the curve of moments drawn for the given vertical load, and let it be so drawn as to pass through A, B, and C, which is evidently always possible. Then, if Y be the ordinate of the curve,

$$M = H \cdot Y.$$

Therefore, by substitution,

$$\mu = H(Y - y).$$

So that the bending moment at each point of the rib is represented graphically by the vertical intercept between the rib and the curve of moments. In the figure, the curve $AZCB$ is the curve of moments, and KZ is the intercept in question.

Arched ribs in practice are rarely, if ever, hinged, and the straining actions on them occasioned by a distribution of the load not corresponding to their form depend, therefore, upon the relative flexibility of the several parts of the rib, and other complicated circumstances. If the position of the virtual joints be known, or the bending moments at any three points, the graphical construction just given can be applied.

Instead of a rigid arch, from which a flexible platform is suspended, we may have a stiff platform suspended from a chain. This is the case where a suspension bridge is adapted to a variable load by means of a stiffening girder. For this case it will be sufficient to refer to Ex. 3, page 87.

40. *Structures of Uniform Strength.*—In any framework structure without redundant bars, the stress on each bar may be determined as in Chapter I., by drawing a diagram of forces for any given load, W, and expressed by the formula

$$H=kW,$$

where k is a co-efficient depending on the distribution of the load. If A be the sectional area of the bar we find by division the stress per sq. inch, which must not exceed a certain limit, depending on the nature of the material as explained in Part IV. of this work. When the structure is completely adapted to the load which it has to carry, the stress per square inch is the same for all the bars, and it is then said to be of Uniform Strength. Uniformity of strength cannot be reached exactly in practice, but it is a theoretical condition which is carried out as far as possible in the design of the structure. Other things being equal, the weight of a structure of uniform strength is less than that of any other. Such a structure is therefore less costly, for weight is to a great extent a measure of cost.

Whenever the load is known, the weight of a structure of given type, and of uniform strength can be calculated thus. Suppose A the sectional area of a piece, H, the stress on it, f, a co-efficient of strength, then

$$H=fA.$$

Next let w_0 be the weight of a unit of volume, usually a cubic inch, and assume

$$\lambda=\frac{f}{w_0},$$

then λ is a certain length, being in fact the length of a bar of the material which will just carry its own weight. Its value in feet for various materials is given in Chapter XVIII. Then assuming the piece prismatic and of length s, its weight is

$$w_0As=\frac{Hs}{\lambda},$$

and therefore the weight of the whole structure must be for the same value of λ,

$$W_0=\frac{\Sigma Hs}{\lambda},$$

the summation extending to all the pieces in the structure, and being performed by integration in a continuous arch or chain. It will be observed that s is the length of any line in the frame-diagram, and H that of the corresponding line in the diagram of forces; we have only then to take the sum of the products of these lines and divide by λ, the result will be the weight of the structure. It is, however, generally necessary to find the weights W_1, W_2, of the parts in compression and in tension separately, because the value of λ is generally different in the two cases.

A remarkable connection was shown by the late Prof. Clerk Maxwell to exist between W_1 and W_2. Let us take a structure of the girder class and suppose the total load upon it G, and the height of the centre of gravity of that load above the points of support h. Imagine this structure to become gradually smaller without altering either its proportions or the magnitude and distribution of the load G, then G descends and does work during the descent in overcoming the resistance (T) of the bars in compression to diminution of length, while at the same time the bars in tension (P) do work during contraction. The values of T and P do not alter, for the diagram of forces remains the same, and therefore if we conceive the process to continue till the structure has shrunk to a point,

$$Gh = \Sigma Ts - \Sigma Ps = \lambda_1 W_1 - \lambda_2 W_2.$$

In particular, if the centre of gravity of the load lies on the line of support, and if the co-efficients be the same, the weights of the parts in compression and tension will be equal. A corresponding formula may be obtained for structures of the arch-class by taking into account the thrust.

The weight of an actual structure is always greater than that found by this method. First, an addition must be made to allow for joints and fastenings. Thus, for example, in ordinary pin joints the eye of the bar weighs more than the corresponding fraction of the length of the bar, and in addition there is the weight of the pin. Secondly, in all structures there is more or less redundant material necessary to provide against accidental strains not comprehended in the useful load. Thirdly, there are local straining actions in the pieces occasioned by their own weight and other causes.

41. *Stress due to the Weight of a Structure.*—The total load on any structure consists partly of external forces applied to it at various points and partly of its own weight: the total stress on any member is therefore the sum of that due to the external load and of that due to the weight of the structure itself. As that stress cannot exceed a certain limit, depending on the strength of the material, it necessarily follows that the stress due to the weight is so much deducted from the strength. Thus the consideration of the weight of a structure is an essential part of the subject, even if we disregard the question of cost.

The weight of each member is, of course, distributed over its whole length, and so also may be a part or the whole of the external load. Applying the general method of reduction explained in Art. 37, we suppose an equivalent load applied at each joint, and drawing a diagram of forces, we determine the mean stress, H, on the member. If the unsupported length of the bars be not too great, a matter to be considered presently, this stress will be the principal part of the straining action on the bar, and the bending may be neglected as in the preceding article.

Now, consider two structures similar in form and loaded with the same total weight, distributed in the same way, so that the only difference in the structures is in size: then the stress on corresponding bars must be the same, for the structures have the same diagram of forces. That is to say, in the formula

$$H = kW,$$

the co-efficient k depends on the type of structure and the distribution of the load upon it, but not on its dimensions. Dividing by the sectional area, the intensity of the stress is

$$p = k\frac{W}{A}.$$

Next let W_0 be the weight of the structure itself, and suppose the relative sectional areas of the several pieces the same, then

$$W_0 = w_0 . cAl,$$

where c is a co-efficient depending on the type of structure, and l a length depending on the linear dimensions of the structure. For example, in roofs and bridges l may conveniently be taken as the span. Then if k_0 be the value of k, which corresponds to the distribution of the weight of the structure, which will be the same whether the structure be large or small,

$$p_0 = k_0 . \frac{W_0}{A} = w_0 k_0 cl$$

will be the stress due to the weight of the structure. In other words, the stress due to the weight of similar structures varies as their linear dimensions.

Since p_0 cannot exceed f, it follows at once that there must be a limit to the size of each particular type of structure, beyond which it will not carry its own weight. If L be that limit given by

$$L = \frac{\lambda}{k_0 c},$$

the stress due to the weight of any similar structure of smaller dimensions will be simply

$$p_0 = f . \frac{l}{L},$$

and $$f' = f - p_0 = f . \frac{L - l}{L}$$

is the strength which may be allowed in calculations made irrespectively of weight. If the structure be of uniform strength throughout under its own weight, the value of p_0 will be the same for each member, but this is not necessarily the case, and there may be a different value of f' for each member. The actual limiting dimensions of the structure will, of course, be the least of the various values corresponding to the various members.

The conclusion here arrived at is obviously of the greatest importance, for it immediately follows that in designing a roof, bridge, or other structure of great size, the weight of the structure is the principal thing to be considered in estimating the straining actions upon it, while a certain limiting span can never be exceeded. On the other hand, in small structures the straining actions due to the weight are unimportant; it is the magnitude and variations of the external load which have the greatest influence. This remark also applies to the local straining actions which produce bending in the pieces, their relative importance increases with the size of the structure, and it is necessary to provide against them by additional trussing. A large structure is therefore generally of more complex construction than a small one, as is illustrated by the various types of roof-trusses considered in Chapter I.

The difference of type of large structures and small ones, as well as the circumstances mentioned at the close of the last article, render tentative processes generally necessary in calculations respecting weight. If the type of structure and the distribution of the total load, W, be supposed known, the value of the co-efficients k and c will be known for some given member. By assuming the stress on that member equal to the co-efficient of strength f, we find

$$W_0 = W . ck . \frac{l}{\lambda},$$

a formula which gives the weight of the structure in terms of the load, but the co-efficients will generally vary according to the span. Among the circumstances on which they depend, the ratio of the vertical to the horizontal dimensions of the structure is most important. For a given span k diminishes when the depth is increased, while on the other hand c generally increases, so that for a certain ratio of depth to span, the weight of the structure is least. In ideal cases c may remain the same (Ex. 10, p. 88), but in actual structures the redundant weight of material necessary to give stiffness and lateral stability increases, so that the most economical ratio of depth to span is generally much less than would be found by neglecting such considerations. These points are illustrated by examples at the end of this Chapter and Chapter XII., where the question is again considered briefly; but for detailed applications to actual structures the reader is referred to works on bridges, in the design of which it is of the greatest importance.

42. *Straining Actions on a Loaded Structure in General.*—The results obtained in the last chapter for the case of parallel forces acting on a structure possessing a plane of symmetry in which the forces lie, may be readily extended to structures which have an axis of symmetry acted on by any forces passing through that axis and perpendicular to it. This is the case, for example, of a beam acted on by a vertical load, and also by some horizontal forces arising say from the thrust of a roof or from wind pressure. We have then only to consider the vertical and horizontal forces separately. Each will produce shearing and bending in its own plane, which may be represented by polygons as before. The total straining action will be simply shearing and bending, and will be as before independent of the particular structure on which the forces operate. The magnitude of the straining action, whether shearing or bending, will be the square root of the sum of the squares of its components, and may therefore be readily found by construction and exhibited graphically by curves. In shafts such cases are common, and some examples will be given hereafter.

Another entirely different kind of straining action sometimes occurs in structures proper (roofs, bridges, etc.), and in machines is one of the principal things to be considered. Imagine a structure of any kind to be divided by an ideal plane section into parts *A* and *B*, and to be acted on by forces parallel to that plane. Let the forces acting on *A* reduce to a couple the axis of which is perpendicular to the section, the forces on *B* are equal and opposite, and the two equal and opposite couples tend to cause *A* and *B* to rotate relatively to each other. As already stated in Art. 16, this effect is called Twisting, and the magnitude of the twisting action is measured by the magnitude of either of the couples which form its elements.

Simple twisting sometimes occurs in practice, for example, when a capstan is rotated by equal forces applied to all the bars, but it is generally combined with shearing and bending. It is then necessary to know about what axis the twisting moment should be reckoned, which will depend on the nature of the structure. In shafts and other cases to be considered hereafter the geometrical axis is an axis of symmetry which at once determines this.

When twisting exists the shearing and bending are determined by the same method as before, for they are independent of the axis of reference. Should, however, the structure be subject to a thrust or a pull (Art. 16), the axis about which the bending moment should be reckoned, must be known, for it will depend on the nature of the structure.

These general observations will be illustrated hereafter, and are only introduced here to show how far straining actions can be regarded as depending solely on the external forces operating on the structure without reference to any other circumstances.

43. *Framework with Redundant Parts.*—In a complete frame, without redundant bars (pp. 11, 50), suppose a link applied to any two bars, one end attached to each. Let the link be provided with a right and left-handed screw, or other means of altering its length at pleasure, then by screwing up the link a pull may be produced in the link of any magnitude we please, while a corresponding stress will be produced in each bar of the frame which will bear a given ratio to the pull. Such a link may be called a straining link, and by its addition we obtain a frame with one redundant bar. The stress-ratio on the parts of a frame of this kind is completely definite, but the magnitude of the stress may be anything we please. Instead of one straining link we may have any number, and if the stress on each of these links be given, the same thing will be true. Thus it appears that a frame with redundant parts may be in a state of stress even though no external forces act upon it. This is of practical importance on account of the effect of changes of temperature. If all the bars of a frame with redundant parts are equally heated or cooled, the frame expands or contracts as a whole, but no other effect is produced; any inequality, however, causes a stress which may, under certain circumstances, be very great. This (at least theoretically) is one of the reasons why redundant parts are a source of weakness. The necessity of providing against expansion and contraction is well known in large structures resting on supports. The ground connecting the supports suffers little change of temperature, and the structure, therefore, cannot be attached to the supports, but must be enabled to move horizontally by the intervention of rollers. The magnitude of the stress produced when changes of length are forcibly prevented will be considered hereafter (Chapter XII.).

There is no essential difference between a frame the stress on the parts of which is due to the action of straining links, and a frame acted on by external forces; for every force arises from the mutual action between two bodies, and may therefore be represented by a straining link connecting the bodies. Even gravity may be regarded as a number of such links connecting each particle of the heavy body with the earth. Accordingly, if we include in the structure we are considering, the supports and solid ground on which it rests, we may regard it as a frame under no external forces, but including a number of straining links screwed up to a given stress. If the original frame be incomplete, its parts will be capable of motion, and it becomes a machine, as will be explained in Part III. of this work.

44. *Concluding Remarks.*—Various other questions relating to framework remain to be considered, especially with reference to the joints by which the parts are connected, but these, involving other than

purely statical considerations, do not come within the present division of our work, but are referred to at a later period.

EXAMPLES.

1. In Ex. 4, page 19, if the weight be supposed uniformly distributed, find the thrust, shearing force, and bending moment at each point of each rafter, and exhibit the results graphically by drawing curves.

Diagrams of shearing force will be sloping lines crossing each rafter at the centre.

Max. shearing for short rafter = 91 lbs.

" " long " = 158·5 "

Diagrams of bending moment will be parabolas.

Max. moment at centre of short rafter = 117 ft.-lbs.

" " long " = 290 "

2. A triangular frame *ABC*, supported at *A* and *C*, with *AC* horizontal, is constructed of uniform bars weighing 10 lbs. per foot, the length being—*AB*=3 feet, *BC*=4 feet, and *AC*=5 feet. Suppose, further, that *AB* and *BC* each carry 50 lbs. in the centre. Draw curves of thrust, shearing force, and bending moment for each bar.

3. The platform of a suspension bridge is stiffened by girders hinged at the centre and at the piers. The chains hang in a parabola, and the weight of the platform, chains, and suspending rods, may be regarded as uniformly distributed. Find the bending moment, at any point of the stiffening girder, and exhibit it graphically by a curve when a single load *W* is placed (1) at the centre of the bridge, and (2) at quarter span.

First case. On account of *W* each half of the girder will tend to turn downwards about the ends, and will be supported by the uniform upward pull of the suspending rods. ∴ total upward pull for each $\frac{1}{2}$ girder = *W*, because the centre of action is at $\frac{1}{4}$ span. Thus each $\frac{1}{2}$ girder will be in the state of a beam loaded uniformly with *W*, and supported at the ends.

Max. moment at middle of each half = $\frac{1}{8}W\times$ half span.

Second case. The upward pull of the suspending rods will still be uniform, but for each half girder will now be only $\frac{1}{2}W$, found by assuming an equal action and reaction at the centre joint, and taking moments of each half about the ends. For the half girder which carries the weight the bending moment will be the difference between that due to *W* concentrated in the centre and $\frac{1}{2}W$ distributed uniformly.

∴ Max. = $\frac{3}{16}W\times$ half span.

On the other half it will be due simply to a distributed load of $\frac{1}{2}W$. Max. = $\frac{1}{16}W\times$ half span.

4. A timber beam 24 feet span is trussed by a pair of struts 8′ apart, resting on iron tension rods forming a simple queen truss 3′ deep without a diagonal brace. The beam is loaded with 5 tons placed immediately over one of the vertical struts. Find the shearing force and bending moment at any point of the beam, supposing it jointed at the centre, and the centre only.

The thrust on each strut must be $2\frac{1}{2}$ tons; therefore, curves of shearing force and bending moment for each half of the beam are the same as those for a beam 12 feet long loaded at a point 4 feet from one end with $2\frac{1}{2}$ tons.

The problem should also be treated by the method of sections. Results should also be obtained for the case where one half the beam is uniformly loaded.

5. A beam uniformly loaded is fixed horizontally at the two ends, and jointed at two given points. Draw the diagrams of shearing force and bending moment. Show that the beam will be strongest when the distance of each point from centre is rather less than $\frac{3}{8}$ span.

6. The platform of a bowstring bridge of span $2a$ is suspended from parabolic arched ribs hinged at crown and springing. One half the platform only is loaded uniformly with w lbs. per foot-run. Show that the greatest bending moment on the ribs is $\frac{1}{16}wa^2$.

7. In the last question, if a weight of W tons travel over the bridge, how great will be the maximum bending moment produced?

$$Ans.\ \frac{Wa}{3\sqrt{3}}.$$

8. A girder is continuous over three equal spans, and is hinged at points in the centre span midway between centre and piers. Find the virtual joints in the end spans when uniformly loaded throughout.

$$Ans.\ \frac{13}{16} \text{ span from end.}$$

9. The weight of the chains, platform, and suspension rods of a suspension bridge may be treated as a uniform load per foot-run which at the centre of the bridge is double the weight of the chain. The dip of the chain is $\frac{1}{13}$th the span. The weight of iron being 480 lbs. per cubic foot, and the safe load per square inch of sectional area of chain being 5 tons, find the limiting span, and deduce the sectional area of chains for a load of $\frac{1}{2}$ ton per foot-run on a similar bridge, 300 feet span.

If A = sectional area of chains at centre in sq. ins., then $\frac{20}{3}A$ = weight of bridge per foot-run in lbs.

$$\text{Horizontal tension} = \tfrac{65}{6}AL = 5 \times 2240 \,.\, A,$$
$$\therefore\ L = 1034 \text{ feet.}$$

If A' = area of one chain of the bridge 300 feet span,

$$\text{Whole load on chain} = (\tfrac{20}{3}A' + \tfrac{2240}{4})\,300,$$
$$\text{Horizontal tension} = \tfrac{1}{8}(\tfrac{20}{3}A' + \tfrac{2240}{4})\,300 \times 13 = 5 \times 2240A',$$
$$\therefore\ A' = 34{\cdot}4 \text{ square inches for each chain.}$$

Remark.—By the use of steel wire ropes and by lightening the platform and other parts of the structure, as much as possible, the limiting span of suspension bridges is much increased, there being several examples of a span of 1250 feet and upwards.

10. In a girder with booms parallel and of uniform transverse section the weight of the web is equal to the weight of the booms. Assuming a co-efficient of strength of 9000 lbs. per sq. inch, and the weight of a cubic inch $\frac{5}{18}$ths of a lb., show that the limiting span in feet is

$$L = 5400N,$$

where N is the ratio of depth to span.

11. The weight of a rib of parabolic form, span l, rise nl, with transverse section varying for uniform strength under a uniformly distributed load W, is

$$W_0 = \left(\frac{1}{8n} + \tfrac{2}{3}n\right)W\frac{l}{\lambda}.$$

This is least when $n = \frac{\sqrt{3}}{4} = {\cdot}433$, then $W_0 = {\cdot}577\,W\frac{l}{\lambda}$.

The formula fails if W_0 be nearly equal to W, for the external load would then have to be partly acting upwards to secure uniform distribution of the total load.

PART II.

KINEMATICS OF MACHINES.

PART II.–KINEMATICS OF MACHINES.

45. *Introductory Remarks.*—The object of a machine is to enable the forces of nature to do work of various kinds. In this operation some given resistance is overcome, which is accompanied by a given motion, while the driving force is accompanied by some other given motion, often at a distant place. Hence a machine may be regarded as an instrument for converting and transmitting motion. When considered under this aspect it is called a Mechanism, or sometimes a Movement, a Motion, or a Gear, the first being the scientific term, and the others occurring in practical applications.

Every mechanism consists of a set of pieces possessing one degree of freedom, that is to say, they are so connected together that when one changes its position all the rest do so too in a way precisely defined by the nature of the mechanism. Thus, for example, when the piston of a steam engine moves through any fraction of a stroke, the connecting rod, crank shaft, and the parts of any machine which it may be driving, all shift their position in such a way that the connection between the various changes is completely determinate, and can be studied without reference to the work which the engine is doing, or the speed at which it is running. This branch of study is called the Kinematics of Machines.

The changes of position may be of any magnitude we please, and if they are very small are proportional to the velocities of the moving parts, hence a part of the subject, and generally an important part, is the consideration of the comparative velocities, or, as they are usually called, the velocity-ratios, of the moving parts. Further, since the comparative velocities are fixed by the nature of the machine, the same must be true of the rates of change of these velocities, that is to say, the accelerations. Hence the general question is to study completely the comparative motions of the several parts of a machine, so that when the position, velocity, and acceleration of any piece are given, those quantities may be known for every other piece. It is the positions and velocities which are chiefly considered.

The converse problem is to discover the mechanisms by which any required motion may be obtained, and for this purpose the connection which exists between different mechanisms is considered. The subject therefore forms an introduction to the science of Descriptive Mechanism in which existing machines in all their vast variety are classified and studied systematically.

AUTHORITIES.

The principal treatises on the theory of mechanism are—

WILLIS. *Principles of Mechanisms.* Longman.
RANKINE. *Millwork and Machinery.* Griffin.
REULEAUX. *Kinematics of Machinery.* Macmillan.

The modern form of the theory is due to Professor Reuleaux, whose nomenclature and methods are followed with some modifications in the present work. The treatise referred to is a translation from the German by Professor A. B. Kennedy.

METRIC MEASURES.

When metric measurement is employed the unit of length is the *metre* decimally subdivided into decimetres, centimetres, and millimetres; the kilometre (1000 metres) being employed for long distances. The kilometre is 3281 feet, or about five-eighths of a mile, and for units of velocity we have therefore

One metre per second =3·281 feet per second
=197 feet per minute (nearly).
One kilometre per hour =54·68 feet per minute
=$\frac{5}{8}$ mile per hour (nearly).

The unit of acceleration will be 1 m.s., or 3·281 f.s. per second. Thus the acceleration due to gravity (g.) which, in British measures is 32·2 nearly, is in metric units 32·2/3·281 or 9·81 nearly.

CHAPTER V.

LOWER PAIRING.

SECTION I.—ELEMENTARY PRINCIPLES.

46. *Definition of Lower Pairs.*—Each piece of a mechanism is in direct connection with at least one other, and constitutes with it what is called a PAIR, of which the two pieces are said to be the Elements. The whole mechanism may be regarded as made up of pairs, and its nature depends on the nature and mode of connection of the pairs of which it is constructed.

In the present chapter we consider exclusively mechanism composed of pairs of rigid elements which are in contact with each other, not merely at certain points or along certain lines, but throughout the whole or part of the area of certain surfaces. Such pairs are of peculiar importance from the simplicity of the relative movement of their elements, from their resistance to wear when transmitting heavy pressures, and from their tightness under steam and water pressure. They are called Lower Pairs, and in many cases this kind of pairing is alone admissible.

In order that two rigid surfaces may be capable of moving over each other while continuing to fit, they must either be cylindrical, including under that head all surfaces generated by the motion of a straight line parallel to itself, or surfaces of revolution, or screw surfaces. In the first case the relative motion of the elements is one of translation along the line, in the second of rotation about the axis of revolution, in the third the motion of translation and rotation are combined in a fixed proportion. Hence there are three kinds of lower pairs, known as Sliding Pairs, Turning Pairs, and Screw Pairs. In each case one of the surfaces is hollow, and wholly or partly encloses the other which is solid, and the motion depends on the surfaces only, and not on the other parts of the elements which assume very various forms, according to the purpose of the mechanism. Either element may be fixed and the other move, or both elements may move in any way whatever, the relative motion is still of the same kind.

As an example of a sliding pair may be taken a piston and cylinder, in which either the cylinder may be fixed and the piston move, or the piston be fixed and the cylinder move, as in some steam hammers, or both cylinder and piston move, as in the oscillating engine. The relative motion is always a simple translation. Velocities of translation are most conveniently measured in feet per 1″ or feet per 1′, but miles per hour and knots are also used, as to which it is convenient to remember that 1 mile per hour is 88 feet per 1′, and one knot, that is, one nautical mile per hour, approximately 101 feet per 1′. For metric measures of velocity see page 92.

As examples of turning pairs may be taken a cart and its wheel, a shaft and its bearing or a connecting rod and crank pin. The relative motion here is one of simple rotation, which may be measured by the number of revolutions (n) per unit of time, or by the speed of periphery (V) of a circle of given radius (r), or by the angle (A) turned through per unit of time. The first two modes of measurement are common in practice, the third is used for scientific purposes only. When employed the angle is always expressed in circular measure, and the three methods are therefore connected by the equations

$$V = Ar = 2\pi nr.$$

When angular velocity is used as a measure of speed of rotation, the unit of time is nearly always 1″, but the minute and hour are common in other cases.

A screw pair consists of a screw and its nut, and the relative motion consists of a motion of translation along the axis of the screw combined with a rotation about that axis. The motion of translation is often called the "speed of the screw," and is equal to np, where p is the pitch, that is to say, the space traversed in one revolution, and n the revolutions in the unit of time. Strictly speaking, the two first lower pairs are limiting cases of the screw pair: in the turning pair the pitch is zero, and in the sliding pair infinite.

In all three cases the motion of either element relatively to the other is identically the same, and the rate of that motion may properly be called the Velocity of the Pair, whether the movement considered be translation or rotation. When the velocity of a sliding pair and a turning pair are compared, rotation may be measured by the speed of periphery of a circle of given diameter; it is the velocity with which bearing surfaces of that diameter would rub each other. The radius of this circle may be called the "radius of reference." The velocity of a screw pair may be measured by the rate either of its translation or its rotation.

In these three simple pairs the motion of one element relatively to the other is completely defined, each point describing a definite curve. Such a pair is called a "complete" or "closed" pair, but we may have pairs in which the motion is not defined unless further constraint be applied, and the pair is then said to be "incomplete." An incomplete pair cannot be used in mechanism without employing such constraint, and this process is called "closing" the pair. A pair may be incomplete, because there is nothing to prevent the disunion of its elements, as, for example, a shaft and its bearing when the cap is removed, but it also may be incomplete in itself. Lower pairing is sometimes, though not very frequently, incomplete in this latter sense; there are three possible cases, first, when the surfaces are spherical, as in a ball and socket joint; second, where a rod fits into a hole, and is free to move endways as well as rotate; third, where a block fits in between parallel plain surfaces. The methods of producing closure will be considered hereafter.

It may be here remarked, in anticipation of what will be said hereafter, that cases of lower pairing may be imagined in which the elements are not in contact over an area but along a line. For example, a rod may fit into a square hole. It is the simplicity of the relative motion which is the essential characteristic.

The motion of the elements of a pair may be prevented by a pin key or other fastening removable at pleasure; the pair is then said to be "locked." In capstans and windlasses, provided with ratchet wheel and pawls, we have pairs which are locked in one direction only.

47. *Definition of a Kinematic Chain.*—It has been already said that a machine consists of a number of parts so connected together as to be capable of moving relatively to one another in a way completely defined by the nature of the machine. Each part forms an element of two consecutive pairs, and serves to connect the pairs so that the whole mechanism may be described as a chain, of which the parts form the links. Such a series of connected pieces is called a Kinematic Chain.

The motion of any piece may be considered either relatively to one of the pieces with which it pairs, or with reference to any other piece which we may choose to regard as fixed. In the first case the rate of movement has already been defined as the Velocity of the Pair. In the second, the fixed piece is usually the frame of the machine, which unites the rest of the pieces, and is commonly attached to the earth or some structure of large size, such as a vessel. For pieces which pair with the frame the velocity of the pair is the same as the velocity of the moving element, and this element alone need be mentioned. In some common practical cases the speed of an element means the speed of one of the pairs of which it forms part. For example, the speed of piston of an oscillating engine would be understood to mean its velocity relatively to the cylinder, in other words, the speed of the "cylinder-piston pair." In the present chapter we consider exclusively chains of closed lower pairs, so that the motion of the pairs is a simple translation, rotation, or screw motion. The motion of some of the pieces relatively to the frame may be much more complex, but this is a subject for subsequent investigation: it is the motion of the pairs alone we consider at present. We shall first direct our attention to the very common and important piece of mechanism employed in direct-acting steam engines. An example is shown in Fig. 1, Plate I., p. 108, which represents a direct-acting engine of the vertical inverted cylinder type which is common in marine engines and often occurs in other cases.

Let us consider the pairs of which this mechanism is constructed. We have, *first*, a cylinder, to which are rigidly attached guides for the crosshead, and bearings for carrying the crank shaft. The cylinder-guide bars and crank shaft bearings all form one part rigidly connected together, and must be considered as being one piece or link of the kinematic chain. It may conveniently be called the frame. *Secondly*, there is a piston, which fits and slides in the cylinder. To the piston a rod and crosshead are rigidly attached, forming practically one piece. Not only is the piston guided in the cylinder, but the crosshead also between the guide bars, and the piston rod in the stuffing box; but yet, since there are practically two pieces only which move relatively to one another, we must look on the cylinder, stuffing box, and guide bars as altogether forming the hollow element of a sliding pair, and the piston, rod, and crosshead as together forming the solid element of the pair. *Thirdly*, there is a connecting rod which is attached by a gudgeon or crosshead pin to the piston-rod head. These two parts will together compose a turning pair. At the other end the connecting rod embraces the crank pin, forming a second turning pair with it. The crank pin is one of the elements contained in the *fourth* piece of the mechanism. This piece consists of the crank pin, crank arms, and shaft with its journals. The journals turn in the bearings of the fixed frame of the machine, the first link mentioned, and so form a third turning pair. Thus the chain is complete. It consists of four links forming one sliding pair and three successive turning pairs.

The same mechanism, in a different form, is shown in Fig. 2 of the same plate which represents the air-pump of a marine engine worked, as is not unusual, by a large eccentric keyed on the crank shaft. The crank pin is here enlarged so as to become an eccentric; and to save room, the piston rod is replaced by a trunk within which the eccentric rod vibrates. We have, however, exactly the same pairs arranged in the same way, and the difference between the mechanisms is therefore merely constructive, the motions of the parts being identical.

48. *Mechanism of Direct-Acting Engine—Position of Piston.*—This is such an important piece of mechanism that we will examine its motion somewhat fully.

First, as to the relative positions of the crank in its revolution and the piston in its stroke. The position of the piston in its stroke will compare exactly with the position of the crosshead, so instead of introducing the length of the piston rod into the diagram, we may just as well determine the relative positions of the point D (Fig. 46) in its straight-line path, and P in its circular path.

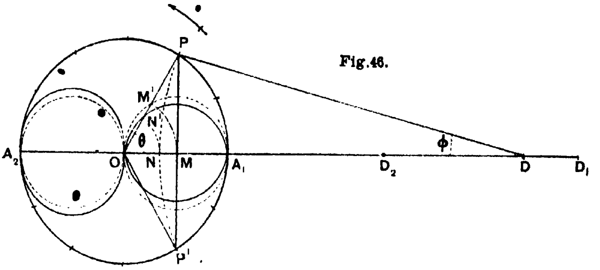


Fig. 46.

Suppose the line of stroke to pass through the centre of the crank-pin circle. Let OP = length of the crank arm, and PD the length of the connecting rod. When the crank arm is in the line of stroke, away from the piston, the piston will be in one extreme position, and when the crank is in the line of stroke towards D, the piston will be in its other extreme position. The points A_1, A_2 on the crank-pin circle are called the dead points. If we take distances A_1D_1 $A_2D_2 = PD$, the length of the connecting-rod, the points D_1, D_2 represent the ends of the stroke of the piston. If now we place the crank in any position OP we obtain the corresponding position of the piston by cutting the line of stroke with a circular arc of radius $= PD$ and with centre P. DD_1, DD_2 will be the distances of the piston from the ends of its stroke. Since $A_1A_2 = D_1D_2$, the length of the stroke, it will be convenient to find the point in A_1A_2 which corresponds to the position of the piston in its stroke. This may be readily done by striking a circular arc PN with centre D. N will be the point, for $A_1D_1 = PD = ND$, therefore $A_1N = D_1D$, and the point N is the same distance from A_1 and A_2 as the piston is from the ends of its stroke.

We may just as easily solve the converse problem of finding the position of crank corresponding to any given position of the piston in its stroke. Let D be any position, cut the crank-pin circle by a circular arc of which D is the centre and DP the radius, then OP or OP' will be the corresponding position of the crank. Let the direction A_1, PA_2 be the ahead direction of the crank, and let us call the motion D_1D_2 towards the crank the forward stroke, and D_2D_1 the back or return stroke of the piston, then when the piston is at D in the forward stroke the crank will be at OP, and again when the piston is at D in the return stroke the crank will be at OP'. Drop a perpendicular PM on to the line of stroke. Then the longer the connecting rod the smaller NM will be, and the more nearly the circular arc PN will coincide with the perpendicular PM. Hence in the limiting case of an indefinitely long connecting rod, M will be the position of the piston corresponding to the position OP of the crank. M being the position, neglecting the effect of the obliquity of the connecting rod, and N the true position, MN is what we may call the error, or deviation due to obliquity.

In general the slide valve is worked by an eccentric, the radius of which is set at a particular angle on the shaft, so that the cut-off takes place when the crank occupies a certain angular position in its revolution, and it consequently follows that the fraction of stroke completed before cut-off takes place will be affected by the obliquity of the connecting rod, so that in the ordinary setting of the slide valve the rates of cut-off will be different in the two strokes. This is well illustrated by Ex. 4, page 103.

We may obtain a convenient approximate expression for MN, the error due to obliquity. Referring to Fig. 46,

$$NM = DN - DM = DN(1 - \cos\phi).$$

Now the length of the connecting rod may be conveniently expressed as a multiple of the length of the crank radius a or stroke s.

$$DN = na \text{ suppose} = \tfrac{1}{2}ns.$$

$$\therefore\ NM = n\frac{s}{2}(1 - \cos\phi) = ns\sin^2\frac{\phi}{2}.$$

In the triangle POD, the sides being proportional to the sides of the opposite angles,

$$\sin\phi = \frac{OP}{DP}\sin\theta = \frac{1}{n}\sin\theta.$$

Now, the angle ϕ is in all practical cases a small angle, so we may write approximately

$$2\sin\frac{\phi}{2} = \sin\phi.$$

$$\therefore\ NM = n\,.\,s\,.\,\frac{\sin^2\theta}{4n^2} = \frac{s}{4n}\sin^2\theta.$$

This is greatest when $\theta = 90$. $NM_{\text{max.}} = \frac{s}{4n}$.

If the connecting rod is four times the crank, the greatest error due to obliquity $= \frac{1}{16}$ stroke.

We see that, in the forward stroke, the effect of the obliquity of the connecting rod is to put the piston in advance of the position due to an infinitely long connecting rod, and, in the return stroke when the piston moves from the crank, the piston will be behind that position.

The relative positions of piston and crank may be very conveniently represented by a curve in this way. Divide the crank-pin circle (see Fig. 46) into a number of equal parts, and supposing the crank pin at the points of division P, find the corresponding positions of the piston N. If then we take along the crank arm a distance ON' equal to ON, the distance of the piston from the centre of its stroke, and do this for a number of positions, we shall find the points N' will lie on a double-looped closed curve, shown in full lines in the figure. This may be called a curve of position of the piston. If we had supposed the connecting rod to be indefinitely long, and had taken a distance OM' along $OP = OM$, the curve of position in such a case would have been a pair of circles, dotted in the figure, on OA_2 and OA_1, as diameters. The true curves of position will deviate from these circles more the shorter the connecting rod. For the half stroke nearer the crank the curve will lie outside the dotted circle, and for the further half stroke inside. In Zeuner's valve diagram the obliquity of the eccentric rod is neglected, and the circles employed to show the position of the slide valve.

49. *Velocity of Piston.*—We will now pass on to the question of the relative velocity of the piston and crank pin.

We will suppose the crank to turn uniformly at so many revolutions in the unit of time. If n = number of revolutions and a = length of crank arm, s = stroke.

Velocity of crank pin $V_0 = 2\pi an = n\pi s.$

Now, as the crank pin moves with uniform velocity, the piston undergoes continual changes of velocity, from being zero at the ends to a maximum at about the centre of the stroke. What is commonly spoken of as the speed of piston is the mean speed. If in the unit of time a complete number of revolutions are performed at a uniform rate, the mean speed will be the actual distance traversed by the piston in the unit of time. In each revolution the piston will complete a double stroke, so that speed of piston $= V = 2ns$. This may be compared with the speed of crank pin V_0.

$$\frac{V_0}{V} = \frac{\pi ns}{2ns} = \frac{\pi}{2}.$$

Next, as to the actual velocity of the piston at any point of its stroke. The piston and crank pin are joined together by a connecting rod of invariable length; one end of this rod has the velocity of the piston and the other that of the crank pin. In Fig. 47b let *ab* be a rod, the ends of which move with velocities V_a, V_b in given directions. If one of these velocities be given, the other can be determined. For in Fig. 47a draw *Oa* parallel and equal to V_a, and *Ob* parallel to V_b to meet a line *ab* which is perpendicular to the line *ab* of the first figure; then, if

Fig.47a.

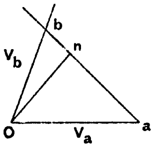
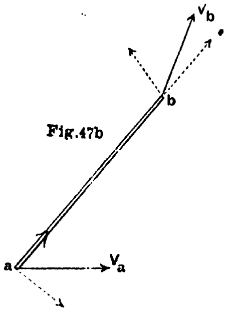
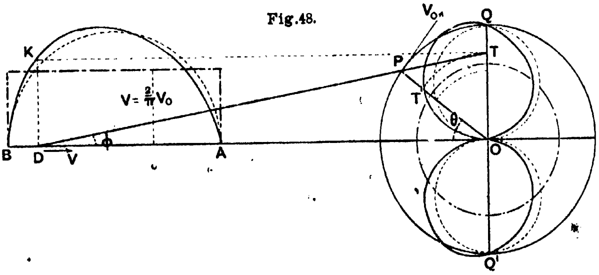


Fig.47b



we drop a perpendicular *On* on *ab*, this will be parallel to *ab* of the first figure, and must represent the resolved part of the velocity V_a along the rod. But the velocities of *a* and *b* resolved along the rod must be equal, because the length *ab* of the rod is invariable; hence *On* also represents the resolved part of V_b along the rod, and consequently *Ob* must represent that velocity in magnitude as well as in direction. The figure *Oab* is called the Diagram of Velocities of the rod, and from it we can find the velocity of any point we please either in, or rigidly connected with, the rod. We shall return to the properties of this diagram frequently hereafter: it will be sufficient now to remark that the triangle *Oab* determines the velocity-ratio of the two ends. In drawing the triangle it is generally convenient to turn it through 90°, so that the lines *ab* in the two figures become parallel, while the sides *Oa*, *Ob* become perpendicular to the velocities they represent.

Fig.48.



In Fig. 48, *OP* is the crank arm, *PD* the connecting rod; through *O* draw *OT* at right angles to the line of stroke to meet the connecting rod produced in T, then P moves perpendicular to OP, and D to OT, therefore OPT is a triangle of velocities, so that if V be the velocity of the piston, V_0 that of the crank pin,

$$\frac{V}{V_0}=\frac{OT}{OP}.$$

This simple construction enables us very conveniently to draw a curve of piston velocity. In the first place, set off along OP a length $OT'=OT$, and do this for a number of positions of the crank. The points T' will be found to lie on a pair of closed curves, shown in full lines in the figure, passing through O and also through Q, Q', the upper and lower ends of the vertical diameter of the crank circle. Had the connecting rod been indefinitely long, the points T' would have been found to lie on a pair of circles, of which the diameters are OQ and OQ', shown in dotted lines. On account of the obliquity of the connecting rod, the curve of actual velocity lies outside the circle on the cylinder side of the crank, and inside the circle when the crank lies away from the cylinder.

When the crank is at right angles to the line of dead centres, the velocity of the piston is the same as that of the crank pin, and neglecting the obliquity of the connecting rod this will be the maximum velocity of the piston. If the obliquity is taken into account, the greatest velocity of piston occurs when the crank is inclined a little towards the cylinder; it is very approximately when the crank is at right angles to the connecting rod, and the maximum velocity will a little exceed the velocity of the crank pin.

The curve just described is a polar curve, the magnitude of the velocity being represented by the length of the *radii vectores* of the curve. But we may draw a curve of velocity in a different way, thus —from the end of the connecting rod which represents the position of the piston when the crank is at OP, set up an ordinate $DK=OT$, and do the same thing for a number of positions of the piston, the curve of velocity AKB will be obtained; the ordinate of which will represent the velocity of the piston when at any point of stroke AB. The longer the connecting rod the more nearly does the curve approximate to the dotted semicircle of which AB is the diameter. The effect of the obliquity is to make the true curve of velocity lie outside the semicircle in the first half of the stroke of the piston towards the crank, and inside for the second half of the stroke.

The mean velocity of the piston may be conveniently represented by an addition to the diagram, thus:—On the same scale that OP, the length of the arm, represents the velocity V_0 of the crank pin, take a length to represent $\bar{V}=\frac{2}{\pi}V_0$.

In the polar diagram draw a circle with O as centre and radius of this length. Where the circle cuts the polar curve of velocity the positions of the crank are given at which the actual speed of the piston is equal to its mean speed. In the second diagram of velocity, set up an ordinate to represent $\overline{V}$, and draw a line parallel to the line of stroke. It will cut the curve of piston velocity in two points.

An approximate expression for the velocity of the piston may be determined thus:

$$V=V_0\frac{\sin OPT}{\sin OTP}=V_0\frac{\sin(\theta+\phi)}{\cos\phi};$$

or expanding the numerator,

$$V=V_0\{\sin\theta+\cos\theta\tan\phi\}.$$

Since ϕ is in all practical cases a small angle, $\tan\phi$ may be written $=\sin\phi$ without sensible error.

$$\therefore\ V=V_0\{\sin\theta+\cos\theta\sin\phi\}.$$

$$\text{Now}\ \frac{\sin\phi}{\sin\theta}=\frac{OP}{PD}=\frac{1}{n}.$$

$$\therefore\ V=V_0\left\{\sin\theta+\frac{1}{n}\sin\theta\cos\theta\right\}.$$

$$=V_0\left\{\sin\theta+\frac{1}{2n}\sin 2\theta\right\}.$$

By differentiation with respect to the time t we obtain the acceleration of the piston. Let a be the length of the crank, then

$$\frac{dV}{dt}=\frac{dV_0}{dt}\left\{\sin\theta+\frac{1}{2n}\sin 2\theta\right\}+\frac{V_0^2}{a}\left\{\cos\theta+\frac{1}{n}\cos 2\theta\right\}.$$

If the length of the connecting rod be infinite, and the crank turn uniformly, we obtain a simple harmonic motion, the deviation from which is therefore, approximately, assuming n large and dV_0/dt small,

$$\textit{Deviation}=\frac{V_0^2}{na}\cos 2\theta+\frac{dV_0}{dt}\sin\theta.$$

A graphical construction for the acceleration when the crank turns uniformly will be found in the Appendix. See also Ex. 10, 11 next page.

EXAMPLES.

1. The driving wheels of a locomotive are 6 feet in diameter, find the number of revolutions per minute and the angular velocity, when running at 50 miles per hour. If the stroke is 2 feet, find also the speed of piston.

Revolutions per minute $=233\frac{1}{3}$.
Angular velocity $=24\frac{4}{9}$ per second.
Speed of piston $=933{\cdot}6$ feet per minute.

2. The pitch of a screw is 24 feet, and revolutions 70 per minute. Find the speed in knots. If the stroke is 4 feet, find also the speed of piston in feet per minute.

Speed of screw $=16{\cdot}58$ knots.
,, piston $=560$ feet per minute.

3. The stroke of a piston is 4 feet, and the connecting rod, is 9 feet long. Find the position of the crank, when the piston has completed the first quarter of the forward and backward strokes respectively. Also find the position of the piston when the crank is upright.

Ans. The crank will make, with the line of dead centres, the angles 55° and 66°.

When the crank is upright the piston will be $2\frac{2}{3}$ inches from the middle of its stroke.

4. The valve gear is so arranged in the last question as to cut off the steam when the crank is 45° from the dead points both in the forward and backward strokes. Find the point at which steam will be cut off in the two strokes. Also when the obliquity of the connecting rod is neglected.

Ans. Fraction of stroke at which steam is cut off is—

·175 in forward stroke,
·118 in backward stroke,
·146 neglecting obliquity.

5. Obtain the results of the two last questions for the case of an oscillating engine, 6 feet stroke, the distance from the centre of the trunnions to the centre of the shaft being 9 feet.

Ans. Angles 51° and 68°: Cut-off ·2 and ·115.

6. In Ex. 3 construct both curves of piston velocity. If the revolutions be 70 per minute, find the absolute velocity of the piston in the positions given. Find also the maximum velocity of the piston.

Ans. $\frac{1}{4}$ stroke forward, velocity = 810 feet per 1′.
$\frac{1}{4}$,, back, ,, = 730 ,,
Maximum ,, = 900 ,,

Find also the points in the stroke at which the actual speed of piston is equal to the mean speed.

Ans. $4\frac{3}{4}$ in. from commencement of forward stroke.
$6\frac{3}{4}$ in. ,, end ,,

7. The travel of a slide valve is 6 in., outside lap 1 in. Find, in feet per second, the velocity with which the port commences to open when the revolutions are 70 per minute.

Ans. Port commences to open when the valve is 1 in. from the centre of its stroke. Neglecting the obliquity of the eccentric rod, velocity of valve is then 1·72 feet per second.

8. Show that the maximum velocity of the piston occurs when the crank is nearly at right angles to the connecting rod, the difference being a small angle, the sine of which is $\frac{1}{n(n^2+2)}$ nearly, where n is the ratio of connecting rod to crank.

9. Referring to Fig. 48, p. 100, show that when the crank rotates uniformly the angular velocity of the connecting rod is proportional to PT. Draw a curve representing it. With the notation of Art. 49, p. 102, show that approximately

$$\text{Angular velocity of rod} = \frac{V_0}{na} \cdot \cos\theta.$$

$$\text{Angular acceleration} = -\frac{V_0^2}{na^2} \cdot \sin\theta.$$

10. At any point K of a linear curve of velocity (such as BKA in Fig. 48, p. 100) draw an ordinate KD to meet the base line in D and a normal KZ to meet this line in Z. Show that the acceleration of the piston or other moving piece is proportional to DZ.

Note.—This well-known construction is due to Pröell. It is perfectly general, but difficult to apply with accuracy, because the exact direction of the normal is generally unknown.

11. Referring to Fig. 48, page 100, draw additional lines as follows. (Mohr's construction)—

(1.) TG horizontal to meet the crank QP produced in G.
(2.) GS vertical to meet the connecting rod PD in S.
(3.) SZ at right angles to the rod to meet the line of centres in Z.

Prove that when the crank turns uniformly

$$\frac{\text{Acceleration of Piston}}{\text{Acceleration of Crank Pin}} = \frac{OZ}{OQ}.$$

SECTION II.—EXAMPLES OF CHAINS OF LOWER PAIRS.

50. *Mechanisms Derived from the Slider-Crank Chain.*—In the investigation just given it has been supposed, for simplicity, that the crank turns uniformly, but if this be not the case the curve constructed will show the ratio of the velocities of the piston and crank pin. In all cases it is the velocity-ratio of two parts, not the velocities themselves, which are determined by the nature of the mechanism. The velocities are of course reckoned relatively to the frame, but as both piston and crank pair with the frame, they are also the velocities of the piston-frame pair and the crank-frame pair (see p. 95), the crank being the radius of reference. The velocities of the other pairs will be determined presently, but in this mechanism are of less importance. We will now direct our attention to other examples of the simple chain of lower pairs, of which the direct-acting engine is only a particular case. In Fig. 49, *D* is a block capable of sliding in the slot of the piece *A*. By means of a pin this block is connected with one end of the link *C*. *B* is a crank capable of rotating about a pin attached to the piece *A*, and united to *C* by another pin. Each of the four pieces of which this mechanism is composed, together with either of the adjacent pieces, constitutes a "pair," of which there are *four*, viz., three turning pairs, *AB*, *BC*, *CD*, and a sliding pair, *DA*. This simple combination of pairs is known, in the modern theory of machines, as a Slider-Crank Chain.

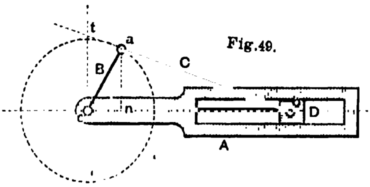


Fig. 49.

Since the relative motions of the parts depend solely on the form of the bearing surfaces of the pairs and the position of their centres, not on the size and shape of the pieces in other respects, we may vary these at pleasure, and thus adapt the same chain to a variety of purposes. Especially we may interchange the hollow and solid elements of the pairs, a process which occurs constantly in kinematic analysis, and is called "inversion of the pair."

Again, any one of the four pieces may be fixed and the other move, so that we can obtain four distinct mechanisms from the same chain, simply by altering the link which we regard as fixed, a process called "inversion of the chain."

(1.) Let *A* be fixed, then we obtain the mechanism of the direct-acting engine already fully considered. In this, however, the connecting link C is much longer than the crank B; by supposing them equal we obtain a mechanism well known in various forms. In Fig. 50, C is prolonged beyond the crank pin a to a point d, such that $ad=ac$, a circle struck with centre a then passes through c, d, and the centre of the block, thus cd is at right angles to the line of stroke, so that d, when the crank turns, describes a straight line. This property renders the mechanism applicable to a parallel motion. It has also been used in air-compressing machinery. (See page 116.)

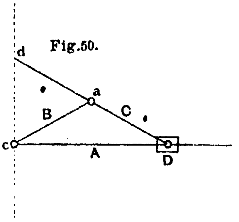


Fig. 50.

The various forms of the well-known toggle joint, some of which will be referred to hereafter, are examples of the same mechanism with different proportions of C to B.

(2.) Instead of A, let us suppose C to be the fixed link, so that A and the other pieces have to take a corresponding motion. With this, by a change in the shape of the pieces, we are able to derive a mechanism well known in two forms. C being fixed, and B caused to rotate, A will have given to it an oscillating motion about the block D, and, at the same time, will slide to and fro on the block, the block itself having a vibrating motion about the other end of the piece C. Now, the relative movement of the parts of this mechanism is identical with that of the oscillating steam engine, and by a suitable alteration in the shape of the pieces, that mechanism may be derived. Thus, suppose, in the first place, the hollow element of A to become the solid one, in the shape of a piston rod and piston, whilst the block D is enlarged into a cylinder to surround the piston, and so becomes the hollow element of the pair. The cylinder D will oscillate on trunnions, in bearings in the fixed piece C, which must be so constructed as to be a suitable frame for carrying the engine, and have bearings in which the crank shaft and crank B can turn.

The oscillating cylinder is in general mounted on bearings, the centre line of which coincides with the centre of the stroke of the piston, so that the distance apart of the shaft and trunnion bearings is equal to the length of the piston rod. An example is shown in Fig. 4, Plate I.

Next let us consider the relative motions of the parts. Returning to Fig. 49 above, suppose a, b, c to be the centres of the turning pairs, and draw ct, an perpendicular to the line of centres bc, to meet C and A in t and n, then it was shown above (page 100) that the velocity-ratio of the pairs DA, BA in the direct-acting mechanism was ct/ac, and as fixing a link makes no difference in the relative motions, this must also be the ratio of the speed of the piston of the oscillator in its cylinder, to the speed of the turning movement of the crank *relatively* to the piston rod. Again when *C* is fixed, as in the oscillator, the link *A* (Fig. 49) slides on the block *D* with a velocity the direction of which is perpendicular to *an*, while the point *c* in it moves perpendicular to *ac*. Hence it follows that the triangle of velocities is *acn*, and therefore the velocity-ratio of piston and crank pin is *an*/*ac*. The curve of piston velocity can be drawn as before; it differs little in form from that of the direct actor, but the maximum velocity of the piston is equal to that of the crank pin, instead of being somewhat greater. Once more, remembering that fixing a link does not alter the relative motions, it appears that, in all cases, the velocity-ratio of the pairs *DA*, *BC* must be *an*/*ac*, so that we have determined the ratio of the speed of piston in the direct actor to the speed of the turning movement of the crank *relatively* to the connecting rod.

Comparing our results, we see that the velocity-ratio of the turning pairs *BC*, *BA* must be *ct* : *an*, or what is the same thing, *bt* : *ab*. Since the three angles of the triangle *abc* are always together equal to 180°, it is clear that the sum of the speeds of the three turning pairs must be zero, due regard being taken of the direction of rotation, and it follows, therefore, that in any slider-crank chain the speeds of the three turning pairs are as *at* : *ab* : *bt*. By the introduction of a suitable radius of reference, we may compare these velocities with that of the sliding pair. The most convenient radius to take is that of the crank, then assuming, as before, $ab = n \, . \, ac$, the velocities of the pairs are shown by the annexed table:—

VELOCITY-RATIOS IN A SLIDER-CRANK CHAIN.

Pair,	*DA*	*BA*	*BC*	*DC*
Velocity,	*ct*	*ac*	$\frac{bt}{n}$	$\frac{at}{n}$

In the oscillator the angular velocity-ratio of the cylinder and crank is the velocity-ratio of the pairs *CD*, *CB*, and is therefore *at* : *bt* or *cn* : *bc*. This can readily be constructed by drawing *nz* parallel to *ac*, then, since *ab* is constant, the angular velocity of the cylinder is proportional to *az*, as may be verified by an independent investigation. The result may be exhibited by a polar curve similar to the curve of piston velocity already drawn. In Fig. 51, *c* is the crank pin, *TST'S'* the crank circle, *az'* is set off along the crank radius equal to *az*, then a curve with two unequal loops is obtained, which shows the law of vibration of the cylinder. The motion of the cylinder is such that, in the swing to the left, whilst the crank pin moves along the arc *T'ST*, the angular velocity is much greater than in the return swing to the right, whilst the crank pin moves along the arc *TS'T'*. Supposing the crank to revolve uniformly, the times occupied by the forward and return swings are as the arcs *T'ST* and *TS'T'*, which are proportional to the angles subtended by them. By measuring or otherwise estimating these angles, the mean angular velocities in the forward and backward oscillation may be determined. This peculiar vibration, rapid one way and comparatively slow the other, has been made use of to obtain a quick return motion of a cutting tool in a shaping machine. The velocity with which a tool will make a smooth cut in metal is limited, and since in general the tool is made to cut in one direction only, time is saved by causing the return stroke to be made more quickly. One construction of such a quick return motion may be thus described. A slotted lever *D* vibrates on a fixed centre in the frame-piece *C*, its motion being derived from the revolution of a crank *B* on another fixed centre in the same frame-piece *C*. The crank pin of *B* turns in the block *A*, which slides in the slotted lever *D*. There is in addition a connecting rod, by means of which a to-and-fro motion of a headstock carrying the cutting tool is communicated from the oscillating lever, the headstock sliding in a guide.

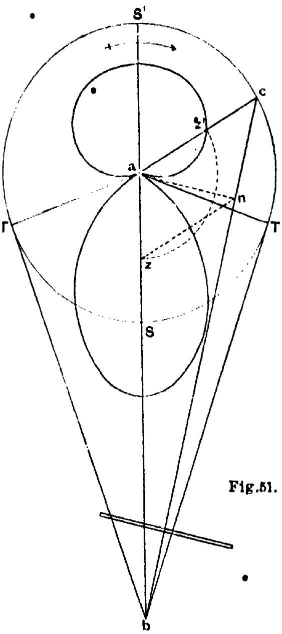


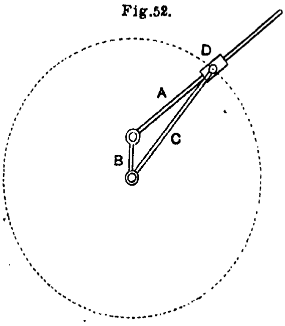
Fig. 51.

Omitting the connecting rod, we have the same kinematic chain, with the same fixed link *C*, as in the oscillating engine. There has been a change made only in the form of some of the pieces. What was the oscillating cylinder is now the slotted lever, and instead of a piston and rod, we have here the simple block *A* sliding in the slot. The crank *B* and frame-link *C* remain practically unaltered. The slotted lever will vibrate according to the same law which we have investigated for the oscillating cylinder, and thus with a uniform rotation of the crank, a quick return motion of the tool will be obtained. This mechanism is shown in Fig. 5, Plate I., in a form employed for giving motion to the table of small planing machines.

(3.) Let us next take an example in which *B* is the fixed link, and becomes the frame, its form being of course modified to suit the new conditions.

A crank arm *C* (Fig. 52) turns on a fixed centre in the frame-piece *B*; so also does another arm *A* on a second fixed centre, *D* slides on *A*, being connected by a pin to the second end of *C*. Both *A* and *C* may make complete revolutions. If we suppose *C* to turn with uniform angular velocity, *A* will rotate with a very varying angular velocity, the movement of *A* in the upper part of its revolution being much more rapid than in the lower. This device has been employed by Whitworth to get a quick return motion of a cutting tool in a shaping machine. When separated from the rest of the machine, the construction may be thus described:—A spur wheel *C* which derives its motion through a smaller wheel from the engine shafting, revolves on a fixed journal *B*, of large dimension. Standing from the face of the journal is a fixed pin placed out of the centre of the journal. On this fixed pin a slotted lever *A* rotates, in which a block *D* slides, a hole in the block receiving a pin which stands out from the face of the spur wheel. A second slot in *A*, on the other side of the pin, contains another block, which, by a screw, can be adjusted and secured at any required distance from the centre of rotation, so as to give any stroke at pleasure. This mechanism, omitting the adjustment by which the stroke is varied, is shown in Fig. 6, Plate I. The same mechanism in a somewhat different form is often employed in sewing machines to give a varying motion to the rotating hook.

Fig. 52.



(4.) The fourth possible mechanism which can be derived from the slider crank chain is obtained by fixing the block *D*. This case is not so common as the three preceding, but in Stannah's pendulum pump, shown in Fig. 3, Plate I., we find an example. In a simple oscillating engine driving a crank shaft and fly-wheel, suppose the cylinder *D* fixed instead of the piece *C* which carries the cylinder and crank shaft. The crank and fly-wheel *B* has become the bob, and the link *C* the arm of the pendulum, from which the mechanism derives its name; *D* is a fixed cylinder, and *A* is a piston and rod.

Plate. I.

Fig. 1.

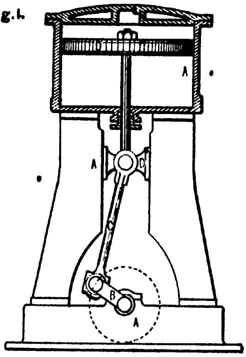


Fig. 2.

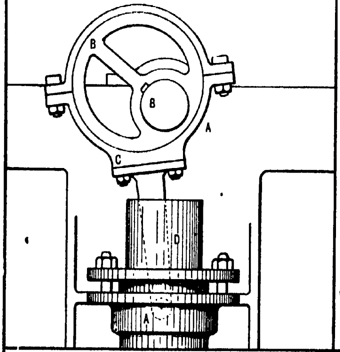


Fig. 3.

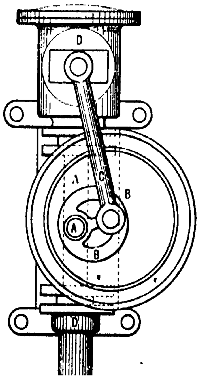


Fig. 4.

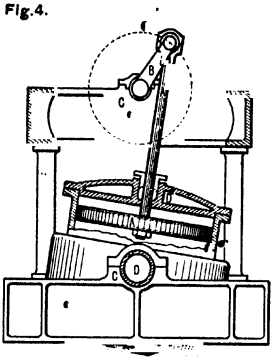


Fig. 5.

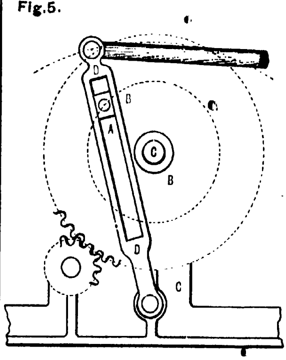
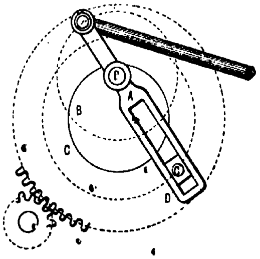


Fig. 6.



To face page 109.

As the crank rotates the crank pin moves up and down, while its centre vibrates in the arc of a circle.

(5.) The four mechanisms here described are all which can be obtained from the simple slider-crank chain, but an additional set may be derived by supposing that the line of stroke of the slider does not pass through the centre of the crank. A common example is found in the chain communicating motion from the piston to the beam in a beam engine.

Although the mechanisms derived by inversion from a given kinematic chain may be described as distinct, it must be carefully observed that there is in reality no kinematic difference between them, the distinction consisting merely in a different link being chosen to reckon velocities from. If we consider the velocities of the pairs which constitute the chain, those velocities are always related to each other in the same way, and the same machine may be regarded sometimes as one mechanism and sometimes another. For example, suppose a direct-acting engine working on board ship; the ship may be imagined to roll so that the connecting rod of the engine is at rest relatively to the earth, and the engine becomes an oscillator to an observer outside the ship. Dynamically and constructively, however, there is a great difference, for the fixed link is the frame, and is attached to the earth or other large body, the predominating mass of which controls the movements of all bodies connected with it. To illustrate and explain the inversion of a slider-crank chain, Plate I. has been drawn. The six examples which have just been described are here placed side by side with the same letters *A B C D* attached to corresponding links so that they may readily be recognized. It will be seen that each link assumes very various forms; thus, for example, the link *A* is the frame and cylinder in Figs. 1 and 2, a piston and rod in Figs. 3 and 4, a block in Fig. 5, and a rotating arm in Fig. 6. The relative motions of corresponding parts are, however, always the same.

51. *Double Slider-Crank Chains.*—We now pass on to the consideration of a kinematic chain consisting of two turning pairs and two sliding pairs. We will commence by showing how this chain may be derived from that previously described. Suppose the piece *D*, instead of being simply a block, is a sector shaped as shown in Fig. 1, Plate II., having a slot curved to the arc of a circle of centre *O*, while the piece *C*, which was before the connecting rod, is compressed into a block sliding in the curved slot. The law of relative motion of the parts of this mechanism will be precisely the same as in the direct-acting engine, for the block *C* will move just as if it were attached by a link, shown by the dotted line, to a point *O*, a fixed point in the piece *D*. The piece *D* will slide in *A*, just as if there were a connecting link from *C* to *O* and no sector—that is, it will slide just as the piston does in the cylinder of a direct-acting engine. Moreover, there are in reality exactly the same pairs in this as in the mechanism of a direct-acting engine, for *C* and *D* together make a turning pair, although only portions of the surfaces of the cylindric elements are employed.

This being so, let us now imagine the radius of the circular slot in the piece *D* to be indefinitely increased, so that the slot becomes straight, and is at right angles to the line of motion *D*. In such a case the pair *CD* would be transformed into a sliding pair, and the mechanism would consist of two turning pairs, and two sliding pairs, and is known as a *double slider-crank chain.*

The most important example of this kinematic chain is that found in some small steam pumping engines. (Fig. 4, Plate II.) The pressure of the steam on the piston is transmitted directly to the pump plunger. The crank *B* and sliding block *C* serve only to define the stroke of the piston and plunger, and, by means of a fly-wheel, the shaft of which carries an eccentric for working the slide valve, to maintain a continual motion. The law of motion of piston and crank pin may be readily seen to be the same as that in a direct-acting engine, in which the connecting rod is indefinitely long. *P* being the position of the crank pin, *M* will represent the position of the piston and reciprocating piece, and *PM* will represent the velocity of the piston at the instant, *OP* being taken to represent the uniform velocity of crank pin. (See Fig. 46, p. 97.) In this case the polar curve of velocity would consist of a pair of circles. This motion, shown in dotted lines in Fig. 48, is called a simple *Harmonic motion*, because the law is the same as that of the vibration of a musical string.

By a change of the link which is fixed, we may now derive other well-known mechanisms from this kinematic chain.

Instead of *A*, which forms part of a sliding and part of a turning pair, being fixed, let *B* be the fixed frame-link. *B* contains the elements of two turning pairs, so that the frame must contain two bearings or journals. An example of such a mechanism is that known as Oldham's coupling, Fig. 5, Plate II., used for connecting parallel shafts, which are nearly but not quite in the same straight line, and which are required to turn with uniform angular velocity-ratio. Each shaft terminates in a disc, in the face of which a straight groove is cut. The two discs, *A* and *C* in the figure, with the grooves, face each other, and are placed a little distance apart, with the grooves at right angles to each other. Filling up the space between them is placed a disc *D*, on the two faces of which are straight projections at right angles to one

Plate. II.

FIG. 1.

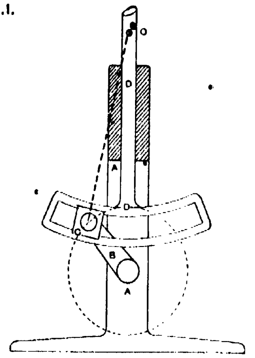


FIG. 4.

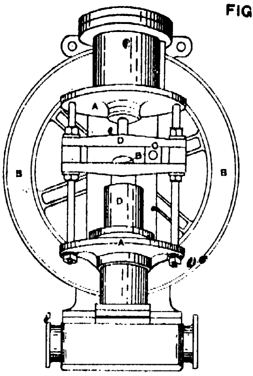


FIG. 2.

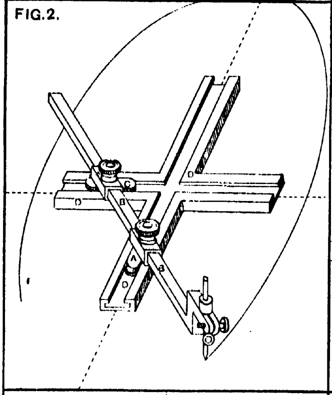


FIG. 5.

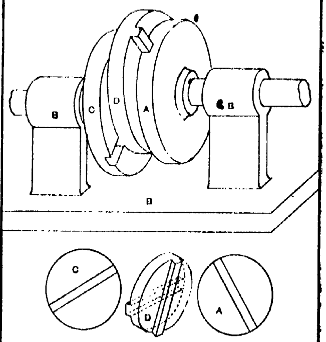


FIG. 3.

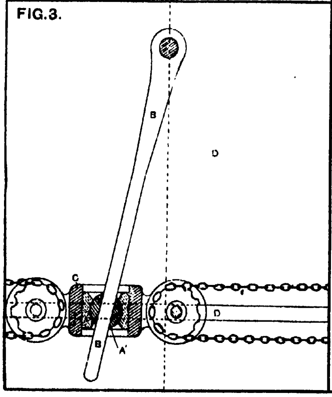
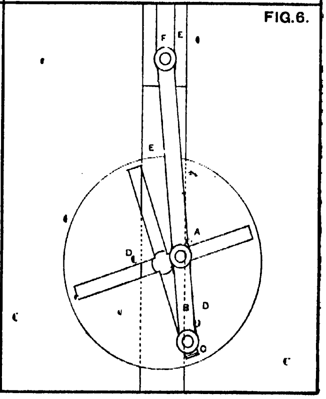


FIG. 6.



To face page 111.

another, which fit into the grooves in the shaft discs. In the revolution of the shafts each of these projections slides in the groove in which it lies, and rotates with it. The two grooves are, therefore, maintained always at right angles to one another, and the two shafts rotate one exactly with the other.

Next, let the fixed link of the chain contain the elements of two sliding pairs, which would be obtained if we made D the frame-piece. An interesting example of this is the instrument sometimes employed in drawing ellipses. (Fig. 2, Plate II.) Two blocks slide in a pair of right-angled grooves. By means of clamp-screws a rod unites them at a constant distance from one another. Pins fitting in holes in the block allow the rod to rotate relatively to the blocks. Any point in the rod will describe an ellipse, as indicated in the figure.

If the link C be fixed, the resulting mechanism does not differ from that derived by fixing A, and the three mechanisms just described are therefore all which can be obtained by inversion of a double-slider chain. In Figs. 2, 4, 5 of the plate referred to they are shown side by side with the same letters attached to corresponding links, as in Plate I.

The directions of motion of the two sliding pairs have been supposed at right angles, but any other angle may be assumed, and mechanisms obtained which we need not stop to examine. A more important change is to suppose that the sliding pairs and turning pairs alternate, so that each link forms an element of one sliding and one turning pair. A mechanism known as "Rapson's Slide," employed as a steering gear in large ships, will furnish an example. Fig. 3, Plate II., shows one way in which it is applied. A' is an enlarged pin made in two pieces between which the tiller B slides while turning about an axis fixed in the ship D. A' is carried by the piece C, which slides in a groove fixed transversely to the ship, being drawn to port or starboard by the tiller chains passing round pulleys mounted on C, as shown in the figure. The further the tiller is put over the slower it moves (Ex. 8, p. 124), and therefore the greater the turning moment (Ch. VIII.), a property of considerable practical value. Another example occurs in the motion of the compensating air cylinders employed in the Worthington direct-acting pumping engine. In this kinematic chain the same mechanism is obtained whichever link is fixed.

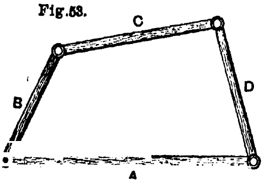
The mechanism shown in Fig. 6 of this Plate is a compound chain, to be referred to hereafter.

51a. *Wedge Chain.*—A chain also may be found which consists of sliding pairs alone: the number of pairs being 3, and the directions of sliding parallel to the same plane.

This chain consists of two sliding pairs, AB and AC, having a common element A. A block attached to B slides in an oblique slot cut in C, thus forming a third sliding pair, BC. The effect of this arrangement is that the pairs AB, AC are connected with uniform velocity-ratio. It is employed when it is desired to alter the direction and magnitude of a sliding motion. An incomplete form occurs in the strap and cotter employed to tighten the brasses of a bearing as they wear. The action of a wedge or the raising of a weight by drawing it up an incline furnishes another example of the same chain, here reduced by omission of one link which, as in various other instances, is replaced by force-closure (p. 123). We may describe it as a Wedge Chain; only one mechanism can be derived from it.

52. *Crank Chains in General.*—Instead of having a chain of sliding pairs or of turning pairs, connected by one or two sliding pairs, we may have turning pairs alone. The number will be four, and their axes must meet in a point or be parallel. Taking the second case, the chain in its most elementary form consists of four bars united by pin joints at their extremities, as in Fig. 53. It is called a crank or four-bar chain, and from it may be derived the slider-crank chain already considered, in the same way as from that chain we derived the double-slider chain. All the mechanisms hitherto considered may therefore be regarded as particular cases of it. In its present form, however, many new mechanisms are included, some of which will be briefly indicated, referring for descriptions and figures to works specially devoted to mechanism.

Fig. 53.



Assuming A the fixed link, B and D which pair with it are called for distinction cranks or levers, according as they are or are not capable of continuous rotation, while C, the connecting link, is called for shortness the *coupler*.

(1.) Let B be a crank and D a lever, then the mechanism is a "lever-crank," an example of which occurs in the common beam engine, D being the beam, B the crank, C the connecting rod, and A the entablature, foundation, and all other parts connected therewith.

(2.) The links B and D may be equal, and C may be equal to A. This may be called "parallel cranks" when B and D are set parallel, as in the coupled outside cranks found in locomotives, or "anti-parallel cranks" when they are set crosswise, a case to be hereafter referred to (page 163).

(3.) The links D and B may still both be cranks if C be greater than A, provided that the difference between B and D be not too great. The mechanism is called "double cranks," and occurs in the common draglink coupling, and also in the mechanism of feathering paddles.

(4.) If the coupling link be too short, neither B nor D will be capable of a complete rotation. The mechanism is then a "double lever," and an example occurs in the common parallel motion to be considered hereafter.

(5.) A number of additional mechanisms may be derived by supposing the axes of the four turning pairs to meet in a point, instead of being parallel; we thus obtain a "conic crank chain." Hooke's joint is a particular case of this, but in general these mechanisms are of less importance.

53. *Screw Chains.*—We have hitherto considered only chains of turning pairs and sliding pairs, but screw pairs also occur in a great variety of mechanisms which we can only briefly indicate.

(1.) In the Differential Screw, there are two screw pairs with the same axes but of different pitch, combined with a sliding pair, forming a three-link chain. The connection between the common velocity of rotation of the screws and the velocity of translation of the sliding pair is the same as that between the rotation and translation of a screw, the pitch of which is the difference between the pitches of the actual screws. The arrangement has often been proposed for screw presses, a mechanical advantage being obtained, at least theoretically, with screws of coarse pitch, which would otherwise require thread so fine as to be of insufficient strength. The right and left-handed screw is an example in common use.

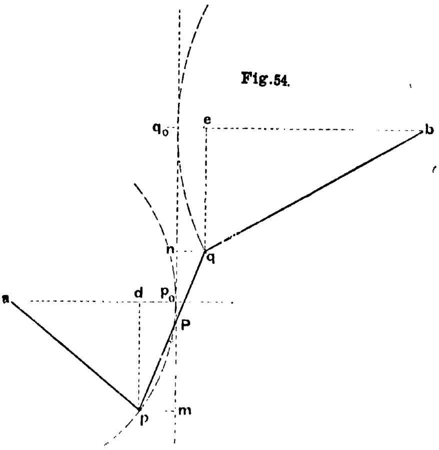
(2.) In the Slide Rests of lathes and other machine tools, the traversing motion of planing machines, and many other cases, we find a three-link chain, consisting of a screw pair, a turning pair, and a sliding pair. This may be regarded as a particular case of the preceding, the pitch of one of the screws being zero.

(3.) In presses, steering gear, and many other kinds of machinery, we find a simple screw chain employed to work a slider-crank chain. Some examples will be given hereafter.

54. *Parallel Motions Derived from Crank Chains.*—In beam engines the connecting rod by which the reciprocating motion of the piston is communicated to the vibrating beam is necessarily short, in order to diminish the height of the machine, and therefore, if guides are employed to retain the end of the piston rod in a straight line, there

will be considerable lateral pressure which is difficult to provide against, and which involves a large amount of friction. The guides may then be replaced with advantage by some linkwork or other mechanism. Such a mechanism is called a Parallel Motion, and in the early days of engineering was employed more extensively than at the present time. In its most simple form it consists of two levers capable of turning about the fixed centres a and b (Fig. 54). The ends of the levers are connected by a coupling link pq, then, so long as the angular movement of the levers is not too great, there is a point in the link pq which will describe very approximately a straight line. In the first instance let us suppose the link so set

Fig.54.



that when ap_0 and bp_0 are parallel, p_0q_0 is at right angles to them. Let $apqb$ be the extreme downward movement of the levers, then p lying to the left and q to the right, there will be some point P in pq which in this extreme position lies in the straight line p_0q_0. In the upward extreme position the same point of pq will, approximately, also lie in this line. If, then, p_0q_0 be the line of stroke, and the point P be selected for the point of attachment of the piston-rod head, then this point will be exactly in the line at the middle and bottom of the stroke, and at other points will deviate but little from it.

To find the point where pq intersects p_0q_0, we must first obtain expressions for the amount that the point p deviates to the left of p_0 and q to the right of q_0; these amounts being the versines of the arcs in which the points move, and shown by dp_0 and eq_0, where pd and qe are drawn perpendicular to ap_0 and bq_0. By supposing the circle of which q is the centre to be completed, it is easy to see that

$$(ad + ap_0)dp_0 = pd^2,$$

$$\therefore\ dp_0 = \frac{pd^2}{ad + ap_0}.$$

If the angle p_0ap is not greater than 20°, we may write

$$dp_0 = \frac{pd^2}{2 \cdot ap_0},$$

the error not being greater than 1 per cent. Now, neglecting the small effect due to the obliquity of the connecting link when in the extreme positions, $pd = \frac{1}{2}$ stroke: therefore, supposing $ap = r_a$ and $bq = r_b$,

$$pm = dp_0 = \frac{(\text{stroke})^2}{8r_a},$$

$$qn = eq_0 = \frac{(\text{stroke})^2}{8r_b}.$$

Now P being the point where pq intersects p_0q_0, we have similar triangles in which

$$\frac{pP}{qP} = \frac{pm}{qn} \quad \text{and} \quad \therefore = \frac{r_b}{r_a}.$$

Thus the point P, which has most correctly the straight-line motion, is such that it divides the coupling link into segments which are inversely proportional to the lengths of the levers. If the levers be placed into all possible positions, then in the motion the connecting link will be inverted and the point P will trace a closed curve resembling a figure of 8. There are two limited portions of this curve which deviate very little from a straight line.

We may approximate still more nearly to a straight line by a little alteration in the setting of the levers. Suppose the centres of vibration, a, b, are brought a little nearer together so that the line of stroke bisects the two versines, dp_0 and eq_0. Then when the levers are parallel, the link slopes to the left upwards, whereas at the ends of the stroke the link will slope to the right upwards. At two intermediate positions about quarter stroke from the ends, the link will be vertical. If we choose the point P as previously described, the maximum deviation will be only about one-fifth of its former amount. In practice, the final adjustment of the centres of motion is performed by trial.

In steam engines the use of parallel motions is almost exclusively confined to beam engines. In that case bq will be the half length of the beam, and in order that the angle through which the beam vibrates should not exceed 20° above and below the horizontal, the length of the beam should not be less than three times the stroke.

The radius rod may be somewhat shorter than the half beam, but should not be less than the stroke, or the error in the motion of P will be too great. This mechanism will, therefore, occupy a considerable space. To economize space, and also to provide a second straight-line path to guide the air-pump rod, a modification of the mechanism is made use of.

In Fig. 54a, be being the half length of beam, a point q is chosen so that

$$\frac{bq}{be}=\frac{\text{stroke of air pump}}{\text{stroke of piston}},$$

and a parallelogram of bars $qeQp$ provided, united by pins. The point p is jointed to the end of the radius rod ap vibrating on the fixed centre a. Consequently there will be some point P in the back link qp which will describe very nearly a straight line. This point is such that

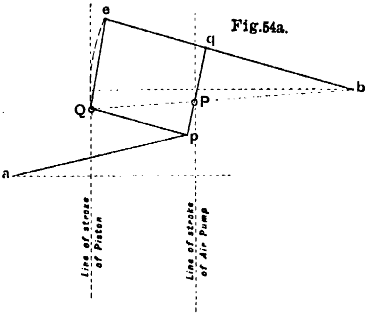


Fig. 54a.

$$\frac{Pp}{Pq}=\frac{bq}{ap}.$$

Now, if the proportions of the links are such that bPQ is a straight line, bQ/bP will be constant, and therefore the path described by Q will be an enlarged copy of the path described by P. That is to say, if P moves approximately in a straight line, then Q will do so also. If then the radius rod is of suitable length we provide a point Q for the attachment of the piston rod, and also a point P for the attachment of the air-pump rod. To find this length we have

$$\frac{bq}{pQ}=\frac{qP}{pP},$$

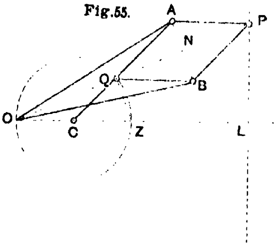
whence multiplying by the preceding equation

$$bq^2=pQ\times ap,$$

$$\text{or Length of radius rod}=\frac{(bq)^2}{ea}.$$

The parallel motion just described which was introduced by Watt is that chiefly used in practice, but there is another form which possesses great theoretical interest because it is exact and yet involves only turning pairs. Scott Russell's parallel motion (Fig. 50, page 105), modified by attaching D to the end of a long vibrating lever, is known as a "grasshopper" parallel motion, but then is only approximate. In its original form it is exact, but as it involves a sliding pair its accuracy depends on the exactness with which the guides of the slides are constructed. Now, a straight edge or a plane surface can only be constructed by a process of copying from some given plain surface or by trial and error, whereas a circle can be described by a pair of compasses independently of the existence of any other circle. Hence an exact parallel motion, with turning pairs only, enables us theoretically to trace a straight line in the same way that a circle is traced with compasses. It has long been known that this could be done by a circle rolling within another twice its diameter, but this method does not satisfy the necessary conditions, and it was not till 1872 that Col. Peaucellier invented a linkwork mechanism for the purpose. This mechanism consists of two equal bars, OA, OB, jointed to each other at O, and at A, B to a parallelogram of equal bars, $APBQ$, so that OQP are in a straight line (Fig. 55). This being so then, however the bars are placed, there will always be some fixed relation existing between OQ and OP. Thus drop a perpendicular AN on OP, then $OQ=ON-QN$ and $OP=ON+NP$. Also, since $AQ=AP$, $QN=NP$,

Fig. 55.



$$\therefore\ OQ\,.\,OP=ON^2-QN^2.$$

But $ON^2=OA^2-AN^2$ and $QN^2=QA^2-AN^2$, therefore $OQ\,.\,OP=OA^2-QA^2$, and is a constant quantity for all positions; that is to say, if we cause Q to move over any curve, then P will describe its reciprocal.

We can now show how this mechanism may be employed to draw a straight line. Let O be a fixed centre and PL be the straight line which it is required to describe. Draw the perpendicular OL on PL. Then the mechanism being placed in any position with P at any point on the line to be drawn, draw QZ at right angles to OQ. Bisect OZ in C and attach Q to C by means of a jointed rod which can turn on the fixed centre C. The circle which Q describes during the motion of the bars will have OZ as a diameter, for OQZ is a right angle, and therefore the angle in a semicircle. We observe now that we have similar triangles OQZ and OLP.

$$\therefore\ OL = \frac{OP \,.\, OQ}{OZ};$$

but $OZ = 2 \,.\, OC$ is a constant quantity and so is the product OP, OQ.

$$\therefore\ OL \text{ is constant.}$$

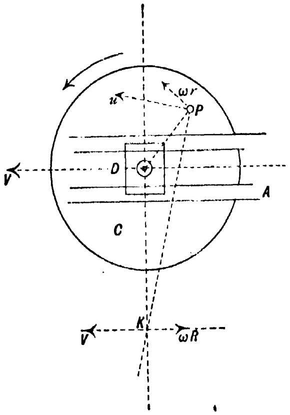
That is to say, wherever P is, the length of the projection of OP on the perpendicular OL is a constant quantity. This can be true only so long as P lies in the perpendicular line PL. Thus, by the constrained motion of Q in a circle passing through O, P is caused to move perfectly in a straight line.

This mechanism has been applied to a small engine used for ventilating the House of Commons.

SECTION III.—KINEMATICS OF LINKWORK MECHANISMS.

55. *Combination of a Sliding Pair and a Turning Pair.*—The motion of the connecting rod in the mechanism of the direct-acting engine (Arts. 48, 49) may be considered as a combination of a motion of rotation due to a turning pair, with a motion of translation due to a sliding pair, and we now propose to consider the effect of such a combination more generally.

Fig 56a.



In Fig. 56a, D is a block sliding in guides attached to the frame-piece A, C is a wheel turning in bearings attached to D, the whole forming a chain of two pairs DA, CD. The block slides with velocity

V, while the wheel turns with angular velocity ω about an axis perpendicular to the plane of the paper. In consequence of the rotation, any point *P* at a distance *r* from the axis has a velocity ωr perpendicular to the radius, while at the same time it is carried onwards in the direction of the sliding with velocity *V*. In a line perpendicular to the direction of sliding and the axis of rotation take a point *K* distant *R* from the axis, and if outside the wheel, as in the figure, suppose it rigidly connected with it by an arm. Then evidently *K* moves forward with velocity *V* in consequence of the sliding, and backwards with velocity ωR in consequence of the rotation. Thus *K* moves forwards with a velocity $V-\omega R$ which may be reduced to zero by taking *R* so that

$$V=\omega R.$$

It appears therefore that it is always possible to find a point *K* which when rigidly attached to the wheel *C* will be for the moment at rest. Join *KP* and observe that the motion of translation of *P* is perpendicular to *DK* and equal to $\omega . DK$, while its motion of rotation is perpendicular to *DP* and equal to $\omega . DP$, then by a well-known kinematical principle it follows that the actual velocity of *P* consequent on the combination must be perpendicular to *KP* and equal to $\omega . KP$. Thus the velocity of *P* is the same as if *C* were rotating with its actual angular velocity ω about an axis through *K*, and this will be true for any point in *C* or rigidly attached to it, the effect of combining a motion of translation with a motion of rotation being simply to shift the axis of rotation through the distance $R=V/\omega$. The point *K* does not remain fixed, but moves so as to be always in the perpendicular, and the axis through it is therefore described as the Instantaneous Axis of the moving piece *C*. Its position is completely represented by the point *K* which is often spoken of as an "instantaneous centre."

If, as in the figure, the rotation be in the opposite direction to the hands of a watch and the translation be from right to left, the point *K* lies below *D*, but if either motion be reversed it will lie above, as in Fig. 56b, p. 120.

Further, there is nothing in the demonstration just given which renders it necessary that the direction of the sliding motion should remain unaltered, and the construction will therefore be the same if the block *D* slide in a slot which is circular instead of straight, or be attached to a piece turning in bearings on *A*. That is, the point *K* can be found in the same way for a combination of two turning pairs, and the effect of the combination of two rotations about parallel axes is to produce a rotation about an instantaneous axis parallel to the former and in the same plane. A particular case is when the rotations are equal and opposite, the instantaneous axis is then at an infinite distance, and the effect of the combination is a motion of translation, the direction of which continually changes. A locomotive coupling-rod (p. 112) is a common example which should be carefully considered, as a useful illustration of the meaning of this theoretical proposition.

Fig. 56b.

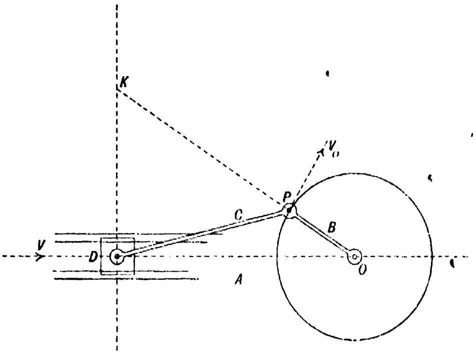


Fig. 56b shows in skeleton the mechanism of the direct-acting engine already considered at length. In this case, *C* is a rotating piece connected as just described with both the sliding block *D* and the rotating crank *B*. In consequence of the first connection it has an instantaneous centre *K* in the perpendicular through *D*, and in consequence of the second an instantaneous centre in the prolongation of the crank *OP*. Hence *K*, the intersection of these two lines, must be that centre, and with the same notation as before

$$\frac{V}{V_0}=\frac{KD}{KP}.$$

If all points in *C* lying in a plane perpendicular to the axis be joined to *K*, the corresponding instantaneous centre, the set of radiating lines may be considered as a diagram of velocities, but, as in the case of stress diagrams, it is generally far preferable to draw a separate diagram on some suitable scale. This may be done, as previously described on page 100, by selecting a pole *O* and drawing *Oa*, *Ob* perpendicular and proportional to the velocities of *a* and *b*, two given points in *C*. Two figures may thus be drawn, corresponding to the two positions into which the triangle *Oab* (Fig. 47a) may be turned by a rotation through 90°. In the first, *ab* is parallel to the corresponding line in *C*, and points in the same direction; the diagram is now similar and similarly situated to the set of lines radiating from K. In the second, ab is also parallel to the corresponding line in C, but it points in the opposite direction, and the diagram may be described as "reversed." In plotting a point p in a reversed diagram which corresponds to a given point in C we have only to draw ap, bp parallel to the corresponding lines in C. The pole O, of course, always corresponds to the instantaneous centre K.

56. *Diagram of Velocities in Linkwork.*—A simple construction has already been given, by means of which the velocity-ratios of the parts of a slider-crank chain are determined, and we will now consider this question for any case of linkwork in which the axes of the pairs are parallel.

Fig. 57a represents a chain of links, $zOabcd$ ..., united by pins so as to form a succession of turning pairs. The first link, Oz, is fixed, so

Fig. 57a.

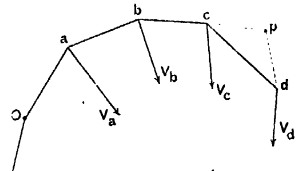
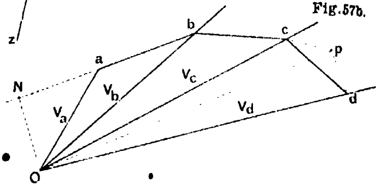


Fig. 57b.



that the second turns about a fixed point, O, as centre, and therefore a moves perpendicularly to Oa, with a velocity V_a, which we may suppose known. The other points, b, c, d ..., move in directions which we suppose given, and with these data it is required to find the magnitudes of the velocities. In Fig. 56b from a pole O draw radiating lines perpendicular to the given directions, and set off on the first Oa to represent V_a, then draw ab, bc, cd ... parallel to the links of the chain to meet the corresponding rays, then the lengths of those rays represent the velocities.

For drop a perpendicular ON from O on to ab, or ab produced, then ON represents the component of V_a in the direction of the second link, but this must also be the component of V_b in that direction, since ab is of invariable length; that is, Ob must represent V_b.

Similarly all the other rays must represent the velocities of the corresponding points.

The figure thus drawn may be called the Diagram of Velocities of the chain. It may be constructed equally well, if the magnitudes of the velocities be given, instead of their directions, also any of the turning pairs may be changed into sliding pairs. If both ends of the chain be attached to fixed points, the diagram will evidently be a closed polygon. Its sides, when divided by the lengths of the corresponding links of the chain, represent their angular velocities, for each side is the algebraical difference of the velocities of the ends of the link perpendicular to the link.

In the four-link chain (Fig. 58a), consisting of two links turning about fixed centres, a, d, coupled by a link bc, the diagram of velocities is a simple triangle, Obc (Fig. 58b), the sides of which when divided by the lengths of the links to which they are parallel, represent the angular velocities of the links. Through a draw aZ parallel to cd, and prolong bc to meet it in Z, and the line of centres in T, then, since the triangle Zab is similar to the triangle of velocities, the angular velocities of the levers cd, ab will be proportional to Za/cd and ab/ab. The last fraction is unity, and therefore we have

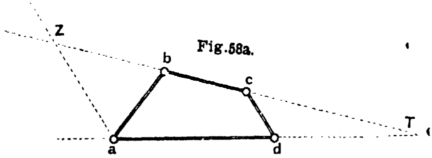


Fig.58a.

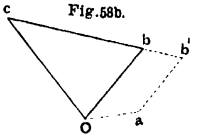


Fig.58b.

$$\text{angular velocity-ratio} = \frac{Za}{cd} = \frac{aT}{dT},$$

showing that the ratio in question is the inverse of the ratio of the distance of T from the centres.

If, instead of the link ad being fixed, the chain of four bars be imagined to turn about one joint such as d, the diagram of velocities would be a quadrilateral $Oab'c$, with sides parallel to $abcd$.

Returning to the general case, let p be any point rigidly connected with one of the links of the chain, say *cd*, in the figure; then if we lay down on the diagram of velocities a point *p*, similarly situated with respect to the corresponding line *cd* of that diagram, it follows at once, by the same reasoning, that the ray, *Op*, drawn from the pole *O*, must represent the velocity of *p* in the same way that the other rays represent the velocities of the point *a*, *b*.... Thus it appears that for any linkwork mechanism, consisting of pieces of any size and shape connected by pin joints, the axes of which are parallel, a diagram may be constructed which will show the velocities of all points of the mechanism. By constructing the mechanism and its diagram of velocities for a number of different positions, curves of position and velocity may be drawn, such as those described in preceding articles for special cases.

56A. *Closure of Kinematic Chains. Dead Points in Linkwork.*—A kinematic chain, like a pair (p. 95), may be "incomplete," that is, the relative moments of its links may not be completely defined. It then cannot be used as a mechanism without employing some additional constraint, a process called "closing" the chain. In order that a chain may be closed it must be endless, and the number of links must not be too great; for example, in a simple chain of turning pairs with parallel axes we cannot have more than four links. If there be five the motion of any one link relatively to the rest will not be definite, but may be varied at pleasure.

So also a chain may be "locked" either by locking one of the pairs of which it is constructed; or by rigidly connecting two links not forming a pair; it then becomes a frame, such as was considered in a previous part of this book.

As an example of an incomplete chain may be taken the combination of a sliding pair and a turning pair considered in Art. 55, and shown in Fig. 56a (p. 118). The relation between the sliding and the turning is here undefined until the chain is closed by the addition of another pair as in Fig. 56b, or in some other way.

A chain is often incomplete or locked for special positions of its links, though closed and free to move in all other positions; this, for example, is the case at the dead points which occur in most linkwork mechanisms. A well-known instance is that of the mechanism of the steam engine, in which the chain is locked and the direction of motion of the crank indeterminate when the connecting rod and crank are in the same straight line. This instance further shows that it is necessary to distinguish between the two directions in which motion may be transmitted through the mechanism, for the dead points in question would not occur if the crank moved the piston instead of the piston the crank. A piece, then, which transmits motion is called a "driver," in relation to the piece in which motion is transmitted, which is called a "follower," terms which will be frequently used hereafter in cases where both pair with the fixed link. (Compare Arts. 47, 86.) The dead points in a mechanism may be passed either by the union of two similar mechanisms, with dead points in different positions, as in a steam engine with a pair of cranks at right angles, or by aid of the inertia of the moving parts. This last method involves what is called "force-closure," a term which will be explained presently.

EXAMPLES.

1. The stroke of an oscillating engine is 6 feet, and the distance between the centre line of the trunnions and the centre of shaft is 9 feet. Find the maximum and mean angular velocity of the cylinder in each of its two oscillations as compared with that of the crank. Find also the velocity of the piston at half stroke as compared with the speed of piston.

Ans. Maximum angular velocity-ratios of cylinder and crank, $\frac{1}{2}$ and $\frac{1}{4}$.
Mean ,, ,, ·277 and ·178.

$$\frac{\text{Velocity of piston at } \frac{1}{2} \text{ stroke}}{\text{Mean speed of piston}} = \frac{1{\cdot}54}{1}.$$

2. The travel of the tool of a shaping machine is to be 9 inches, and the maximum return, three times the maximum cutting, velocity. The connecting link is horizontal in the extreme positions of the lever and is attached to a point in it which is on a level with the crank pin when the crank is upright, find the proper proportions of the quick return motion (Fig. 5, Plate I.). Find also the revolutions per minute for a maximum cutting velocity of 6 inches per second, and compare the times of cutting and return.

Ans. The length of slotted lever = 9 inches.
Distance apart of centres = twice length of crank. $\frac{\text{Time of cutting}}{\text{Time of return}} = \frac{2}{1}.$
Revs. per minute of crank = 19·1.

3. In Whitworth's quick return motion find the proportions that the maximum return may be three times the maximum cutting velocity, and compare the times of cutting and return.

4. In Example 1, draw curves showing the angular velocity and position of the piston for any position of the crank.

5. A reciprocating movement is given to the table of a small planing machine by a uniformly rotating crank below connected by a rod to a projecting arm so that the rod is horizontal when the crank is upright. Find by graphic construction the ratio of the times of a forward and a backward movement. Also find the velocity of the table in any position.

6. In question 1, p. 102, supposing two pairs of driving wheels coupled, the lengths of cranks 1 foot, find the velocity of the coupling rod in any position. First, relatively to the locomotive; second, relatively to the earth.

7. In Ex. 6, p. 103, find in feet per second the maximum and minimum velocity of rubbing of the crank pin, assuming its diameter 12 in., and the revolutions 30 per 1′. Draw a curve showing this velocity in any position of the crank.

8. In Rapson's Slide (p. 111), if the tiller be put over through an angle θ, show that the velocity-ratio of tiller and slide varies as $\cos^2\theta$, and draw a curve of velocity.

9. In a draglink-coupling the shafts are 6 in. apart, the draglink 1 foot long, and the cranks each 3 feet long. By construction, determine the four positions of the following

crank when the leading crank is on the line of centres, and at right angles to the line of centres.

10. The length of the beam of an engine is three times the stroke. Supposing the end of the beam when horizontal is vertically over the centre of the crank shaft at a height equal twice the stroke, and the crank also is then horizontal, find the length of connecting rod and the extreme angles through which the beam will sway. Adjust the crank centre so that the beam may sway through 20° above and below the horizontal.

Length of rod=2·06 stroke. The beam sways $22\frac{1}{2}$° above the horizontal, and 17° below.

11. The depth of the floats of a feathering paddle wheel is $\frac{1}{6}$th the diameter of the wheel, and the immersion of the upper edge in the lowest position $\frac{1}{4}$th the depth of the float. Assuming the stem levers $\frac{3}{5}$ths the depth of the floats, find the position of the centre of the collar to which the guide rods are attached. Determine the length of the rods, and draw the float in its highest position.

If *O* be centre of wheel, *K* centre of collar, *OK*=·054 of diameter of wheel, and is horizontal (very approximately).

Length of guide rods=1·01 radius of wheel.

12. In Ex. 9, find the angular velocity-ratio of the shafts when the cranks are in the positions mentioned.

13. In Oldham's coupling, show that the centre of the coupling disc revolves twice as fast as the shafts, and hence show how to give two strokes of a sliding piece for one revolution of a shaft.

14. In a simple parallel motion the length of the levers are 3 feet and 4 feet respectively, and the length of the connecting link is $2\frac{1}{2}$ feet. Find the point in the link which most nearly moves in a straight line, and trace the complete curve described by this point as the levers move into all possible positions, the motion being set so that, when the levers are horizontal, the link is vertical.

Ans. The required point in link is $17\frac{1}{7}$ in. from the 3-feet lever.
$12\frac{6}{7}$ in. ,, 4-feet ,,

15. In a beam engine the stroke of piston is 8 feet, of air-pump $4\frac{1}{2}$ feet, length of beam 24 feet, the front and back links of the parallel motion being 4 feet. Find the proper length of radius rod, and the point in the back link where the air-pump rod should be attached.

Ans. Length of radius rod=8 feet $8\frac{1}{8}$ inches.
Point of attachment of air-pump rod=2 ,, 3 ,, below beam.

16. Suppose in last question the parallel motion set for least deviation from a straight line, find the correct positions of the centre lines of air-pump and piston, and the position of the centre of motion of the radius rod.

Ans. Horizontal distances from centre of beam—
Line of stroke of piston, · 11 feet 8 inches.
,, air-pump, · 6 ,, $6\frac{3}{4}$,,
Centre of motion of radius rods, 15 ,, $1\frac{1}{8}$,,

17. Draw diagrams of velocity (Art. 56) for any position of the mechanism—
(1) In the beam engine of Ex. 10;
(2) In the quick return movement of Ex. 2;
(3) In the Peaucellier parallel motion (Fig. 55).

18. In the last question, draw curves showing in case 1 the velocity of the piston for any position of the crank, and in case 2 the velocity of the tool at any point of the cutting and return strokes, assuming in each case that the crank rotates uniformly.

REFERENCE.

A good collection of linkwork and other mechanisms, some of which do not occur in the larger works cited on page 92, will be found in the later editions of Professor Goodeve's *Elements of Mechanism*. Much valuable information on the details of machine design is contained in a treatise on Machine Design by Professor W. C. Unwin, M.I.C.E. (Longman).

CHAPTER VI.

CONNECTION OF TWO LOWER PAIRS BY HIGHER PAIRING.

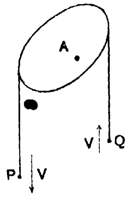
SECTION I.—TENSION AND PRESSURE ELEMENTS.

57. *Preliminary Remarks. Tension Elements.*—If one of the elements of a pair be not rigid, or if the contact be not of the simple kind considered in the preceding chapter, the pairing is said to be "higher," because the relative motion of the elements is more complex. Higher pairing is seldom employed alone; it is generally found in combination with lower pairs, the elements of which it serves to connect. The most important case is that where a chain of two lower pairs is completed by contact between their elements or by means of a link which is flexible or fluid. Motions may thus be produced in a simple way which are impossible or difficult to obtain by the use of lower pairing alone. The present chapter will be devoted to mechanisms derived from chains of this kind, the fixed link being generally a frame common to the two lower pairs. The velocities of each of the pairs are thus the same as those of their moving elements. We commence with the case of non-rigid elements.

A body which was incapable of resistance to any kind of change of form and size would of course be incapable of being used as part of a machine, for it could not furnish any constraining force whereby the motion of other pieces could be affected, but if it resists any particular kind of change it will supply a corresponding partial constraint which may be supplemented by other means. The first case we take is that of a flexible inextensible body, such as is furnished approximately by a rope, belt, or chain. This is called a Tension Element, being capable of resisting tension only, and it is plain that when any two points are connected by it, their distance apart, measured along the element itself, must be invariable so long as the rope remains tight. If the rope be straight, it may be replaced by a link, and we obtain the mechanisms already considered, but we now suppose it to pass over a surface of any form.

In Fig. 59a, let A be a fixed body of any shape, round which an inextensible rope PQ passes, the ends hanging down. If P moves downwards with velocity V, Q moves upwards with the same velocity, the rope slipping over A at all points with velocity V. In practice A is generally circular, and is mounted on an axis, upon which it revolves. We have then a "pulley block," of which A is the "pulley" or "sheave," and the rope causes it to rotate instead of slipping over it, but this makes no difference in the motion, and the only object of the arrangement is to diminish friction and wear.

Fig.59a.



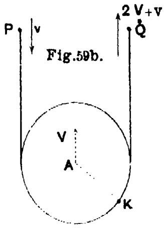
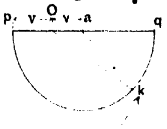


Fig.59b.

Fig.59c.

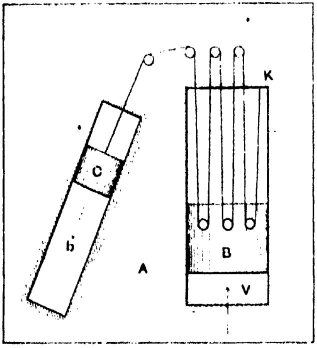


Next suppose the pulley movable (Fig. 59b), and imagine P attached to a fixed point, while Q moves upwards with the same velocity V relatively to A as before. Then A must move upwards with velocity V, because its motion relatively to the fixed point P is unaltered, and hence Q moves with velocity $2V$. More generally, if P, instead of being fixed, moves downward with velocity v, Q must move upwards with a velocity $2V+v$, or to express the same thing otherwise—*the difference of velocities of the two sides of the rope is twice the velocity of lifting*—a principle applicable to all questions relating to pulleys. The velocity of rotation of the pulley is $V+v$, its radius being the "radius of reference" (Art. 46). The motion of rope and pulley may be represented by a diagram of velocity. Thus, in Fig. 59c, describe a semi circle with radius equal to $V+v$, then the radius of that circle represents the velocity of rotation or the velocity of any point in the rope relatively to the centre of the pulley. The actual velocity of any point K in the rope is found by compounding this with V, the velocity of the centre of the pulley. The pole of the diagram is therefore a point O, distant V from the centre of the circle, so that if k be the point in the diagram corresponding to the point K of the rope, Ok represents the velocity of K.

58. *Simple Pulley Chain. Blocks and Tackle.*—We have now a simple means of solving one of the most important problems in mechanism—namely, to connect two sliding pieces with a constant velocity-ratio.

In Fig. 60a, *B*, *C* are pieces sliding in guides attached to a frame-piece *A*, thus forming two sliding pairs with one link common. In *B* a number of pins are fixed, and in *A* an equal number placed as in the figure, so that a rope passing round them as shown may form a number of plies parallel to *B*'s motion.*

Fig. 60a.

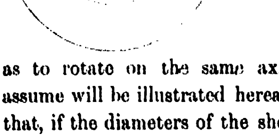


The rope is attached at one end to *C*, and led to the nearest fixed pin, over a guide pin placed so that this part of the rope may be parallel to *C*'s motion, while the other end is attached to a fixed point *K*. The effect of this arrangement is that when *C* moves in the direction of the arrow, *B* also must move with a velocity which is readily found by the principle just explained, for the difference of velocities of the two parts of each ply must be the same, being twice the velocity of *B*. Thus reckoning from the fixed end, if *B*'s velocity be *V*, the velocities of the several parts of the rope must be

$$0,\ 2V,\ 2V,\ 4V,\ 4V,\ 6V,\ 6V\ldots,$$

so that if there are *n* pins in *B*, the velocity of the other end of the rope must be $2nV$, and the velocity-ratio $2n:1$. The diagram of velocities consists of a number of semi circles (Fig. 60b), the lower set struck with centre *a* and the upper with centre *O*, where *O* is the pole and *Oa* the velocity of lifting.

Fig. 60b.



The simple kinematic chain here described may be inverted, by fixing *B* or *C* instead of *A*. In the blocks and tackle so common in practice, the pins are replaced by movable sheaves, usually, but not always, of equal diameters, and placed side by side so as to rotate on the same axis. Some of the various forms they assume will be illustrated hereafter. The diagram of velocities shows that, if the diameters of the sheaves are proportional to the diameters of the circles shown in the diagram, they will have the same angular velocity, and may therefore be united into one, an idea carried out in White's Pulleys.

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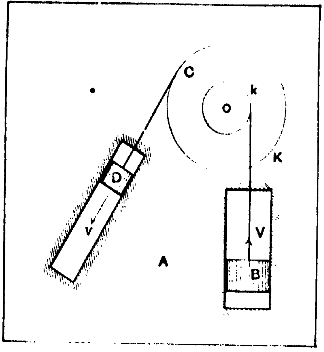
* This figure is taken, with some modifications, from the second edition (1870) of Willis's Mechanism.

In all cases the mechanism which we have been considering (Fig. 60a) is a closed kinematic chain only so long as the rope remains tight. One method of securing this would be to supply a second rope passing under another set of pins below *B* (not shown in the figure) and led to the other side of *C* by a suitably placed guiding pulley; we should then, by tightening up the ropes, have a self-closed chain similar to those considered in the preceding chapter. In practice, however, forces are applied to *B* and *C* which produce tension in the rope; thus, for example, when employed for hoisting purposes, the weight which is being lifted keeps the rope tight. This is the simplest example of what is called force-closure, where a kinematic chain, which is not in all respects closed, is made so by external forces applied during the action of the mechanism. In practical applications the principle of force-closure is carried still further, for the guides which compel the pieces *B* and *C* to move in straight lines are usually omitted. In the case of *B* the weight and inertia of the load which is being raised or lowered supply sufficiently the necessary closure, while in the case of *C* the end of the rope may be guided by the hand.

59. *Wheel and Axle.*—When mechanical power is employed for hoisting purposes, the end of a rope is frequently wound round an axle the rotation of which raises or lowers the weight, and this leads us at once to a different and equally important method of employing tension elements,—namely, by attaching one end to a fixed point in the cylindrical surface of an element of a turning pair. The rope in this case passes over the surface and is guided by it, but does not slip over it as it does over the pins of the previous arrangement. The most useful case is that where the transverse section of the surface is a circle, and the direction of the rope always at right angles to the axis of rotation; then it is clear that the motion of the surface is the same as the motion of the rope.

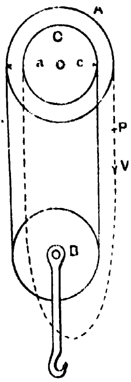
The well known Wheel and Axle is a combination of two chains of this kind. In its complete ideal form it consists of two sliding pairs *AB*, *AD*, with planes parallel and one link *A* (Fig. 61) common. A rope is attached to *D* and, passing partly round a wheel, is attached to it at a fixed point *K* in its circumference; a second rope is attached to *B*, and passing partly round an axle, is attached to a fixed point *k* in its circumference, the two ropes lying in parallel planes. The wheel and axle are fixed together, and form with *A* the turning pair *AC*. We have thus a second means of connecting two sliding pieces so that their velocity-ratio may be uniform, for the velocities of *B* and *D* must be inversely as the radii of the wheel and the axle. As before, the ropes must be kept tight, also the guides of the pieces *B* and *D* may be omitted and replaced by force-closure, and this will be necessary if the wheel is to make more than one revolution, for then a lateral movement is required to enable the rope to coil itself on the surfaces.

Fig. 61.



In practical applications the second rope is generally omitted and the wheel turned by other means; the lateral movement is sometimes provided for by permitting the axle to move endways in its bearings, but more often, in cases where the load is not free to move laterally, the effect of a moderate inclination of the rope to the axis is disregarded. We may, however, escape this difficulty by the use of force-enclosure of a different kind. Instead of attaching the rope to a fixed point in the surface, let it be stretched over it by a force at each end, there will then be friction between the rope and the surface, which will be sufficient to prevent slipping if the tendency to slip be not too great.

Fig. 62.



The Differential Pulley is a good example of the application of these principles. As is shown in Fig. 62, there are two blocks, of which the upper, which is fixed, carries a compound sheave, consisting of two pulleys *A* and *C*, of somewhat different diameters, fixed to one another. The lower block carries a single sheaf *B*, the diameter of which should theoretically be a mean between those of *A* and *C*, in order that the chain may be vertical. The chain is endless, and passes round the pulleys in the manner shown, so that when the side *P* is hauled downwards with a given velocity *V*, it will raise the lower block *B* with a velocity which we will now determine.

In passing around *A* and *C* the chain is not capable of slipping. To ensure its non-slipping the periphery may be recessed to fit the links of the chain. In passing around *B* the slipping is immaterial;

the raising of B would take place with the same velocity, whether there were an actual slipping of the chain round the circumference, or whether B were a rotating pulley.

When the point P is hauled downwards with velocity V, it necessitates the rotation of A, and with it of C. Thus the left-hand portion of the chain passing round B will be hauled upwards with the same velocity as the point P downwards, and the right hand will descend with a velocity which is less in the ratio of the radii, c, a, of the united pulleys, and thus on the whole there will be an ascending motion given to B. Now, since the upward velocity of B is half the difference between the velocities of the two portions of the chain,

$$v = \frac{1}{2}\left(V - \frac{c}{a}V\right) = \frac{V}{2}\left(1 - \frac{c}{a}\right) = \frac{a-c}{2a}V.$$

Thus, by making the difference between a and c small, the relative velocity of B to P may be made as small as we please.

This apparatus, in a somewhat modified form, is much employed. It is called Weston's Differential Pulley Block, and possesses the valuable property that the weight will not descend when the hauling force is removed, for reasons which will be explained hereafter (Ch. X.).

60. *Pulley Chains with Friction Closure. Belts.*—A tension element may also be employed to connect the elements of two turning pairs. The most important case is that where two shafts are connected by an endless belt passing over a pair of pulleys and stretched so tightly that the friction between belt and pulley is sufficient to prevent slipping. If the belt were absolutely inextensible the speed of centre line of the belt would be the same at all points, and therefore the angular velocities of the pulleys would be inversely as their radii each increased by half the thickness of the belt. This mode of connection is unsuitable where an exact angular velocity-ratio is required, for even though the belt may not slip as a whole, yet it will be seen hereafter (Ch. X.) that its extensibility causes a virtual slipping to a greater or less extent. In the case of leather belts, the error in the angular velocity-ratio due to this cause is said to be about 2 per cent.

There are two ways in which the belt may be wrapped around the pulleys, being either *crossed or open*. If the belt is crossed, the pulleys will revolve in opposite directions. The crossed belt embraces a larger portion of the circumference of the pulleys than the open belt, and there is thus less liability to slip.

There is a proposition of some importance connected with the length of a crossed belt, which it will be useful to give here.

AC and BD (Fig. 63) being radii, each drawn at right angles to the straight portion of the belt CD, will each make the same angle θ with the line of centres. Hence the portion of the belt in contact with the pulley $A = (2\pi - 2\theta)\, r_A$ and that in contact with the pulley $B = (2\pi - 2\theta)\, r_B$. The length not in contact $= 2 \,.\, CD = 2(r_A + r_B) \tan\theta$.

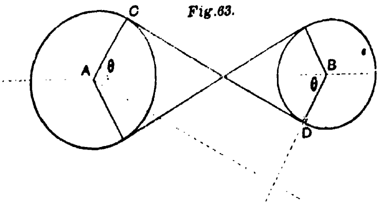


Fig.63.

$$\text{Thus whole length of belt} = 2(\pi - \theta + \tan\theta)(r_A + r_B).$$

$$\text{But } \cos\theta = (r_A + r_B)/AB,$$

and consequently, if the distance AB between the centres is a constant quantity, and if, further, the sum of the radii $r_A + r_B$ is constant, then the angle θ will be constant. That being so, the total length of the belt will be a constant quantity.

This property is made use of when it is desired to connect two parallel shafts with an angular velocity-ratio, which may be altered at pleasure. A set of stepped pulleys, such as are shown in Fig. 1, Plate III., are keyed to each shaft, and the belt being shifted from one pair to another of the pulleys, the angular velocity-ratio is altered at will. If the belt is crossed, then the same belt will be tight on any pair of pulleys, if the sum of the radii is the same for each pair. This does not hold good for open belts. The actual length of belt required in any given example is best found by construction.

The tightness of the belt necessary to effect closure by friction of this kinematic chain may be produced simply by stretching the belt over the pulleys so as to call into play its elasticity, but the axis of rotation of one pulley is sometimes made movable, so that the belt may be tightened by increasing the distance apart of the shafts, while in other cases an additional straining pulley is provided. The belt may then be tightened and slackened at pleasure; a method frequently used in starting and stopping machines.

In order that the belt may remain on the pulleys they must be provided with flanges, or, as is more common in practice, they must be slightly swelled in the middle, for when the shafts are properly in line, a belt always tends to shift towards the greater diameter. Great care, however, is necessary in lining the shafts that each side of the belt lies exactly in the plane of the pulley on to which it is advancing. Thus, for example, if the shafts be in the same plane, they must be exactly parallel, otherwise the belt will shift towards the point of intersection. This remark, however, does not apply to the receding side of the belt, and the shafts may make a considerable angle with each other, subject to the above restriction.

Friction-closure is always imperfect, because the magnitude of the friction is limited, but this is often a great advantage, since it permits the chain to open when the machine encounters some unusual resistance, which would otherwise produce fracture. By the use of grooved pulleys provided with clips the friction may be increased to any extent, so that great forces may be transmitted, but these devices are only suitable for low speeds, as in steam-ploughing machinery. Slipping may be avoided altogether by the employment of gearing chains, the links of which fit on to projections on the pulleys; force-closure is here replaced by chain-closure, and the action is in other respects analogous to toothed gearing. The speed is limited, as will be seen hereafter.

61. *Shifting of Belts. Fusee Chain.*—By the use of drums of considerable length as pulleys, the belt may be shifted laterally at pleasure. This principle is much employed in practice, as for example—

(1) To stop and set in motion a machine.—The drum on one of the shafts is divided into two pulleys, one fast and the other loose on the shaft.

(2) To reverse the direction of motion.—The drum is divided into three pulleys, the centre one fast, the two end ones loose on the shaft. Two belts, one crossed and the other open, are placed side by side. By shifting the belt either is made to work on the fast pulley at pleasure.

(3) To produce a varying angular velocity-ratio. The drums are made conical instead of cylindrical. The fusee employed in watches to equalize the force of the main spring is a common example.

The kinematic character of these devices will be considered in the next chapter.

62. *Simple Hydraulic Chain. Employment of Springs.*—Incompressible fluids may be employed to connect together two or more rigid pieces forming a class of elements which may be called "pressure elements," since they are capable of resisting pressure only. The pressure must be applied in all directions, and the fluid must therefore be enclosed in a chamber which pairs with the different pieces to be connected. For constructive reasons lower pairing must generally be adopted, and almost all cases are included in the following investigation.

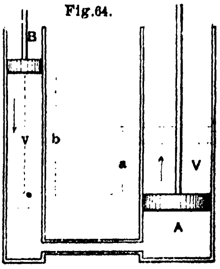


Fig. 64.

Suppose two cylinders, each fitted with a piston (*A* and *B* in

Fig. 64), to be connected by a pipe, the space intervening between the pistons being filled with fluid. Then when the piston B moves downwards with velocity v, the piston A will rise with velocity V, which is easily found by considering the spaces traversed by the two pistons in a given time. Let A, B be the areas of the pistons, a, b the spaces traversed, then, since the volume of the fluid remains the same, we must have $Aa = Bb$, and therefore,

$$\frac{V}{v} = \frac{a}{b} = \frac{B}{A}.$$

The chain here considered, in which the elements of two sliding pairs are connected by a fluid, is kinematically identical with the arrangement of Fig. 60a, p. 128, the replacement of a tension-element by a pressure-element constituting merely a constructive difference between the mechanisms. In the hydraulic press, in pumps, in water-pressure engines driven from an accumulator, and in other cases this kinematic chain is of constant occurrence, and will be frequently referred to hereafter. Combinations of an hydraulic chain with blocks and tackle are common in hydraulic machinery. Some examples will be found in Chapter XX.

Springs, compressible fluids, and even living agents, are employed in mechanism, not only in a manner to be explained hereafter as a source of energy, by means of which the machine does work, but also in force-closure, and especially for the purpose of supplying the force necessary to shift pieces which open and close, or lock and unlock kinematic chains, and so produce changes in the laws of motion of the mechanism. The force of gravity, which, as has already been shown, frequently produces closure, should be regarded as the tension of a link of indefinite length connecting the frame-link of the mechanism with the link we are considering. The inertia of moving parts likewise gives rise to forces which are not unfrequently applied to similar purposes. Examples will be given in a later section.

EXAMPLES.

1. A shaft making 90 revolutions per minute carries a driving pulley 3 feet in diameter, communicating motion by means of a belt to a parallel shaft, 6 feet off, carrying a pulley 13 inches diameter. Find the speed of belt and its length 1st, when crossed, and 2nd, when open. Find also the revolutions of the driven shaft, allowing a slip of two per cent.

Speed of belt = 847·8 feet per minute.
Length when crossed = 19 feet 2 inches.
,, open = 18 ,, 8 ,,
Revolutions of the follower = 244¼.

2. Construct a pair of speed pulleys to give two extreme velocity-ratios of 7 to 1 and

3 to 1, and two intermediate values. The belt is to be crossed and the least admissible diameter is 5 inches.

Velocity-ratios	$\frac{21}{3}$	$\frac{17}{3}$	$\frac{13}{3}$	$\frac{9}{3}$.
Diameter of pulleys	5 35	6 34	7½ 32½	10 30.

3. The diameters of the compound sheave of a differential pulley block are 8 inches and 7 inches respectively; compare the velocities of hauling and lifting.

Velocity-ratio = 16 to 1.

4. In a pair of ordinary three-sheaved blocks compare the velocity of each part of the rope with the velocity of lifting.

5. In a hydraulic press the diameter of the pump plunger is 2 inches and that of the ram 12 inches, determine the velocity-ratio. *Ans.* 36.

SECTION II.- WHEELS IN GENERAL.

63. *Higher Pairing of Rigid Elements.*—We next consider pairs of rigid elements in which the relative motion is not consistent with continuous contact over an area. The elements then touch each other at a point or along a line which is not fixed in either surface, but continually shifts its position. The form of the surfaces is not then limited as in lower pairing, but may be infinitely varied, with a corresponding variety in the motion produced.

This kind of pairing occurs when a chain of two lower pairs is completed by simple contact between their elements. In the double slider-crank chain shown in Fig. 4, Plate II., of the last chapter, let us omit the block *C* and enlarge the crank-pin so as just to fill the slot. By so doing the relative motions of the remaining parts will be unaltered, but we shall have three pairs instead of four, the turning pair *BC* and sliding pair *CD* being replaced by a single higher pair *BD*. This process is called *Reduction* of the chain, and when higher pairing is admissible the reduced chain serves the same purpose as the original, but with fewer pieces. The crank-pin and slot are in contact along a line only which during the motion continually shifts its position. In practice, the elements not being perfectly rigid, the contact extends over an area, but this area is of very small breadth, and consequently, if heavy pressures are to be transmitted at high velocities, the wear is excessive. If we trace the development of pieces of mechanism we observe that in the earlier stages higher pairing is much employed for the sake of simplicity of construction, but is gradually replaced by lower pairing. Nevertheless, where the object of the machine is mainly to transmit and convert motion rather than to do work, or where the velocity of rubbing is low, higher pairing may be employed In many cases it is necessary, because the required motion cannot be produced by any simple combination of lower pairs.

Higher pairing of rigid elements may be divided into two classes according as the surfaces in contact do or do not slip over one another, just as in the case of tension elements considered in the last section. In the first case the contact is spoken of as Sliding Contact and in the second as Rolling Contact. In rolling contact the difficulty of wear does not occur, and friction is greatly reduced, so that it is always used when possible. *When a roller rests on a hard plane surface the points in contact lie on a line which, if there be no slipping, remains for an instant at rest as the roller moves.* On reference to Art. 55, page 118, it will be seen that the motion of the roller is completely represented by a turning about this instantaneous axis, the point K (Fig. 56a) being in this case on the periphery of the wheel of which R is now the radius. The same is true when one circle B rolls within or without another fixed circle A, a case to be considered further on: the motion at B at the instant is a simple rotation about the point of contact. We first however consider the simple and important case in which both surfaces move, the line of contact being fixed.

64. *Rolling Contact.* Rolling contact may be employed for the communication of motion between two shafts, the centre lines of which are either parallel or intersect, by means of surfaces rigidly attached to the shafts. In the first case the surfaces are cylindrical and in the second conical, the apex of the cone being the intersection of the shafts. By far the most important case, and the only one we shall

Fig.64a.

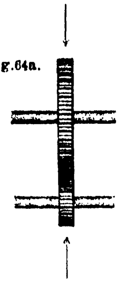
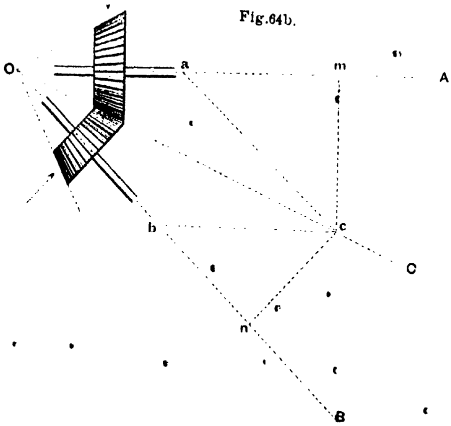


Fig.64b.



here consider, is that in which the transverse sections of the surfaces, are circular. Portions of the surfaces are used, as in Figs. 64a, 64b, and are pressed together by external forces, so that sufficient friction is produced to prevent the slipping of the surfaces. In other words, force-closure is necessary, as in the case of connection by a belt. *This being supposed, it will immediately follow that the velocity of the two surfaces at the points of contact is the same*, and hence, as before, the angular velocity-ratio of the shafts is inversely proportional to the radii of the wheels. In the case of intersecting shafts, the surfaces are frustra of cones called "bevel," or, if the semi-angle of the cone be 45°, "mitre-wheels," and their radii may be reckoned as the mean of that at the inner and outer periphery. The shafts revolve in opposite directions, unless one of the surfaces be hollow so that the other may be inside it, in which case the corresponding wheel is said to be "annular." When it is inconvenient to use an annular wheel, the same result may be obtained by transmitting the motion through an intermediate or "idle" wheel. If the radius of a wheel be infinite, it becomes a "rack," and the surface a plane.

In the case of bevel wheels the corresponding cones may be found, when the centre lines of the shafts and the angular velocity-ratio are given, by a simple construction. In Fig. 64b, let OA, OB be the centre lines of the shafts, and let distances Oa, Ob be marked off upon them in the ratio of the required angular velocities. Complete the parallelogram $Oacb$, then OC must be the line of contact of the required cones. For drop perpendiculars cm, cn, on OA, OB, then

$$\frac{cm}{cn} = \frac{\sin aOc}{\sin bOc} = \frac{Ob}{Oa},$$

so that the radii of any frustra of the cones employed for wheels will be inversely as the angular velocities of the shafts.

The particular case may be mentioned in which one of the cones becomes a plane; the corresponding wheel is then a "crown" or "face" wheel. The shaft of a wheel which is to work correctly with a crown wheel must be inclined to the plane of that wheel at an angle depending on the angular velocity-ratio required, a restriction not generally attended to, especially in the earlier stages of machinery in which face wheels were of common occurrence.

If, as generally happens, it is required to transmit a working force of a considerable amount, then the friction between the two circumferences will be found not to be sufficient to prevent slipping taking place, unless a considerable pressure to force the shafts together is employed, which involves an excessive friction on the bearings. In what is known as "frictional gearing," this is partially avoided by the use of wheels with triangular grooves fitting each other as the thread of a screw fits into its nut; but, in general, to prevent slipping, teeth are cut on the two peripheries, and the motion is transmitted by the gearing together of the teeth. Since this is a substitution for the rolling contact of two surfaces, it is required to so design the number and form of the teeth that the wheels on which they are cut shall turn one another with the same constant angular velocity-ratio as that due to the two original surfaces. If recesses are cut in each wheel, and projections be added between the recesses so as to fit into the corresponding recesses of the other wheel, then the two wheels may be placed to gear together at such a distance that the two original surfaces would have been in contact and would have rolled together. In the case of a pair of toothed wheels, such a pair of imaginary surfaces which will roll together with the same angular velocity-ratio as that obtained from the toothed wheels, are called *pitch surfaces.* Considering first the case of parallel shafts, the transverse sections of these surfaces are called *pitch circles*, and their point of contact is called the *pitch point.* The radii of these pitch circles must be to one another in the inverse of the velocity-ratio. The circumference of each circle is to be divided into a number of equal parts, which will include a tooth and a recess. The length of each part measured along the pitch circle is called the *pitch*. Let p = *pitch*, and n = number of teeth, d = diameter, then

$$p = \frac{\pi d}{n}.$$

The thickness of each tooth is made a little less than $\frac{1}{2}p$ to allow the clearance necessary for easy working. The magnitude of the pitch which governs the thickness of the teeth must be determined from considerations as to their strength. If n' = number of teeth in the second wheel, and d' = its diameter, then the pitch being the same for each wheel

$$p = \frac{\pi d}{n} = \frac{\pi d'}{n'}.$$

The distance apart of the shafts is generally adjusted to allow the pitch to be some exact number of inches, half, or quarter inches. The pitch is to be measured along the pitch circle, and is not the chord of the arc, as is sometimes stated.

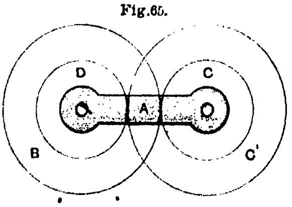
In some small wheels used for spinning machinery, another kind of *pitch* is referred to. The diameter of the pitch circle is divided by the number of teeth, and the result is called the *diametral pitch.* In the smallest class of wheelwork used in clocks, the dimensions of the teeth are stated as so many to the inch. The proper form of teeth will be considered further on.

65. *Augmentation of a Kinematic Chain. Trains of Wheels.*—Another important application of rolling contact is to diminish friction by the intervention of rollers, hence called Friction Rollers. Thus the friction between the elements of a sliding pair, subject to heavy pressure, will be so great as to require a great force to overcome it, but if rollers be placed between the elements the friction is greatly reduced, as will be seen hereafter. In this case sliding friction is wholly replaced by rolling friction; in carriage wheels the sliding velocity which, without the wheel, would be the actual velocity of the carriage, is reduced to that at the periphery of the axle, that is to say, in the ratio of the diameters of the axle and the wheel. The sheaves of an ordinary pulley block are examples of the same principle. In all these cases where additional pieces are added to a kinematic chain, in order to reduce friction or to serve some other non-kinematical purpose, the chain is said to be "augmented."

Chains are frequently augmented for purely constructive reasons; thus, if the velocity-ratio of a pair of shafts is great, the diameters of a single pair of wheels necessary in order to obtain it will be inconveniently large or small. A train of wheels is then resorted to. This is also the case where the shafts to be connected are too near or too far apart; in the latter case bevel wheels and an intermediate transverse shaft may be employed.

When, however, the shafts to be connected are in the same straight line, a train of wheels is kinematically necessary, and forms virtually a new mechanism. This is a common case in practice when a pulley or wheel is loose on a shaft, and it is required to connect the wheel and the shaft so as to revolve with different velocities. Such a train is shown in Fig. 65 in a simple ideal form. B and D are two wheels turning on the same centre but disconnected. C, C' are two wheels gearing with B and D and turning about another centre but united. The two centres are connected by the frame-link A. When B revolves it drives C, and C' drives D. If the numbers of teeth in these wheels be denoted by the letters which distinguish them, and the velocity of B be unity, the velocity of C or C' will be B/C, and that of D will be BC'/DC. Let it now be observed that the wheels B and D form a pair,

Fig. 65.



the velocity of which will be the difference between the velocities of these wheels. We have then altogether four turning pairs in this train of wheels, the relative velocities of which are—

Pair,	*BA*	*CA*	*DA*	*DB*
Velocity,	Unity	$-\frac{B}{C}$	$\frac{BC'}{C'D}$	$1+\frac{BC'}{CD}$

One of the wheels in this train may be annular, and all may be bevel; in either case the wheels *C*, *C'* may be equal, and the train reduced to three wheels, though the number of simple pairs remains as before four. Examples are given in the figures of Plate III.

Either this or any other train of wheels may be inverted by fixing one of the wheels instead of the frame-link, the resulting mechanism is then called an Epicyclic Train; the velocity-ratios of the various pairs are unaltered, and are therefore shown by a table similar to that given above. Should the angular velocity of any wheel be required relatively to the fixed wheel, we have only to add to the velocity of the corresponding pair the velocity of the frame-link. Some examples of epicyclic trains are shown in the figures, but for detailed descriptions we must refer to a work on mechanism. Their use in compound chains will be further referred to in the next chapter.

66. *Wheel Chains involving Screw Pairs.*—In a simple wheel chain (Fig. 66) consisting of a wheel *B*, a pinion *C*, and a frame-link *A*, not shown on the figure, suppose *C* to be of considerable length, then there will be nothing to prevent the endways movement of *B* in its bearings if they be supposed cylindrical. This circumstance is often taken advantage of in machinery in shifting wheels in and out of gear, but the case to be examined here is that in which the endways movement is given by independent means during the action of the mechanism. The simplest example is a three-link chain derived from the train of wheels just considered by changing the turning pair *BA* into a screw pair; *B* then travels endways through the pitch of the screw in each revolution. The pinion *C* sometimes slides on the shaft which carries it, but quite as often it is made long enough to permit the necessary traverse of *B*. A well-known example of this mechanism is that of the feed motion

Fig. 66.

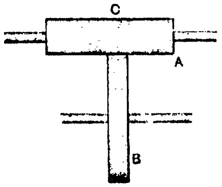


Plate. III.

FIG. 1.

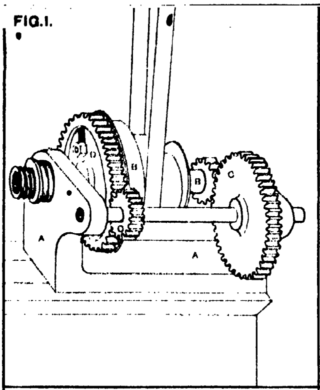


FIG. 2.

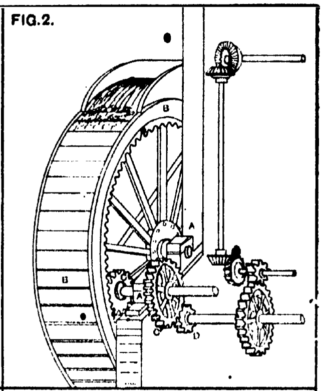


FIG. 3

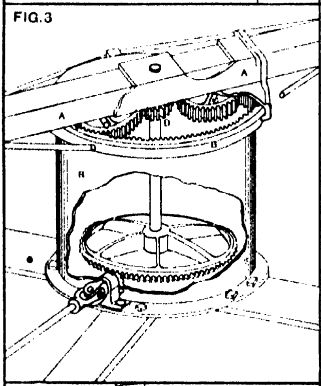


FIG. 4.

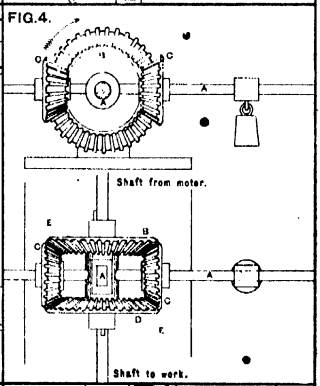


FIG. 5.

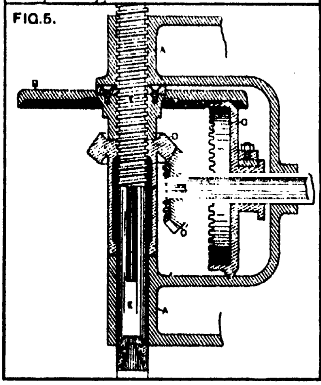
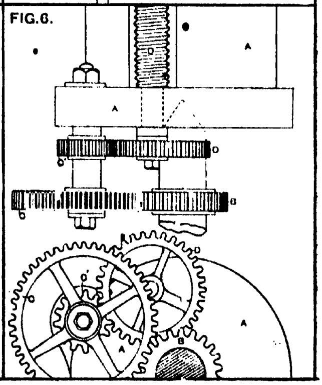


FIG. 6.



To face page 141.

common in drilling and boring machines, in which the train of wheels of the last article is used with B and D nearly equal, so that the velocity of the pair BD is very small. B is attached to the nut and D to the screw, so that BD is a screw pair. D then traverses through B by a space each revolution which may be made very small.

To illustrate and explain preceding articles Plate III. has been drawn, giving examples of trains of wheels, especially of the differential trains of Fig. 65.

Fig. 1 shows the slow motion of a lathe. D is a wheel keyed on the mandrel and connected with B, the driving-pulley, when the motion is not in use. B rides loose on the mandrel, and by means of a pinion gears with C, a wheel on the same shaft with C', which gears with D. $C'D$ being large compared with BC', the speed of the mandrel is much less than that of the pulley. For lighter work CC' are thrown out of gear by an endways movement of the shaft.

Fig. 2 represents the train of wheels by which the slow movement of a water-wheel is multiplied and transmitted to all parts of a factory. B is now an annular wheel attached to the water-wheel gearing with C, C' with D, and so on. A vertical shaft F with bevel wheels transmits the motion to the upper floors. The bearings of the secondary shafting are omitted for clearness, but they all form part of a frame-link A, which is fixed.

In Fig. 3 the kinematic chain is inverted. B is a fixed annular wheel, CC' are of equal diameter and reduce to one wheel, which, however, is in duplicate, in order to balance the driving forces. This epicyclic train is applied to many purposes. In the example shown the frame-link is a long arm, at the end of which a horse is attached, and a rapid motion thus given to the central pinion D. The motion is further multiplied by the bevel gear shown below, and applied to drive a thrashing machine or some similar purpose. The same mechanism is employed as a purchase in capstans and tricycles.

In Fig. 4 the train consists of three bevel wheels BCD, C' and C'' reducing to one, as in the preceding case. The simple chain consists of these wheels and the train arm A. When A is fixed the wheels B and D turn in opposite directions with equal velocities; when B is fixed A revolves with half the velocity of D. The mechanism is much employed, but usually as a compound chain, and as such will be considered in the next chapter. The example shown is a dynamometer.

Fig. 5 represents the feed motion of a drilling machine. A is the frame of the machine in which rotates the vertical drill spindle E driven by a pair of mitre wheels D and C' from a horizontal shaft. A screw thread is cut on the spindle, of which B forms the nut. If B and D rotate at the same speed the drill moves neither up nor down, but any difference will result in a motion of the screw pair BE, and will thus give the necessary feed or raise the drill out of the hole. In the example chosen B is driven by a flat disc gearing by friction with a wheel C'' turning with D (Naish's patent). This wheel, by means of a lever, can be moved along the shaft so as to gear with B at any radius at pleasure, and can therefore be set so as to raise or lower the drill at any required speed. The contact between C, D here is not pure rolling (p. 136); but as C is of small breadth the error is not of practical importance.

In Fig. 6 the same kinematic chain is employed as an epicyclic train to give motion to the cutters of large boring machines. The cylinder to be bored is fixed, and the boring bar rotates on the lathe centres. The wheel B is fixed; D is attached to the end of a long screw, which on turning causes a nut E (not shown in the figure) to traverse slowly, carrying with it the cutting tools. The train arm A rotates and carries on it the wheels C, C', and D.

EXAMPLES.

1. The diameter of pitch circle of a wheel is 4 feet, and the number of teeth 120 Find the pitch.

Pitch = 1¼ inches.

2. Two shafts about 4 feet apart are to be connected by spur wheels, the velocity-ratio being 4 to 1. Find the diameters of the wheels and also the number of teeth, assuming the pitch to be 2 inches.

Ans. The number of teeth in wheels are 30 and 120, and the exact distance apart of the shafts = 47¾ inches.

3. The diameter of the pitch circle of the annular wheel by means of which a water wheel communicates motion to a mill, is to be as nearly as possible 24 feet. The pitch is to be 4 inches. Find the diameter and the number of teeth in the wheel. The velocity of the periphery is to be 5¼ feet per second and the first motion shaft is to make 30 revolutions per minute. Find the necessary diameter of pinion and the number of teeth in it.

Ans. The number of teeth in the annular wheel = 226, and its exact diameter is ¼ inch less than 24 feet. The number of teeth in the pinion is 32, making the revolutions per minute somewhat less than 30. The diameter of pinion = 40¾ inches.

4. A pair of shafts, the centre lines of which intersect at an angle of 60°, are to be connected by bevel wheels so as to revolve, the one at 250 and the other at 90 revolutions per minute. Find the pitch surfaces.

Angles of cones 90° and 30°.

5. Two shafts intersecting at an angle of 75° are connected by a crown wheel gearing with a pinion. What must be the velocity-ratio?

6. The weight of a revolving turret rests on a ring of friction rollers, the axis of rotation of which radiate horizontally from the axis of the turret; find the angle at which the rolling surfaces must be bevelled. Compare the rates of rotation of the ring and the turret.

7. The feed motion of a boring machine consists of a nut working on a screw cut on the spindle of the drill or borer which is raised or lowered whilst the nut turns on it. The nut carries a wheel of 96 teeth which gears with one of 35. When the drill is at work the wheel of 35 teeth is secured to one of 36 on the same axis, and this latter gears with one of 95 teeth secured to the spindle of the drill. The screw has four threads to the inch. Determine the depth of hole bored per revolution.

$$\text{Depth of hole bored per revolution} = \tfrac{1}{4}\text{ inch}\left(1 - \frac{35 \times 95}{36 \times 96}\right) = \cdot0095 \text{ inch.}$$

8. The train of wheels in the preceding question is used as an epicyclic train by fixing the wheel of 96 teeth. Find the direction and number of revolutions of the train arm for each revolution of the spindle.

Ans. For each revolution of 95 wheel forwards, the arm turns backwards through

$$\frac{95 \times 35}{96 \times 36 - 95 \times 35} = 25\cdot4 \text{ revolutions.}$$

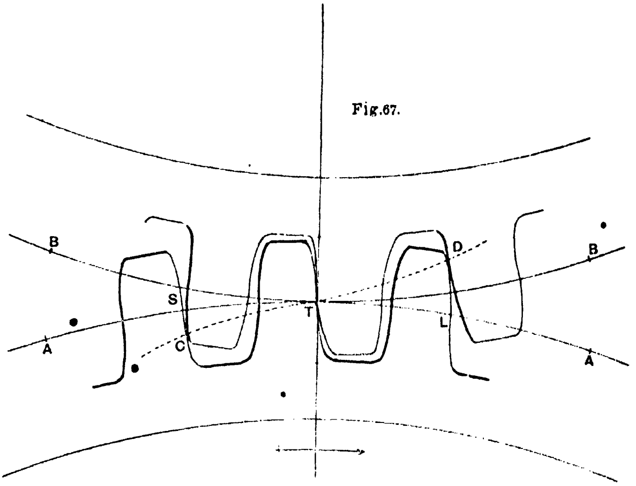
SECTION III.—TEETH OF WHEELS.

67. *Preliminary Explanations.*—Even though the number of teeth in a pair of wheels be such as to give the correct mean angular velocity-ratio due to the rolling together of the pitch circles, yet if they be of improper form they will jam or work roughly.

Theoretically the form of the teeth of one of a pair of wheels may be chosen at pleasure if a proper corresponding form be given to the teeth of the other; the problem of rightly determining the form is therefore one which admits of many solutions. We commence with some general explanations applicable to all forms of teeth.

The diagram (Fig. 67) shows a section of a pair of spur wheels in gear, with three teeth in action, the lower wheel being the driver. *BTB*, *ATA* are the pitch circles in contact at the pitch point *T*. $ST = TL$ is the pitch, being the distance of a point in one tooth from the corresponding point in the next consecutive measured along the pitch circle. The teeth as shown in the figure partly project beyond the pitch circle and fit into corresponding recesses in the other wheel, so that each tooth is divided into two parts, a part within and a part without the pitch circle. The corresponding acting surfaces are called the Flank and the Face of the tooth respectively. In annular wheels the flank is outside and the face inside the pitch circle. The teeth commence action before reaching the line of centres by the flank of a tooth of the driver *A* coming into contact with the face of a tooth of the follower *B*, as shown at *C* in the diagram, and gradually approach that line till after the wheels have turned through a certain arc, which measured on the pitch circle is called the Arc of Approach; they are then in contact at *T* the pitch point. After passing the line of centres they remain in contact till the wheels have turned through a second arc, called the Arc of Recess, and then cease contact as shown at *D*, the face of a tooth of the driver being always in contact with the flank of a tooth of the follower. The sum of these arcs is called the Arc of

Fig. 67.



Action, and must be great enough to permit at least two teeth to be in contact at once. Their magnitudes depend on the projection of the teeth beyond the pitch circle, a quantity which is called the addendum of the corresponding wheel, the arc of approach depending on the addendum of the follower, and the arc of recess of the addendum of the driver.

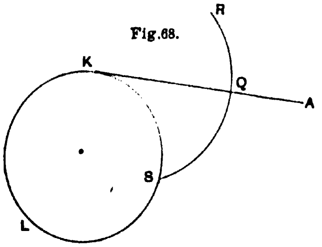


Fig. 68.

68. *Involute Teeth.*—The question of the form of the teeth requires much explanation to render it completely intelligible; we shall only give a brief sketch, referring for full details to the works cited on page 92. Some points will be further considered at a later period. We commence with what are known as Involute Teeth.

Imagine a string *AKL* wound on a cylinder (Fig. 68). If the string be gradually unwound, the string being kept tight all the time, a point *Q* of the string will trace out a curve *SQR* called the *Involute of the Circle*. Instead of causing the string to be unwound around the fixed circle we may if we please move *A* in a fixed straight line and cause the unwinding to take place by the revolution of the circle. If now a piece of paper be fixed to and revolve with the circle, the same involute curve will be traced on it as before.

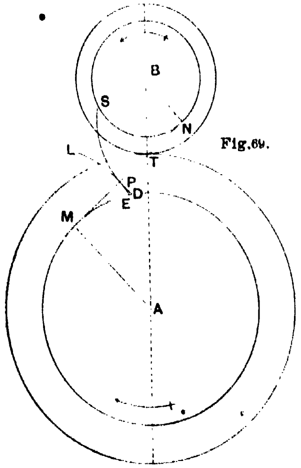


Fig. 69.

Now let *A* and *B* (Fig. 69) be two circles not in contact which are each capable of revolution about its centre. If we connect them by a crossed belt, of which one half is shown in the diagram by the line *MTN*, each will be capable of driving the other with a constant angular velocity-ratio, namely, the inverse ratio of the radii. If, therefore, *T* be the point where the belt crosses the line of centres,

$$\frac{A_A}{A_B} = \frac{r_B}{r_A} = \frac{BT}{TA}$$

Now, with centres *A* and *B* and radii *AT* and *BT*, describe circles which touch one another. These two circles would turn one another by rolling contact with the same angular velocity-ratio as that due to the belt. If we were to form teeth on the two wheels and cause them to turn one another by the gearing of the teeth, then the two circles passing through *T* may be regarded as the pitch circles of the two wheels.

Now to trace the form of the teeth. Attach a pencil (*P*) to any point of the belt, and fix a piece of paper to the wheel *A* so that it may turn with it, then the pencil will trace on the paper the curve *EPL*, being an involute of the circle *A*. Similarly, if we imagine a piece of paper attached to *B*, an involute *DPS* of the circle *B* will be traced on that. These two curves will be in contact at the tracing point *P*, and will always remain in contact as the circles turn. If, therefore, we construct teeth of this form with any given pitch, and then remove the belt, the two toothed wheels will drive one another with the constant angular velocity required. In this form of tooth the face and flank are one continuous curve, which is a property practically confined to involute teeth. From this fact a practical advantage follows. By the continual action of the teeth together they wear and cause a looseness of fit, which may be remedied by bringing the centres of the wheels more nearly together, and this without altering the smooth action of the teeth or the exact uniformity of the angular velocity-ratio. In no other form of tooth occurring in practice is this possible.

The line of action of the mutual pressure between the teeth is always along the tangent line to the two base circles from which the teeth are generated, thus tending always to force the axles apart. If the angle between this line and the common tangent to the two pitch circles, or, as it is called, the "obliquity," be large, much friction in the bearings would result. On this account the obliquity is made as small as possible, not being allowed to exceed 14½° or 15°. With this a limit is introduced to the smallness of the number of teeth which may be used. The action of the teeth must always be along the line *MTN*, and hence cannot extend beyond the point *N*. If it is essential that when two teeth are in contact at the pitch point another pair of teeth should just be coming into action whilst a third pair are just ceasing action, then the length of the arc of the pitch circle which corresponds to an arc on the base circle equal to *TN* will be the greatest length that can be given to the pitch of the teeth, and when the obliquity is 14½° there will be about twenty-five such pitches on the pitch circle, and hence the number of teeth cannot be less than twenty-five.

Having given the pitch circles we first lay off, through the pitch point, the line of oblique action which is to be allowed, and then draw the base circles touching this line. The involutes of the base circles will give us the form of the teeth. The thickness of the tooth is to be taken a little less than half the pitch, and the addenda of the teeth such as to give a sufficient number of teeth in contact at the same time. (Art. 71.)

All involute teeth of the same pitch and obliquity will work together; they have never been much used in practice, although there appears to be no reason why they should not be in cases where it is not necessary to have less than twenty-five teeth. Their wear is said to be greater than that of teeth of other kinds.

69. *Path of Contact the Pitch Circle.*—In involute teeth the tracing point is attached to a belt stretched over pulleys, and therefore describes a straight line on paper, which is fixed to the line of centres so as not to revolve with either wheel. Now, the tracing point is also the point of contact of the two teeth, and therefore the path of this point, or, as it is conveniently called, the "path of contact," is a straight line. Teeth of any shape may be traced by this method if, instead of simply stretching the belt over the pulleys, we pass it over a fixed *curve* between the pulleys, so that the tracing point describes the curve in question instead of a straight line, provided the fixed curve be such that the curves traced on the rotating circles touch one another. In other words, we may assume various "paths of contact" at pleasure and obtain teeth which will work together correctly. We shall next suppose the tracing point attached to the circumference of a rotating wheel, in which case the path of contact is a circle.

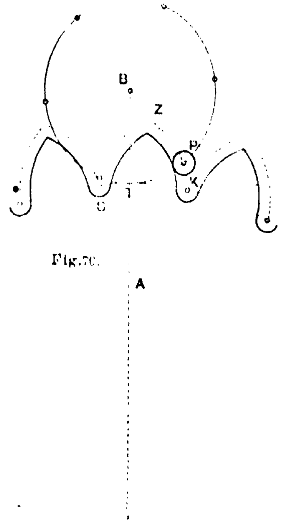


Fig. 70.

In the use of toothed wheels the earliest idea was, for simplicity of construction, to form the smallest wheel of a number of cylindrical pins projecting from a disc. Supposing one of a pair of wheels to be so constructed, it is required to determine the proper form of the teeth for the other wheel.

On the wheel *B* (Fig. 70) let pins be placed at equal distances, with their centres on the pitch circle, and in the first place suppose the pins indefinitely small, being mere points. Now, if at one of the points *P* a pencil be attached, then if *B* be caused to roll without slipping over the surface of *A* kept fixed, the pencil *P* will trace a curve on a piece of paper attached to the wheel *A*. The same curve will be drawn if we cause one wheel to drive the other without slipping, the centres *A* and *B* being fixed, while the paper is attached to *A* and turns with it. If the tracing point started from the pitch point *T*, then the curve *KP* will have been drawn on the paper, which, by the further rotation of the circles, will be produced to *Z*. This curve is called an Epicycloid, and will be the proper form of teeth for the wheel *A* to drive the pinion *B*. For the pin *P* will be always in contact with the tooth *KZ* as the wheels revolve with uniform angular velocity-ratio. We complete the form of the teeth by drawing a similar curve *ZS* for the other face, *SK* being the pitch, in order to enable the wheels to be turned in the opposite direction if necessary. Placing a number of such teeth on the pitch circle *A*, we see they all touch one another at the roots on the pitch circle. The reason is because we have imagined the pins of *B* to have no definite dimensions, but to be mere mathematical points. In practice some definite dimensions must be given to the pins of *B*. In such a case the proper form for the teeth of *A* is derived from the previous construction by drawing a curve which at all points shall be at a distance from the epicycloid, when measured along the normal, equal to the radius of the pin. Below the pitch circle *A* a semi circular recess must be formed, as shown by the full curve in figure.

These teeth possess the peculiar property of having faces but no flanks. The consequence is that, the toothed wheel *A* being the driver, the action of the teeth is wholly after the line of centres; there is no arc of approach, but only an arc of recess. On this account the pin-wheel must always be the follower, for if it be the driver the action of the teeth would be wholly before the line of centres, in consequence of which the friction is said to be more injurious.

The angle which *PT* makes with the common tangent is, as in the case of involute teeth, called the "obliquity"; it is now no longer constant, but varies from zero, when *P* passes the line of centres at *T*, to a maximum value when *P* escapes. It is easily seen that this angle is always one-half the anlge *PBT*, which *PT* subtends at the centre of the pin-wheel, and hence the obliquity increases uniformly as the wheels turn; its mean value may be taken at half the maximum, and is limited in the same way as in involute teeth to about 15°, so that the greatest value of the angle *PBT* may be taken at 60°.

If the two sides of the teeth are alike, as in the figure, the pin then comes to the point of the tooth at Z. This circumstance determines the smallest number of pins which can be used, for one pin must not escape before the next comes to the line of centres; that is to say, PT cannot be greater than the pitch, the pitch then must not be greater than one-sixth the circumference of the pin-wheel, whence it appears that the least number of pins is six.

Pins are now rarely employed except in clock and watch work; they have the great practical disadvantage that the toothed wheel to work with them must be specially designed, as it will work with only one diameter of pinion.

If we imagine a pin-wheel to work with an annular wheel, the teeth may be traced in the same manner as shown in Fig. 71 (p. 149), to which the same letters are attached. The point P now traces out a curve called a Hypocycloid, the general character of which may be seen by joining P to F, the other extremity of the diameter TF of the circle B; for since the angle FPT must be a right angle, the angle APT will be greater than a right angle if, as in the figure, F lies between A and T, and less than a right angle if F lies beyond A. Thus the hypocycloid must reduce to the radius AK if F coincides with A, that is, if the diameter of the pin-wheel be half the diameter of the annular wheel; while, for smaller diameters, it forms a curve always concave towards T. Hence it appears that to work with a pin-wheel of half its diameter the teeth of the annular wheel should be constructed simply by drawing radii of the pitch circle. With a larger diameter of pin-wheel the teeth would be undercut, and therefore weak; the annular wheel must be the driver as before.

In all epicycloids and hypocycloids the normal to the curve at the tracing point P passes through the point of contact T of the circles considered--for, as already shown (Art. 63, p. 136), the motion of the rolling circle is for the instant a rotation about T.

70. *Path of Contact any Circle.*—Teeth traced in the way just described are wholly within the pitch circle, and this circumstance suggests that by a combination with the preceding case, where they were wholly without, a form may be found which may be more suitable for practical use.

In Fig. 71 a third circle C is shown, touching the two others at the same pitch point T. The three circles ABC turn each about its own centre without slipping. Imagine paper attached to A and C and rotating with them, while a pencil P is attached to B as before; then P will trace out two curves as in the case of involute teeth, one outside the circle C, the other inside the circle A. A's curve will be an hypocycloid *KPZ*, starting from *K* in the circle *A*, while *C*'s curve is an epicycloid *K'PZ'* starting from *K'* in the circle *C'*. Now these curves will, as in involute teeth, touch one another, having a common normal *PT*, and hence it follows that, while the circles turn with uniform annular velocity ratio, the curves will always be in contact, and may be taken as face and flank of a pair of teeth. Thus it appears that we can obtain the faces of the teeth of *C*, and the flanks of the teeth of *A*, by causing a third circle *B* of any diameter to rotate within the circle *A*. If the diameter of *B* be half the diameter of *A*, the flanks for *A* will be simply radial lines, but if it be less they will be concave towards *T*, the effect of which is that the teeth will spread out at the root, which is desirable on the score of strength. We can now imagine the faces of the teeth of *A* and the flanks of the teeth of *C* to be traced by another circle *B'* rotating within *C* instead of within *A*. The diameter of this circle need not be the same as that

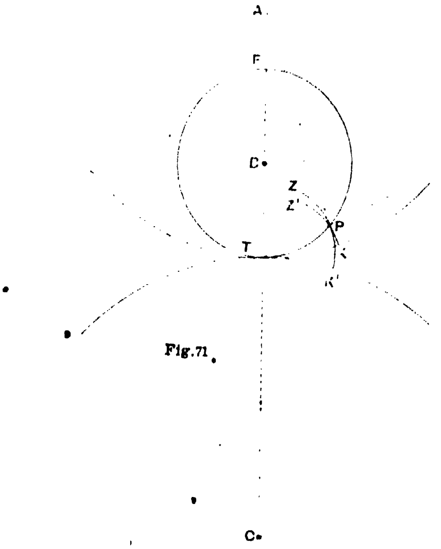


Fig. 71

of *B*; it may, for example, be half the diameter of *A*, while *C*'s diameter is half that of *B*; if so, the flanks of the teeth of both wheels will be radial. Teeth with radial flanks have the disadvantage of weakness, especially when the number of teeth is small, because the thickness at the root is less than that at the pitch circle, and they are, besides only capable of working correctly with wheels specially designed fo them. In order that a set of wheels of this kind may be interchangeable, it is necessary that the circles $B'B$ be of equal diameter and the same for all the set. This diameter should not be larger than half that of the smallest wheel of the set, for, if it is, the flanks of the teeth of the small wheels will be undercut and consequently weak, while, on the other hand, it should be as large as possible, for otherwise the teeth of the large wheels will be too thick at the roots and too thin at the points, a form which is found to be unfavourable to good wearing. Hence the diameter chosen for B is half that of the smallest wheel of the set, the flanks of which will be radial. As B is a pin-wheel, its smallest circumference is six times the pitch (Art. 69), and the smallest wheel of the set has consequently 12 teeth ; but if no wheel is required with so small a number of teeth as this, it will be better, for the reason stated above, to take a large describing circle.

71. *Addendum and Clearance of Teeth.*—In any form of teeth it is clear from what has been said that the point of contact travels along the path of contact DT (Fig. 67, page 143) from the pitch point T to the end of the tooth at D, where the contact ceases. The length of the path of contact thus traversed is equal to the arc of recess in all kinds of cycloidal teeth, and less than that arc in a given ratio in involute teeth. By stepping off a suitable length on the path of contact then, we can find the end of the tooth for any given arc of recess, and the distance of this point from the pitch circle A of the driver is what we have already defined as the "addendum" of that wheel. The position of this point on the flank of the tooth of the follower B gives the working length of flank necessary. Similarly the length of face in the follower and flank in the driver depend on the arc of approach. The depth of the recesses between the teeth, however, must be made greater than is necessary for working length of flank, in order to allow the ends of the teeth to clear; the amount usual in practice appears to be about one-fifteenth the pitch.

The allowance necessary in practice for clearance in the thickness of the teeth depends on the degree of accuracy attainable in construction. The value formerly employed for teeth shaped by hand was one-eleventh the pitch, but the best modern teeth are machine cut, and a much smaller amount is sufficient. Less clearance is required for involute teeth than in teeth of other kinds. The setting out of bevel teeth is not theoretically more difficult than in the case of spur gear, but their accurate execution by a machine is far from easy. If the machine operate by straight cuts like an ordinary shaping machine, the tool must be mounted so that the line of cut always passes through the apex of the pitch cone. Gear cutting machines generally employ revolving cutters formed to fit the space between two teeth. Much ingenuity has been expended on giving the cutter a lateral movement to suit the bevel, but an exact bevel tooth cannot be formed in this way.

72. *Endless Screw and Worm Wheel.*—When two shafts are to be connected which are not parallel, and the centre lines of which do not intersect, it is necessary to resort to skew bevel, or screw teeth. Only one case of this kind need be mentioned here as being of common occurrence, namely, the endless screw and worm wheel employed when the shafts are at right angles, and a slow motion of one of them is desired. In a common screw let the thread be so formed that the longitudinal section of the screw thread shows a range of teeth like those of a rack which would gear with a given spur wheel. Let the teeth of the wheel be set obliquely at an angle equal to the pitch angle of the screw; strictly speaking, they also are screw threads, the pitch angle of which is the complement of the pitch angle of the screw. Then the screw and wheel will gear together, and the wheel moves through one tooth for each revolution of the screw. Like screws in general, this combination is non-reversible unless the pitch of the screw be coarse (Ch. X.), and for this reason, and on account of its simplicity, is much employed in practice. The method of constructing the teeth of a worm wheel is explained in a work by Prof. Unwin, cited on page 125.

EXAMPLES.

1. A pair of wheels have 25 and 120 involute teeth respectively, and the addendum of each is $\frac{3}{10}$ths the pitch. Find the arcs of approach and recess in terms of the pitch assuming the obliquity $14\frac{1}{2}^\circ$, the large wheel being the driver. (See Art. 71.)

Ans. Arc of approach = ·89 × pitch.
Arc of recess = 1·12 × pitch.

2. If the arcs of approach and recess in involute teeth are each to be equal to the pitch, show that the addenda of the wheels should be calculated by the approximate formula

$$\text{Addendum} = \left(\frac{1}{4} + \frac{3}{n}\right) \times \text{pitch},$$

where n is the number of teeth.

3. A pair of wheels have 25 and 120 teeth respectively, the flanks being in each case radial. Find the addendum of each wheel that the arcs of approach and recess may each be equal to the pitch.

Ans.—Addendum of driver = ·283 × pitch.
Addendum of follower = ·178 × pitch.

SECTION IV.—CAMS AND RATCHETS.

73. *Reduction of a Crank Chain by Omission of the Coupling Link.*—A pair of spur wheels in gear form a particular case of a three-link kinematic chain consisting of two lower pairs with parallel axes two, *elements of which are united and generally form the frame-link, while the other two pair by contact.*

Such a chain may be derived from the four-link crank chain of Art. 52, page 112, by omission of the coupling link, a process of reduction which has already been employed on page 135.

In Fig. 72a, *ab*, *dc* are levers turning about fixed centres and connected by a coupling link *bc*, all three links being in one plane as in the article referred to, Imagine how the crank-pins at *b* and *c* enlarged until they touch one another as shown by the dotted circles and then remove the coupling link. Suitable forces being applied to close the chain by keeping the surfaces in contact, the link *bc* may be removed without in any way altering the motion, and therefore the angular velocity-ratio will still be as before $aT : dT$, where T is now the intersection of the common normal at the point of contact, with the line of centres. Now the instantaneous motion of the levers cannot be affected by the shape of the pins except at the point of contact, and it therefore follows that if we replace the pins by any surfaces such as those indicated by the full lines in the figure, which have the same common normal at the point of contact, the result will be the same.

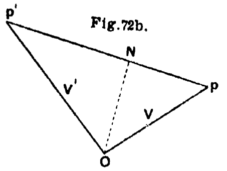


Fig. 72b.

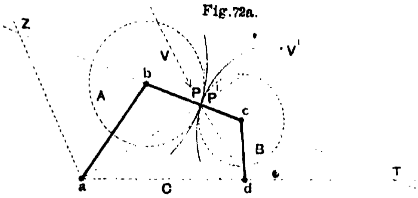


Fig. 72a.

We may reach this conclusion directly by constructing a diagram of velocities for the two pieces in question. For let P, P' be points in the profiles which at the instant considered coincide by becoming the point of contact. Then P's velocity in the direction of the normal must be the same as that of P', for otherwise the surfaces would interpenetrate or move out of contact. If then from a given point O (Fig. 72b) we draw *Op*, *Op′* parallel to the lines *aP*, *dP′*, to meet a parallel to the normal in pp', it follows by the same reasoning as in the case of linkwork that Opp' is a triangle of velocities of which the sides Op, Op' represent the velocities of P, P'. Hence drawing aZ parallel to dP' it appears as before that the angular velocity-ratio of the lines aP, dP' is dT/aT, and these lines are fixed in the rotating pieces so as to have the same velocity-ratio.

The third side pp' of the triangle of velocities represents in this case the velocity with which the surfaces rub against one another, for dropping the perpendicular ON the segments Np, Np' represent the resolved part of the velocities along the common tangent. Suppose A, A' to be the angular velocities of the pieces, V, V' the actual velocities of P, P', then by similar triangles

$$\frac{pp'}{PZ} = \frac{OP}{aP};$$

that is, if r be the velocity of rubbing,

$$\frac{r}{PZ} = \frac{V}{aP} = A,$$

for which we obtain

$$r = A \,.\, PZ = A\,(TZ - PT).$$

But it was shown above that

$$A \,.\, aT = A' \,.\, dT:$$

$$\therefore\ A \,.\, TZ = A' \,.\, PT:$$

hence

$$r = (A' - A)PT.$$

This formula supposes the pieces to turn in the same direction, as in the figure. If they turn in opposite directions, as in a pair of toothed wheels,

$$r = (A' + A)PT,$$

a simple and important result which we shall hereafter verify.

It follows at once that for rolling contact the point of contact must lie on the line of centres, and that for a constant angular velocity-ratio T must be a fixed point. Thus in all forms of teeth for wheels the common normal at the points of contact of the teeth must always pass through a fixed point on the line of centres, as we found to be the case in the examples already considered. The velocity with which the teeth slide over one another is given by the above formula.

The diagram of velocities may when necessary be completed by laying down on it the velocities of all points rigidly connected with either rotating piece as explained before in the case of linkwork.

74. *Cams with Continuous Action.*—In toothed wheels the revolution of one wheel is always accompanied by that of the other in the same or in opposite directions, according as the gearing is inside or outside, or, in other words, the directional relation is always the same. We now pass on to cases in which the directional relation varies, the continuous rotation of one piece being accompanied by an oscillating motion of the other. The rotating piece is then called a "Cam," or sometimes a "Wiper."

Cams are of two kinds. In the first the contact is continuous, and the oscillating motion produced is completely defined by the form of the cam; while, in the second, the contact is only during the forward vibration of the oscillating piece, while the backward vibration is produced by other causes. In both kinds force-closure is common, and sometimes indispensable.

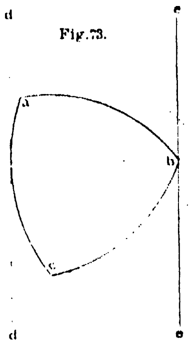
We shall now give some examples of cams of the first kind. Fig. 1, Plate IV. (p 159), represents a sliding piece *C*, to which a reciprocating movement is given by a cam *B*, which rotates about an axis *O*, perpendicular to the direction of the sliding motion, the chain being completed by the frame-link *A*. Suppose, in the first instance, that the cam presses against a pin placed in the piece so that a line joining it to the centre of rotation gives the direction of the sliding motion.

As the cam turns in the direction of the arrow, *C* moves downwards to a certain limiting position, after which contact will cease unless some force be applied to keep it pressed against the surface. With suitable force-closure, however, supplied by the spring shown in the figure, *C* will return upwards to a second limiting position, and so on, continuously oscillating to and fro.

By properly taking the shape of the cam, any required relation may be obtained between the motions of the cam and slider; we have, in fact, only to draw a curve of position such as that constructed in Fig. 46, page 97, showing the position of the sliding piece for each position of the rotating piece. This curve will be the proper profile for the cam. In practice the chain is usually augmented by the addition of a friction roller, and the shape of the cam is modified by cutting away its surface to a depth equal to the radius of the friction roller, as was done in the case of the teeth of a wheel which drives a pin-wheel.

Force-closure, though common, is not necessary for the action of a cam chain of this kind; it may be avoided in two ways, both of which occur frequently in practice, though the mechanism would not always be described as a cam. First, the pin of the last example may be made to work in a slot cut in the face of a cam-plate, the centre line of the slot being formed to the profile of the original cam. Secondly, a slot may be cut in the sliding piece at right angles to the direction of sliding, and the cam may fit into the slot. Thus, for example, the cam may be a pin or an eccentric of any size: the chain is then merely a reduced double-slider crank motion, as explained on page 131. With other forms of cam other kinds of motion may be obtained: a common example is the Triangular Eccentric formed by three circular arcs (Fig. 73), each struck from one of the corners of an equilateral triangle *abc*. Such a curved triangle will fit between the sides *dd*, *ee* of a rectangular slot, and may therefore be used as an eccentric by fixing it to an axis passing through any point in it. In practice a figure would be used with rounded-off corners, derived by striking small circular arcs with centres *a*, *b*, *c*, and uniting them by larger arcs having the same centres, thus obtaining a profile shown by dotted lines in the diagram, possessing the same essential property of uniform breadth, so that it will fit a rectangular slot of somewhat larger size. The mechanism is shown in Fig. 3, Plate IV.: it is sometimes used for a valve motion, the opening and closing of the valve taking place more rapidly than with a common eccentric. It has also been used in the "man engine" employed in mines to enable the miners to reach the surface without the fatigue of ascending ladders.

Fig. 73.



In these, as well as all other cam motions, a triangle of velocities can be constructed by the general method explained in Art 73, and hence curves can be drawn showing the comparative velocities of the cam, the slider, and the rubbing between the two.

75. *Mechanisms with Intermittent Action.*—In all cases of higher pairing by contact between rigid elements, the closure of the chain is imperfect in the absence of external forces, for an exact fit between the surfaces, even if it exist originally, is soon destroyed by wear during the action of the mechanism. Thus, for example, when a pin works in a groove, as in the last article, the smallest looseness of fit will prevent the grooved piece from exactly following the movement of the pin when the contact passes from one side to the other of the groove. The same effect is produced by the clearance necessary for the safe action of the teeth of a wheel. In cam mechanisms, where the contact is continually changing from one side to the other, the chain opens for a short interval at every change unless force-closure be employed as

described above. *The pair of which the oscillating piece forms an element is locked by friction during the interval.*

Suppose now that the groove is purposely made of much greater dimensions than the pin, the oscillating piece will remain at rest for a considerable interval, and will thus have an intermittent motion. The same thing occurs in wheels which work by the successive action of a number of teeth when some of the teeth in one of the wheels are removed. The pair which moves intermittently may be locked during the interval of rest either by friction or by the special means described in the next article.

Intermittent motions of both the cam and wheel class occur frequently in mechanism. Two common examples may be mentioned.

(1) A wheel with one tooth may be employed to turn another wheel with any number of teeth through a small space at each revolution.

(2) A wheel with one or more teeth may move a sliding piece alternately backwards and forwards.

In all cases, during the interval of motion, we have a chain of the kind already described which closes at the commencement of the interval. The closure is accompanied by a shock which renders such mechanisms unfit for the transmission of considerable forces, and limits the speed at which they can be run. (See Ch. XI.)

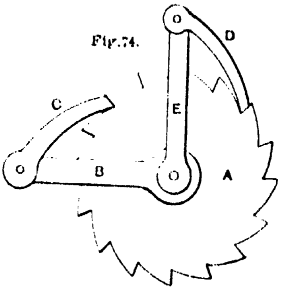
76. *Ratchets.*—The oscillating motion of the piece *C* may be a turning instead of a sliding motion, as is often the case in shearing machines, for example, but no new principle is here involved, and we now proceed to the second class of cam motions in which the forward vibration alone is subject to the action of the cam, while the backward vibration is effected by independent causes, generally by means of springs or of gravity. In such cases the forward vibration follows the same laws as in cams of the first kind, but during the backward vibration the oscillating piece forms a distinct machine by itself, working by means of energy supplied by the cam during the forward movement. In tilt hammers and stampers the work of the machine is done in this way and we need not here further consider them; but the object may be merely to shift the position of the piece and so to lock or unlock a pair, to open or close a kinematic chain. The piece is then called in general a Ratchet, though it may receive other names according to circumstances, and a chain in which it occurs is thus known as a Ratchet Chain.

(1) The shifting piece may lock a turning or a sliding pair in one or both directions. A common latch for example rises to permit

a gate to close and then drops into its place and fastens the gate until again raised by external means.

The piece *C* (Fig. 74) forming a turning pair with a fixed piece *B* fits in the hollows of the teeth of a wheel *A* which also pairs with *B*. The teeth are formed as in the figure so as to permit *A* to move in one direction by raising *C* till it drops by the action of a spring or by gravity into the next hollow. In the other direction the pair *AB* is locked. *C* is then called a pawl, and the arrangement is the ordinary one employed in windlasses, capstans, and lifts to prevent the machine reversing when the hauling power is removed.

Fig. 74.



(2) Two shifting pieces may be employed to lock alternately two pairs which have a common element. This is the ratchet mechanism proper from which the name of the class is derived.

Returning to Fig. 74, *A*, *B*, *C* are the same as in the previous case, *E* is an additional piece which pairs with *B*: in the figure the axis of the pair has been supposed concentric with *A*, but this is not necessary: *D* is the ratchet pairing with *E* and at the same time fitting like *C* into the teeth of the ratchet wheel. If now an oscillating movement be communicated to *E*, the ratchet wheel *A* will be locked alternately with *B* and *E* according to the direction of motion of *E*. Accordingly *A* has an intermittent movement moving with *E* in its forward oscillation and resting in the backward. Instead of a pawl *C*, friction may be relied on to lock *AB* in the backward movement as in the common ratchet brace, but the nature of the mechanism is the same always. It sometimes happens that the pairs *AB*, *BE* are not concentric; the chain *ABED* is then an ordinary four-link chain which opens when moved in one direction and closes when moved in the other, while the pair *CA* unlocks and locks as before, so as to permit *A* to move intermittently. In both cases the movement is single-acting, but two such chains may be employed which move in opposite directions and open and close alternately; the movement may then be described as double-acting. The well-known "Levers of Lagourousse" (Fig. 6, Plate IV.) is a double-acting ratchet mechanism in which the two chains have all the links common except the ratchets. The ratchet wheel then moves continuously in one direction, and the locking pawl *C* may be omitted. The ratchet wheel employed in the case of a turning pair may of course be replaced by a rack when a sliding pair is required, but no new principle is here involved.

(3) The shifting piece may be connected with a pendulum or balance wheel which vibrates in equal times. Time may be thus measured by unlocking a kinematic chain at intervals. In clocks and watches a tooth of the ratchet wheel escapes from the action of the ratchet at each vibration or semi-vibration; the mechanism is therefore called an escapement.

(4) In pumps various kinds of ratchet mechanisms are universal. The common reciprocating pump is a true ratchet mechanism, the column of water being locked and pairing with the plunger alternately; it may be single- or double-acting. It is needless to say that the ratchet is here called a "valve."

77. *Other Forms of Ratchet Mechanism.*—In all the examples of the preceding article the shifting piece is not subject to the action of the rest of the mechanism during its return oscillation, but it may also be worked by a cam movement of the first kind, or by linkwork mechanism; the slide valves of a steam engine are a familiar instance. Also it may be worked by external agency instead of by the machine itself, as in all kinds of starting and reversing gear. The ratchet chains form a large and interesting class of mechanical combinations, but their discussion would be out of place here.

78. *Screw Cams.*—The three-linked chain of Art. 73 may have the axes of its lower pairs inclined at an angle instead of parallel, and a number of mechanisms of the cam class may thus be derived which are analogous to those already considered. Some of these may also be derived from a screw chain, and may here be briefly mentioned.

Let us take a simple screw chain consisting of a sliding pair, a turning pair, and a right-handed screw pair. Let the screw be of several threads, and let a fraction of the pitch be employed. The screw and the nut may then be alike as shown in Figure 2, Plate IV., each resembling a crown wheel with ratchet teeth. When the movement has taken place through the fraction of the pitch in question, the teeth escape and the nut may be moved back endways by force-closure, or by a second screw and nut similar, but left-handed. This movement, which is the only possible cam motion with lower pairing, has been employed to work the shears in a reaping machine,* and is also well known as a clutch.

In its original form the chain consists of a sliding pair *AB*, a screw

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* Journal of the Franklin Institute for March, 1880.

Plate.IV.

FIG.1.

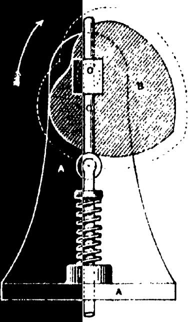


FIG.2.

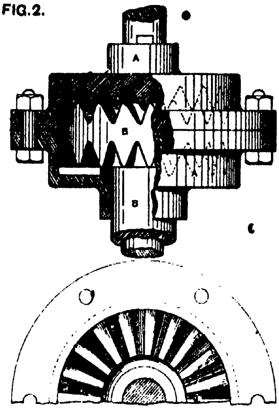


FIG.3.

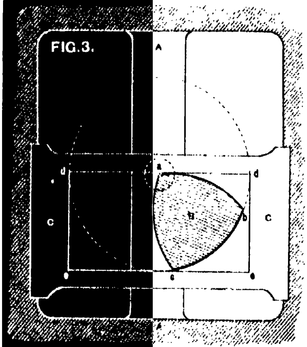


FIG.4.

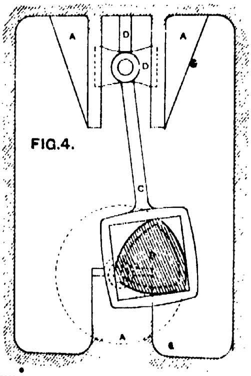


FIG.5.

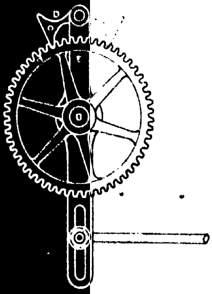
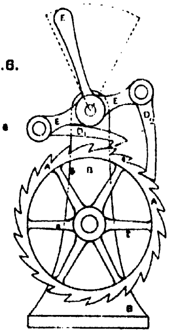


FIG.6.



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pair BC, and a turning pair CA; the piece A may however be omitted, and we obtain a two-link chain consisting of a screw pair BC, the elements of which are united to those of an incomplete lower pair, B and C both sliding and turning during the forward motion and simply sliding during the backward motion. Now imagine one of the screw surfaces replaced by a simple pin, then the other may be made of any form we please, and the elements of the incomplete pair will have a cam motion following any given law. A valve motion common in stationary engines is an example. B is a revolving crown wheel on which is a projection which raises the rod C at the proper time for opening or closing the valve. The "swash" plate usually given in treatises on mechanism is another example.

Plate IV., the figures in which are not taken from actual examples, represents some of the cam and ratchet mechanisms referred to in this section. Fig. 1 is a "heart cam," so called from its shape, in which the sliding and rotating pieces are connected with uniform velocity ratio. Fig. 2 is the screw cam just described. Figs. 3 and 4 are two forms of the triangular eccentric motion (p. 150). Fig. 5 shows a ratchet motion (p. 152) in a form common in the feed motions of machine tools: the direction of movement of the ratchet wheel A is here reversible by putting over the ratchet D into the dotted position. Fig. 6 is referred to on page 171.

EXAMPLES.

1. A reciprocating piece moves in guides under the action of a cam attached to a shaft which rotates uniformly, and the centre of which lies in the line of motion. Trace the form of the cam that the piece may slide uniformly and make one complete movement in each revolution. Suppose a friction roller used of diameter equal to $\frac{1}{6}$ stroke, and suppose also that the least radius of the cam is $\frac{1}{4}$ the stroke.

2. A stamper is raised by a cam such that the rise takes place uniformly during a part of the revolution of a shaft which is distant from the stamper half the rise. Trace the proper form of cam, and find the fraction of the revolution in which the rise takes place.

The best solution is that in which the profile of the cam has the form of the involute of the dotted circle, whose radius is half the lift of the stamper; for then the pressure of the cam on the pin is always in the vertical direction. The rise takes place whilst the cam turns through an angle, the arc of which is equal to twice the radius, or $1/\pi$ of a revolution.

3. Draw a curve of velocity for a reciprocating piece moved by a uniformly rotating triangular eccentric.

CHAPTER VII.

MECHANISM IN GENERAL.

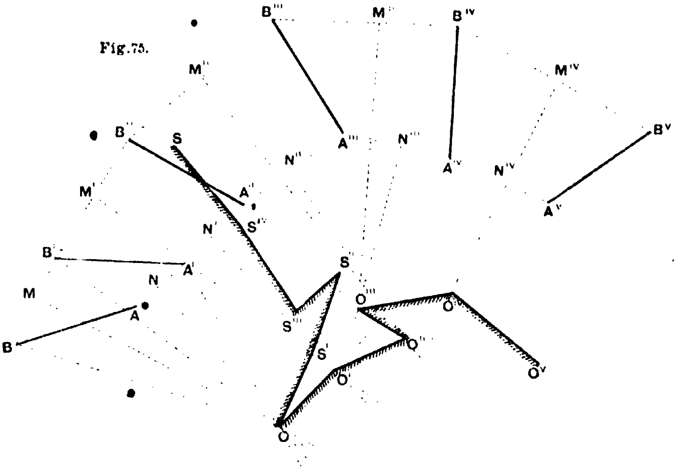
79. *Plane Motion in General. Centrodes.*—In the two preceding chapters the mechanisms considered have been composed either wholly of lower pairs or else of two lower pairs connected by higher pairing. The velocity-ratios of the various lower pairs have been considered and diagrams of velocity have been drawn for the complete mechanism, but the comparative motion of pieces which do not pair with each other, or which form the elements of a higher pair, has only been considered in a few special cases. It will now be necessary to treat this question more generally.

First, suppose the two pieces to move in such a way that a plane attached to one moves parallel to a plane fixed in the other. The motion is then the same as that of a plane area which slides on a fixed plane, and may be called for brevity "plane motion." If the position of any two points in the moving area be given, all the rest can be found, and the motion is therefore completely defined by the movement of the straight line joining these points.

Let AB, $A'B'$, $A''B''$ ... (Fig. 75) be successive positions of such a line. Join AA', BB', and from the middle points of these lines draw perpendiculars NO, MO to meet in O, then $OA = OA'$ and $OB = OB'$, from which it can be proved that $AOB = A'OB'$, so that AB might be moved to $A'B'$ by attaching it to a plane area, and rotating that area about O as a centre. Obtain similar centres O', O'', O''' ... for the succeeding changes of position, then it is clear that the motion of AB, and therefore of the plane area to which it is attached, may be completely represented by the rotation of the area about the centres O, O', O'', ... in succession through certain angles which are given, being the inclinations to each other of the successive positions of AB.

Next, through O draw OS', making it equal to OO' and inclined to OO' at the first angle of rotation, $S'S''$ equal to $O'O''$ and inclined to it at an angle equal to the sum of the first and the second angle of rotation, and so on; we thus obtain a second polygon $OS'S''$ ..., the sides of which are equal to those of the original polygon $OO'O''$... Imagine this polygon rigidly attached to AB so as to move with it,

Fig. 75.



then during the motion the polygon will rotate about O till S' reaches O', then about O' till S'' reaches O'', and so on in succession; that is to say, the changes of position of AB may be produced by the rolling of one polygon upon the other. Thus, by properly determining the polygons, any given set of changes of position of a plane area may be produced at pleasure by rolling the movable polygon on the fixed one.

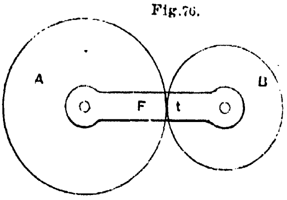
Now imagine the moving area to become fixed in its original position, and let the originally fixed area move by rolling the polygon $OO'O''$... which is attached to it upon the polygon $OS'S''$ which is now fixed. Evidently the two areas take up the same relative positions, and we obtain the very important proposition that any set of changes of relative position of two areas may be obtained by the rolling of one polygon upon another. If the positions are taken at random the polygons may have acute angles as at O'' in the diagrams, but they may also be such as would occur in a continuous motion, and the polygons will then reduce to continuous curves when the positions are taken very near together. Thus every continuous plane motion of two pieces is represented by the rolling of one curve upon another, the point of contact being a centre about which either piece is for the instant rotating relatively to the other. These curves are called Centrodes, and the point is called the Instantaneous Centre. Whenever the directions of motion of two points in a moving piece are known, the instantaneous centre is at once determined by drawing perpendiculars to intersect. During the motion it traces out two curves, one in the moving piece, the other in the fixed piece, which curves are the centrodes of the motion.

80. *Axoids. Elementary Examples of Centrodes.*—Any two bodies moving in the way described may be divided into slices by planes parallel to the plane of motion, the centrodes of which will of course be all similar and equal, so that we may regard them as the transverse sections of cylindrical surfaces in contact, with each other along a generating line. The surfaces are called Axoids, and the line the Instantaneous Axis. The relative motion of the bodies is represented by the rolling of the axoids upon one another, endways motion being supposed prevented.

Any two parts of a mechanism have a relative motion which is completely defined by the nature of the mechanism, as has been sufficiently explained already; and it follows, therefore, that they must have given axoids, the nature of which completely defines the motion of the pieces. In every kinematic chain there are as many sets of axoids as there are sets of two pieces, and these surfaces are the same for all the mechanisms derived from that chain by inversion. These remarks apply even when the motion is not plane, as will be seen further on.

First. Take the case of a pair of spur wheels *AB* in gear, *F* being the frame-link (Fig. 76), forming the three-link chain considered in the last chapter. Let the pitch circles touch at the pitch point *t*, then, as before explained, those circles roll together without slipping, and therefore must themselves be the centrodes, the pitch surfaces being the axoids. Hence the point *t* is the instantaneous centre of *B*'s motion relatively to *A*, or *A*'s motion relatively to *B*. We shall return to this immediately, but for the present merely remark that if the centres of *A* and *B* move up to each other, the pitch circles reduce to points, and the axoids become coincident straight lines, the point *t* is fixed in *A* and *B*, the two pieces then become a turning pair. In lower pairing, then, the axoids are coincident straight lines, which are at infinity if the pair be sliding. The case of a screw pair in which the motion is not plane will be referred to further on.

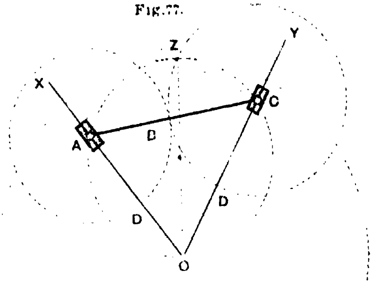
Fig. 76.



Secondly. Take the case of a double-slider chain; there are here four pieces which may be taken two and two in six ways; there are, therefore, six sets of axoids. Four of these, however, are only the four axes of the four lower pairs, and it remains to determine the other two.

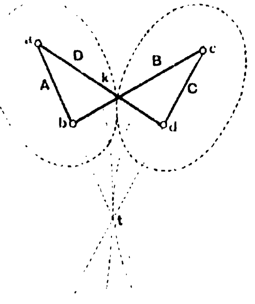
In Fig. 77 the blocks A, C are connected by a link B and slide on a piece D along lines OX, OY, forming the chain described fully in a former chapter. The blocks A, C form two turning pairs with the link B, and the velocities of these pairs are equal because B makes angles with OX, OY, the difference of which is constant. The centrodes for A, C are therefore equal circles, the centres of which are the centres of A, C. Since A and C rotate in the same direction these circles must be of infinite size, and to represent them in the figure equal circles of finite size are employed which give the same motion in opposite directions. Next, to find the centrodes of B, D, through those centres draw perpendiculars to OX, OY to meet in Z, then Z is the instantaneous centre for B when D is fixed, and for D when B is fixed. First, suppose B fixed, then the angle at Z is the supplement of the angle at O, and is therefore constant, so that Z traces out an arc of a fixed circle, of which OZ is the diameter. Next, suppose D fixed, then, since OZ is constant, Z traces out a circle, the centre of which is O. Thus the centrodes of B, D are two circles, one half the diameter of the other; the large circle is fixed to D, and the small circle to B.

Fig. 77.



Thirdly. In the four-link chain A, B, C, D, consisting of four turning pairs with parallel axes, the sections of which are represented by the points a, b, c, d (Fig. 78); suppose opposite links equal, but set so as not to be parallel. This is the case referred to already (page 112) as "anti-parallel" cranks.

Fig. 78.



Joining ac, bd by the dotted lines in the figure, the quadrilateral $abdc$ has two sides and two diagonals equal, hence the triangles bac, cda must be equal in every respect, so that bd is parallel to ac. Hence if k be the intersection of the diagonals, and t

the intersection of the sides, $ak = ck : bk = dk : bt = dt$, from which it appears that $ak + bk = ck + kd = ad = bc$

and $at - dt = ct - bt = ab = cd$.

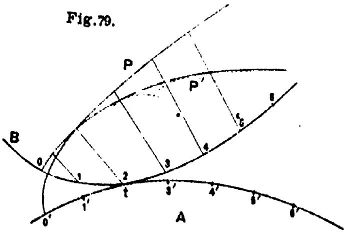
Suppose, now, A to move while C is fixed, then a moves perpendicular to ad, and b moves perpendicular to bc, so that k must be the instantaneous centre for the motion of A relatively to C, or for that of C relatively to A. Now, in the first case, it appears from what has been said that k traces out an ellipse, of which c and d are foci, while, in the second, it traces out an equal ellipse, of which a and b are foci. Thus the centrodes for the motion of A and C are equal ellipses, as shown in the diagram. In like manner the centrodes for the motion of B and D are the equal hyperbolæ traced out by the point t.

The four other pairs of centrodes are the points a, b, c, d, which are the centres of motion of the four turning pairs.

81. *Profiles for given Centrodes.*—Any given motion of one piece relatively to another may be produced in an infinite number of ways. One way of doing this is by rolling contact, for if the motion is given the centrodes will be given, and by forming the profiles so as to represent the centrodes, and applying forces to press the pieces together and cause them to roll on one another without slipping, the given motion may be produced. But if slipping be permitted, the same motion may be produced, at least theoretically, by assuming any form whatever for one profile and properly determining the other.

(1) Let a given profile be attached to the moving piece, and as it rolls into different positions let that profile be traced on paper attached to the fixed piece. If the positions be taken near enough together, a curve may be drawn through their ultimate intersections which will envelop them all, and if a profile formed to that envelope be attached to the fixed piece, the two pieces will fit one another and yet be capable of relative motion of the prescribed kind.

Fig. 79.



(2) To apply the foregoing process a model would be necessary, but by a simple modification, a geometrical construction may be obtained. In Fig. 79, A and B are the pieces, which move so that the centrodes are 0, 1, 2, 3..., 0′, 1′, 2′, 3′..., curves which are shown touching at the point t. P is a profile of given form attached to B; it is required to find a profile attached to A, which will always remain in

contact with *P* and so be capable of moving it in the required way by simple contact.

Divide the centrode of *B* into arcs of equal length, starting from *O*, the point where *P* intersects it, and let 2 be the point of contact at the instant considered. Divide the centrode of *A* into equal arcs, stepping from 2 in both directions, then 0′, 1′, 3′, 4′... are points in *A*'s centrode, which correspond to 0, 1, 3, 4 ... in *B*'s centrode, being during the motion points of contact in succession. From 1, 2, 3 ... drop normals on to the curve *P*, and with these normals as radii trace circular arcs with centres 1′, 2′, 3′ ... ; the envelope of these arcs must be the required profile *P*′.

(3) Instead of assuming one profile and determining the other to suit it, it is generally more convenient to employ some method of determining pairs of profiles which satisfy the required conditions.

In Fig. 80 *A*, *B* are the centrodes as before, *C* is a third curve, theoretically of any form, which rolls on *A* and *B*, always touching these curves at their point of contact *t*. *P* is a tracing point which is attached to *C* and traces out two curves during the motion, one on *A*, the other on *B*. First, suppose *A* fixed, then, since *t* is the instantaneous centre of the motion of *C*, *Pt* must be normal to the curve *NP* traced out on *A*. Similarly, supposing *B* fixed, *Pt* is normal to the curve *MP* traced out on *B*. Thus the two curves touch one another at the point *P*, and therefore may be taken as profiles which will give the required motion. If *A*, *B*, *C* be circles, this construction becomes that already considered when discussing the form of teeth for a wheel. This and the preceding method show clearly that the condition which the two profiles must always satisfy is that the common normal at the point of contact must always pass through the pitch point as already proved otherwise for the special case of wheel teeth.

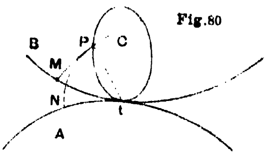


Fig. 80

Not every pair of curves which satisfy the geometrical conditions could actually be used as profiles, either for centrodes, or, in the cases just mentioned, to give a required motion, because there is nothing in the geometrical construction which excludes an interpenetration which would not be physically possible in the areas of which the profiles form the boundaries, but an infinite variety of forms can be found for given centrodes which might be so used.

In all cases in which the centrodes are known for the relative motion of two pieces, one of which is fixed, the velocity-ratio of any two points (a, b) in the moving piece is known for each position of the pieces. For, joining the two points to the instantaneous centre O, the ratio of the distances Oa, Ob must be the velocity-ratio in question, since the moving piece is for the moment turning about O. It is easily seen that the triangle Oab is similar to the triangle of velocities constructed as in Art. 49, p. 100.

82. *Centrodes for a Higher Pair Connecting Lower Pairs.*—Among the infinite variety of profiles which correspond to given centrodes it is frequently possible to find some which are closed curves, one completely surrounding the other. If these curves be used as the external and internal boundaries of two areas, the two pieces thus formed will fit one another and be capable of no motion except that of the prescribed kind without requiring any additional constraint. In Fig. 4, Plate IV., a form of the triangular eccentric motion is shown, which has been occasionally used and which furnishes an example. On reference to Art. 74 it will be seen that such an eccentric will exactly fit a square within which it is enclosed, and therefore forms with it a higher pair which is "complete" in itself.

Complete higher pairs are very unusual in mechanism, higher pairing being employed almost exclusively to complete a chain of lower pairs as in the preceding chapter. It is then generally "incomplete," the necessary constraint being furnished by the rest of the kinematic chain to which it belongs, as for example in the triangular eccentric motion shown in Fig. 3, Plate IV. The general problem in mechanism is not to connect two pieces in a given way, but to convert the motion of a given pair into the motion of a different pair—that is to say, to connect two pairs so as to have a prescribed relative motion. This will be further considered presently, but we must first return for a moment to a question considered in the last chapter.

In the three-link chain of Art. 73 we have two lower pairs AC, BC with axes parallel, connected by simple contact between A and B at the point P (Fig. 72, p. 152). Draw the common normal PT to meet ad in T, then when B is fixed the motion of a is perpendicular to ad, and the motion of P perpendicular to PT, therefore T must be the instantaneous centre for the motion of A relatively to B. Let v be the velocity of rubbing at P; A', A'' the angular velocities of the pairs AC, BC: further let $ad=l$ and $PT=z$; then, since B is fixed and A is rotating round T,

$$\frac{v}{z}=\frac{\text{velocity of } a}{aT}=A'\cdot\frac{l}{aT}.$$

Similarly supposing A fixed,

$$\frac{v}{z} = \frac{\text{velocity of } d}{aT} = A \cdot \frac{l}{dT},$$

from which it appears that

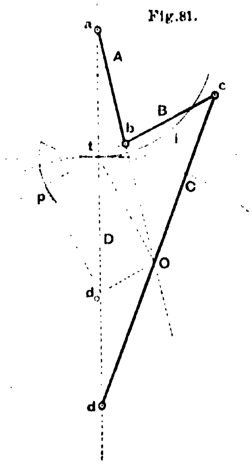
$$\frac{A}{A'} = \frac{dT}{aT}; \quad v = z(A' - A),$$

results which agree with those obtained in the article cited by a different method.

The centrodes in this case, as well as in that of the four-link chain from which it was derived by reduction, may be traced graphically by plotting the position of T for a number of positions of the pieces, but they are known curves only in exceptional cases such as those of Art. 80, and generally have infinite branches which render their use inconvenient.

When the point P lies on the line of centres it coincides with T, and the velocity of rubbing is zero; the centrodes are then no other than the profiles themselves of A and B. The curves are then said to roll together: a particular example is that of the equal ellipses of Art. 80 which are not unfrequently used to connect two revolving shafts with variable angular velocity-ratio. In this case the velocity-ratio is the ratio of the focal distances of the point of contact, but by properly determining the profiles it is theoretically possible to give any velocity-ratio to the shafts at pleasure. The question, however, is not one of much practical interest.

Fig. 81.



83. *Construction of Centres of Curvature of Profiles. Willis's Method.*—In the four-link chain $ABCD$ shown in Fig. 81, D is the fixed link and B the coupling link: a, b, c, d are sections of the axes of the pairs which are supposed parallel.

If the coupling link bc be prolonged to meet the line of centres ad in the point t, and ab to meet cd in O, it appears as in previous cases that O must be the instantaneous centre of B, and that the angular velocity-ratio of A and C is $dt : at$. Join Ot, and imagine bt an actual prolongation of the bar *bc*, so that *t* is rigidly connected with it, then *t*'s motion will be perpendicular to *Ot*. Suppose now that the proportions of the links are taken so that *Ot* is perpendicular to *bt*, then *t* moves in the direction of the length of the rod, and the rod therefore may be imagined to slide through a fixed swivel at *t*.

This reasoning shows that the levers *A* and *C*, when in this position, will move for a short interval with uniform angular velocity-ratio, and the movement of a pair of wheels in gear is thus imitated by a linkwork mechanism.

Let us now form a reduced chain by omission of the coupling-link, and we shall be able to solve the important problem of finding a pair of circular arcs which will serve for the profiles of a pair of teeth in contact. For this purpose, with centres *b* and *c*, strike arcs through any point *p* on *cbt* produced, and let these arcs be rigidly connected with *A* and *C* respectively; the coupling-link may now be removed and *A* imagined to drive *C* by direct contact of the arcs. Evidently wherever *p* is, the pieces will move for the moment with uniform angular velocity-ratio and pitch point *t*. The uniformity, however, is only momentary, because the position of *O* changes, and to trace the profiles with accuracy it would be necessary to perform the construction for a succession of positions of *cbt*, hence the face and flank of a pair of teeth in contact cannot be exactly represented by a pair of circular arcs. When it is sufficiently approximate to do so, the arcs are found by assuming a mean position for the point *p*, and the mean value for the obliquity *i*, found by experience to give good results. The method here described was invented by the late Professor Willis, and the value of *i* recommended by him was $\sin^{-1}\cdot25$, or about 14½°, being about the actual mean value of the obliquity in cycloidal teeth of good proportions. Also the value of *pt* was taken by him as half the pitch, *p* being then about midway between the pitch point *t* and the point of the tooth.

Having made these assumptions, it still remains to fix the position of the point *O*, which may be taken anywhere on a line through *t* inclined at 14½° to the line of centres. This is done by observing that *O* must be the same for all wheels *D* intended to work with a given wheel *A*, and that teeth never should be undercut (Art. 70); that is, *c* and *b* must lie on the same side of *t*. Hence in the smallest wheel intended to work with *A*, *c* is at infinity, so that if d_0 is its centre, d_0O is parallel to *pt*, and therefore perpendicular to *Ot*. The flank of the tooth in this case becomes a radius d_0p. The position of *O* is thus completely determined for all the wheels of a set when the pitch is given.

Willis's method is of great theoretical interest, and has consequently been given here, but the form of teeth obtained is not always sufficiently approximate. It may, therefore, with advantage be replaced by other methods, as to which the reader is referred to a work by Professor W. C. Unwin on Machine Design.

84. *Sphere Motion.*—When a body moves about a fixed point its motion is completly represented by that of a portion of a spherical shell of any radius which fits on to a corresponding sphere, and moves on it just as in the case of plane motion. Everything which has been said respecting plane motion also applies to sphere motion, but the axoids are conical instead of cylindrical surfaces, the centrodes spherical instead of plane curves, and all straight lines are replaced by great circles of the sphere on which the motion is imagined to take place. The corresponding crank chains are called "conic" crank chains, the axes of the pairs lying on a cone instead of a cylinder.

85. *Screw Motion.*—In the plane motion of two pieces, endways motion of the cylindrical axoids is supposed to be prevented by some suitable means. Let us now remove this restriction and imagine the axoids to slide endways, while continuing to roll together, the relative movement will now not be completely defined, but additional constraint will be required. In the first place take the case of a lower pair in which the axoids are coincident straight lines; if endways sliding be permitted we obtain an incomplete pair, unless the nature of the surfaces in contact define the relation between the endways motion and the rolling motion. In the simple screw pair the two are in a fixed ratio, in the screw cams of Art. 78 they have a varying ratio. In every case of non-plane motion with cylindrical axoids, not only must the axoids be given, but also a connection between the endways sliding and the motion of rotation.

In the most general case possible the instantaneous axis changes its direction as in spherical motion, its position as in plane motion, and in addition there may be an endways sliding. This is expressed by the rolling and sliding of certain surfaces on one another, which are now neither cylindrical nor conical. These surfaces are in all cases of the kind known as "ruled" surfaces, being generated by the motion of a straight line, along which they touch each other. The surfaces are still called Axoids, and the line is in the Instantaneous Axis. The hyperboloidal pitch surfaces for wheels connecting two shafts which do not intersect are examples of this kind; but for the discussion of this question, which is not of very common occurrence, the reader is referred to the works already cited.

86. *Classification of Simple Kinematic Chains.*—On observing the action of any mechanism, several of the pieces of which it is constructed may be readily distinguished as having functions different from the rest. These pieces, like the rest, occur in pairs, and may be described as such, though the pairing is not necessarily kinematic. First, one or more perform the operations which are the object of the mechanism; these may be called the Working Pairs, as, for example, the tool and the work in machine tools, the weight raised and the earth in the hoisting machines. Second, one or more form the source from which the motion is transmitted, as, for example, the crank handle and frame of a windlass, the piston and cylinder of a steam engine. These may be called the Driving Pairs. Thirdly, various sudsidiary working pairs carry out various operations incidental to the working of the machine. The object of the mechanism is always to convert the motion of the driving pairs into that of the working pairs.

The simplest case is that in which the motion has only to be transmitted without alteration; a single pair will then suffice. Thus by means of a long rod sliding in guides or turning in bearings, a motion of translation or rotation may be transmitted to a distance only limited by non-kinematical considerations. By use of flexible elements—among which should be included the flexible shafts recently introduced—the direction may be altered at pleasure and any desired position reached.

If, however, the magnitude of the motion is to be altered, a mechanism must be employed in which at least one element of the driving and working pairs is different. The driving pairs are usually kinematic ower pairs, and the working pairs are so very frequently, and this is why so many of the simplest and most important mechanisms are examples of the connection of lower pairs. The peculiar motions of lower pairs being translation and rotation, a number of mechanisms may be classed as examples of the conversion of rotation into translation or rotation and conversely, with uniform or varying directional relation or velocity-ratio. This is especially the case when, as so frequently happens, the driving and working pairs have a common link which is fixed.

It has been shown, however, that many apparently different mechanisms are in reality closely connected, being derived from the same kinematic chain. Mechanisms are therefore to be classed according to the kinematic chains to which they belong. The number of simple chains actually employed in mechanism is limited by the preceding considerations to those already described, which are ranged by Reuleaux in the following classes, the names of which are derived from the most important piece in some example of common occurrence:

(1) Crank chains.
(2) Screw chains.
(3) Pulley chains.
(4) Wheel chains.
(5) Cam chains.
(6) Ratchet chains.

In the first two are included all combinations of sliding, turning, and screw pairs; in the third, all cases where tension or pressure elements are employed; in the fourth, all cases of connection by contact where the directional relation remains the same; in the fifth, all cases where it varies; while in the last, all combinations are included where, by action of a shifting piece, the law of motion is periodically varied.

87. *Compound Kinematic Chains.*—In a complete machine, the motions required are generally too complex to be carried out by a single kinematic chain of this simple kind; it is necessary to combine together a number of such chains, and we conclude this part of the subject with some general remarks on such combinations which may all be regarded as compound chains derived from two or more simple chains by union of their links.

(1) From any two closed chains a third may be derived by uniting two links. The links must have the same relative motion, for otherwise the chains would lock each other, and they generally form a pair.

This is one of the commonest of all combinations. When two machines are driven from the same shaft, or when the same shaft is driven by two separate engines, we have examples in which the driving pairs or the working pairs are common, but the mechanism are otherwise independent. Further, in every complete machine we find, in addition to the principal chain which does the work, a number of auxiliary chains which carry out various operations necessary to the working of the machine. Thus, in the steam engine, besides the slider-crank or other mechanism which turns the crank, we have the valve motion which governs the distribution of steam, the air pump motion which produces the vacuum in the condenser, and frequently others as well. Each of these auxiliary mechanisms has a pair in common with the principal chain which serves as a driving pair, but the chains are otherwise independent. Again, in trains of mechanism which, as previously remarked (page 139), are frequently simple chains augmented for non-kinematical reasons, a number of such chains are arranged so that the working pair of one chain is the driving pair of the next in succession. A train of wheels or the mechanism of a beam engine are examples already referred to, in which one link is common to all the separate chains, but cases occur *in which this is not so, as, for example, the well-known Lazy Tongs.*

The case here considered is that where the movements of various driving pairs have to be transmitted to various working pairs, but no new motion is required in a working pair other than could be produced by a simple chain. All such combinations may be called Trains, and may be divided into "converging," "diverging," and "transmitting" trains.

(2) If two closed chains have only one link common they are completely independent, like two machines standing on the same floor, but disconnected. It might, therefore, be supposed that nothing was obtained that was new. In fact, however, this is a combination which is as common as the preceding, being employed to give a motion to a working pair which is too complex to be produced by simpler means, or which requires to be varied at pleasure. The working pair consists of two elements, one of which is supplied by one chain, the other by the other, and the motion of the pair is thus a combination of the motions of the two independent chains. Completely new motions are obtained in this way, and they may be varied at pleasure by alteration of either or both of the primary motions.

Take, for example, the common planing machine. The working pair consists of the table upon which the work is mounted, and the tool. To the first a reciprocating movement is communicated by means of a suitable kinematic chain connecting it with the driving shaft. The other is mounted on a slide rest, forming an element of a screw chain which gives it a horizontal movement. This chain has one link in common with the principal chain, but is otherwise independent. In the ordinary working of the machine this chain is locked by friction, except at the end of each reciprocating movement of the table when it moves to take the next cut. The tool thus traces out a complete plane surface.

In this example the common link is fixed, but this need not be the case, and in fact in the planing machine a third independent chain is added to adjust the tool vertically, the tool being mounted on a vertical slide having an independent movement. Also, one element of the working pair may be fixed, and both movements given to the other, which is common to both chains. Double and treble chains of this kind occur whenever it is necessary to move the elements of the working pair into all possible positions. In cranes of all kinds we find a treble movement, one to raise and lower the jib, a second to swing the jib round, and a third to raise and lower the load. In traversing cranes the three movements are rectangular, as in the planing machine. In either case we find the methods employed by mathematicians to define the position of a point in space by rectangular or polar co-ordinates exactly imitated by the mechanism.

The elements of the working pair need not be wholly disconnected as we have hitherto supposed, they may form an incomplete kinematic pair. Thus if the axoids be cylindrical, endways motion may still be possible and may be given by an independent chain. A common example is a drilling machine, the working pair in which consists of a table on which the work is mounted, and a spindle carrying the drill which rotates and at the same time descends as the hole is drilled; the two movements may be quite independent, the one proceeding from a driving shaft, the other operated by the workman.

A similar combination is employed when a train is varied by shifting one of the links. Fig. 5, Plate III. (p. 141), represents a case of this kind. The wheel *C* is mounted on a shaft which can be shifted endways by an independent mechanism. The shifting of belts (Art. 61, p. 133) is another example.

Again, the movements of the working pieces may be connected by a transmitting train connecting the chains which produce them. In the self acting feeds of planing and shaping machines the connection is intermittent, but it may also be continuous, and we then have a fertile means of producing complex movements variable at pleasure. In a screw-cutting lathe the tool is mounted on a slide rest moved by a screw, and the work is attached to a rotating mandrel. Connecting these independent chains by a train of "change" wheels, the tool cuts a screw of any pitch.

The principle of all combinations of this kind is the closure of an incomplete or disconnected pair by independent chains. We may describe them as Multiple Chains.

(3) If two closed chains have two or more pairs common, they must be of the same kind, for otherwise the pairs would not have the same relative motion, and the chains would lock each other. The differential mechanisms, examples of which have been already given, are cases of this kind. Thus in the differential pulley (Fig. 62, p. 130), if *A* and *C* be disconnected we have two simple pulley chains with common movable pulley *B* and separate axles. Either of these might be operated independently. In the actual mechanism *A* and *C* are united, and the movement of *B* is the difference of the movements due to each separate chain.

Complex examples of similar combinations occur in the epicyclic mechanisms. Fig. 82 (p. 175) shows a combination of two of the differential trains described on p. 139. *C*, *C′* are wheels turning about the same axis in the frame link *A* and united: *E*, *E′* are also united, but have a different frame-link *A′*. Both gear with the wheels *B*, *D*, which are disconnected, but turn on an axis common to *A* and *A′*. On comparing this with Fig. 65 it will be seen that two trains have been compounded by uniting the wheels *B*, *D*, which are common to both. If now one of the frame-links, say *A′*, is fixed, and *EE′* be rotated, the other frame-link *A* will rotate with a velocity which can be found on the principles of the articles cited. For simplicity, *EE′* have been supposed to gear directly with *B*, *D*, but they may also gear with wheels of other diameters fixed to *B*, *D*, or the wheels may be replaced by a different train of mechanism, all that is necessary being that the motions of the pairs *BA′*, *DA′* should be connected.

Many examples of this mechanism may be found—especially in the case where *C*, *C′* are equal and the train reduces to three bevel wheels (p. 140). In traction engines and tricycles, for instance, a mechanism of this kind is sometimes employed to facilitate turning. *A′* is then the frame of the machine. *B* and *D* are equal bevel wheels attached to the axle, which is divided into halves, each connected with one of the driving wheels. If now the motive power be applied to *A*, *B* and *D* will rotate, but not necessarily with the same velocity, and the machine may therefore be guided in a curve by the front wheel without the slipping which would occur if the driving wheels were fixed to an undivided axle.

Combinations of this class are not essentially different from multiple chains in which the elementary chains are connected by a train, as described above. They may be called Compound Trains; all consisting of simple trains compounded in various ways, either for non-kinematical reasons or to enable the train to be varied at pleasure.

(4) All the preceding combinations are formed of simple closed chains united together in various ways: no new chain is obtained, but merely an aggregation of forms already known. Certain mechanisms, however, occur, which, if taken to pieces by separation of united links, are found to contain one or more chains which are not closed.

Take for simplicity a common slider-crank mechanism, and imagine the crank pin, instead of being fixed to the crank, to be mounted on a slide so as to be free to move to and from the centre. The chain is now augmented by an additional sliding pair, and is no longer closed, so that it cannot be used as a mechanism. If, however, we introduce a screw, which moves the slide, we may lock the sliding pair in any position and thus obtain a closed chain, one link of which can be varied at pleasure. This mechanism is used in practice to obtain a varying stroke in a sliding piece. It is often required to make the stroke increase or diminish at each revolution of the crank. A wheel attached to the screw then comes in contact with a projecting piece and moves through a small space, the screw chain being locked by friction during the rest of the revolution. The mechanism thus varies at intervals its own law of motion.

Fig. 82.

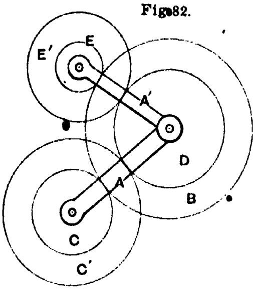
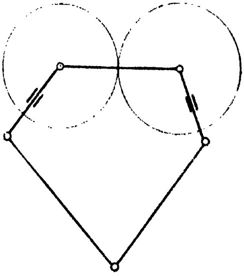


Fig. 83.



By a suitable transmitting train, however, a continuous variation may be produced, and the combination then furnishes us with an entirely new mechanism. An important example is the wheel crank chain (Fig. 83), formed by combining a simple wheel chain with an open crank chain of five links. A number of mechanisms may be derived from this chain by inversion, but for particulars the reader is referred to Reuleaux's work already cited.

Another example is shown in Fig. 6, Plate II. (p. 111), which represents a mechanism employed in sewing machines to give two strokes to a sliding piece for one revolution of a shaft. We have here a closed double slider chain combined with a single slider rendered incomplete by omission of the crank pin. Combinations of this class are called by Reuleaux "true" compound chains to distinguish them from the preceding classes, in which no new mechanism results from the combination. Perhaps the words "higher" and "lower" would more clearly express the meaning.

PART III.

DYNAMICS OF MACHINES.

PART III.—DYNAMICS OF MACHINES.

CHAPTER VIII.

PRINCIPLE OF WORK.

SECTION I.—BALANCED FORCES (STATICS).

88. *Preliminary Explanations. Definition of Work.*—If the principal object of a piece of mechanism be to do some kind of work it becomes a machine. Many mechanisms—as, for example, clocks and watches—are not, properly speaking, machines; for though work is done during their action, yet the object of the mechanism is not the doing of the work, but the measurement of time or some similar operation. Even in these cases, however, the forces in action cannot in general be excluded from consideration, and therefore in all mechanism a study of the manner in which forces are transmitted and modified is essential. This part of the subject is called the Dynamics of Machines.

A body can in general only be moved into a different position or be changed in form or size by overcoming resistances which oppose the change. This process is called doing WORK, and the amount of work is measured by the resistance multiplied by the space through which it is overcome. If there be many resistances, the total work done is the sum of that done in overcoming each resistance separately.

Consider the case of a mass of matter raised vertically. Here the resistance is due to the action of gravity, which is overcome by some external force, and the work done is simply the product of the resisting force and the height through which the mass is raised. The resisting force is commonly described as the "weight" of the mass, and is measured by comparing it with that of a certain quantity of matter, the weight of which is taken as a unit for measuring forces. This mode of measurement has the disadvantage of giving a different unit for different points on the earth's surface, because the force of gravity varies according to the position of the point, and for scientific purposes, therefore, force is measured by the velocity which, when unbalanced, it produces in a given quantity of matter. In practical applications, however, gravitation measure is preferable, especially as the variation is very small, and the measure may be made precise when necessary by specifying the place on the earth's surface at which our operations are taking place. As already stated (p. 3) the unit of force employed in Britain is the weight of a piece of matter called a pound, while the unit of space is generally one foot, so that the unit of work is one pound raised through one foot, or, as it is generally called, 1 foot-pound. Other units, however, such as, for example, "foot-tons," may also be employed for special purposes.

In the United States of America British units are chiefly used, but in other countries metric measures are universally adopted. In the metric system the units of space and force employed by engineers are the metre and the kilogramme, the derived unit of work being the kilogrammetre. These units are connected with the British system (see also p. 92) by the relation

One metre = 3·2809 feet.
One kilogramme = 2·2046 pounds.
One kilogrammetre = 7·2331 foot-pounds.

The question of measurement will be further considered in a later chapter (Ch. X.).

89. *Oblique Resistance.*—In the case just considered, the resistance is directly opposed to the movement which is taking place; if this be not so, it must be resolved into two components, one along and the other perpendicular to the direction of motion. The second of these is balanced by a constraint to which the motion is subject or by the opposition which the inertia of the body offers to a change in its direction at any finite rate; it is the first alone in overcoming which work is done. In Fig. 84 let R be a resistance applied at a point A which moves through a distance AB in a direction inclined at an angle θ to the direction of the resistance, then the work done is $R.\cos\theta.AB$, but if BN be drawn perpendicular to the direction of R to meet that direction in N,

$$AN = AB.\cos\theta,$$

and therefore the work done is $R.AN$.

Now AN is the distance through which A has moved in the direction of the resistance, so we obtain another rule for estimating the work done against an oblique resistance. **It is equal to the product of the resistance into the distance moved in the direction of the resistance.**

Suppose, for example, that a weight is raised, but that instead of being lifted vertically, it is moved in any curved path—there being no friction or other resistance than that due to gravity.

Considering any small portion AB of the path (Fig. 85), the resistance being always vertical, the work done is $W \cdot AN$. So the total work of raising the weight is $W \cdot \Sigma AN$ or $W \cdot h$, which is independent of the path described by the lifted weight, but depends simply on the height through which the weight is raised.

Fig. 84.

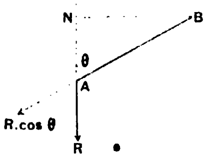
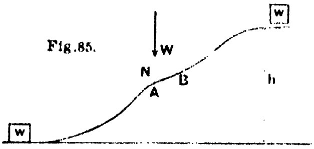


Fig. 85.



If there are a number of weights, each of them raised through different heights, the total work done in raising all the weights is the sum of the works done in raising each weight separately; and the direct method of finding the total work is to add the separate results for each weight. But it may be determined by another method thus—

Let W_1, W_2, W_3 etc. be a number of weights which are at heights y_1, y_2, y_3 etc. above a given datum plane. Now suppose they are raised so that they are at heights Y_1, Y_2, Y_3 etc. above the same plane. The total work done in raising the weights will be the sum of the products,

$$W_1(Y_1-y_1)+W_2(Y_2-y_2)+W_3(Y_3-y_3)+\text{etc.}$$

Now suppose the centres of gravity g and G for the initial and final positions of the weights to be at heights y and Y above the datum plane.

The centres of gravity g and G are such that if all the weights were collected at either centre, the moment of the collected weights about the plane is equal to the sum of the moments of each separate weight, before being collected. about the same plane. This is mathematically expressed thus—

$$y=\frac{W_1y_1+W_2y_2+W_3y_3+\text{etc.}}{W_1+W_2+W_3+\text{etc.}}$$

$$\text{and }\bar{Y}=\frac{W_1Y_1+W_2Y_2+W_3Y_3+\text{etc.}}{W_1+W_2+W_3+\text{etc.}},$$

By subtracting we have

$$\bar{Y}-\bar{y}=\frac{W_1(Y_1-y_1)+W_2(Y_2-y_2)+W_3(Y_3+y_3)+\text{etc.}}{W_1+W_2+W_3+\text{etc.}},$$

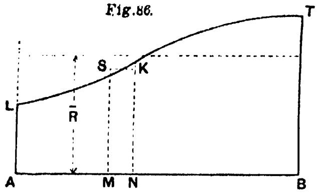
hence

$$\text{Total work}=\Sigma W(\bar{Y}-\bar{y}).$$

That is to say, the total work of raising a number of weights is equal to the product of the sum of the weights by the vertical *displacement of the centre of gravity of the weights.*

90. *Variable Resistance.*—Let us next consider the work required to be done to overcome a variable resistance. The whole distance through which the resistance is overcome must then be divided into a number of parts, each being so small that, for that small space, the magnitude of the resistance may be treated as sensibly uniform. The work of overcoming the resistance through each of the small spaces being thus found, the total work will be the sum. The estimation can generally be most conveniently performed by a graphical construction. We will, for simplicity, take the case in which the direction of action of the resistance is that of the line of motion. Suppose a body moved from *A* to *B* against a resistance the magnitude of which varies from point to point in such a way that it is represented by the ordinates of the curve standing above *AB* (Fig. 86). For the small distance *MN* the resistance will vary slightly, but will have a mean value represented by *SM* or *KN* suppose, and the work of overcoming the resistance through the small space *MN* is $MN \times SM$ or is exactly represented by the area of the curve standing above *MN*; and so for any other small portion of the displacement of the body. Thus the total work of overcoming the resistance through *AB* is represented by the whole area $ALTB =$ mean resistance $\bar{R} \times AB$.

Fig. 86.



The curve *LST* is called a curve of resistance. Two important special cases may be mentioned, both of which frequently occur.

Fig. 87a.

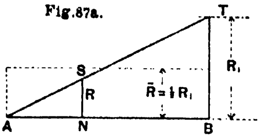
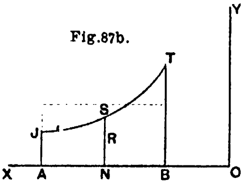


Fig. 87b.



(1) Let the resistance vary uniformly. This is the case of a perfectly elastic spring which is compressed, as will be further explained hereafter. The curve of resistance is a straight line *AST* (Fig. 87a) where *AB* is the compression of the spring, *BT* the corresponding

compressing force R_1. During the compression R is at first zero and gradually increases to R_1, its value at any intermediate point being graphically represented by the ordinate SN corresponding to the compression AN. The work done is the area of the triangle, that is $\frac{1}{2}R_1 . AB$, and the mean resistance is consequently $\frac{1}{2}R_1$.

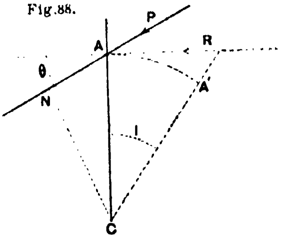
(2) Let the resistance be inversely proportional to the distance of the point of application from a given point O (Fig. 87b).

This applies to many cases of the compression of air and other elastic fluids. In the figure $NS = R$ is the resistance and $ON . NS$ is constant, so that the curve of resistance JST is an hyperbola. Let the ratio $OA : OB$ be called r, this is called the ratio of compression: then from the geometry of the hyperbola we know that the area of the curve is equal to the constant rectangle $ON . NS$ multiplied by $\log_e r$, the logarithm being Napierian, or, as it is often called, "hyperbolic" from this property of the hyperbola. If ON be denoted by V this gives a formula in frequent use for the work done in this kind of compression.

$$\text{Work done} = RV \log_e r.$$

91. *Resistance to Rotation. Stability of a Vessel.*—It often happens that we have to consider the resistance of a body to rotation about an axis. Let A (Fig. 88) be the point of application of a force P which resists the rotation of a body about an axis C perpendicular to the plane of the paper. If the resistance at A be not in the plane of rotation P must be supposed to be the component in that plane; the other component will be parallel to the axis of rotation and need not be considered. Let θ be the angle it makes with the direction of A's motion, then $R = P . \cos\theta$ is the effective resistance, the other component of P merely producing pressure on the axis. As the body turns through an angle i the resistance R will be overcome through the arc AA', and, assuming in the first instance R constant, the work done will be—

Fig. 88.



$$\text{Work done} = R . AA' = R . CA . i.$$

But, drooping a perpendicular CN on P's direction,

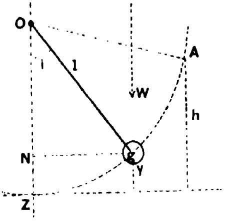
$$CN = CA . \cos\theta,$$

$$\therefore \text{ Work done} = P . CN . i = Mi,$$

where M is the moment of the resistance about the axis of rotation.

If there be many resistances then the same formula will hold if M be understood to mean the total moment of resistance.

Fig. 89a.

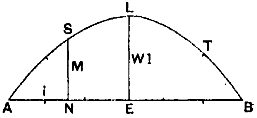


We can readily extend this to the case of a variable moment by the graphical process already described for a linear resistance, the base of the diagram now representing the angles turned through and the ordinates the corresponding moments. As an example take the case of a heavy pendulum swinging about an axis Q (Fig. 89a), let g be the centre of gravity, $Og = l$, and let it be swung through the angle i from the vertical, then the moment of resistance is

$$M = W.gN = W.l\sin i.$$

In Fig. 89b draw a curve on the base AB such that the horizontal ordinate AN at every point represents the angle i on the same scale that AB represents two right angles, while the vertical ordinate represents M. This curve will be the curve of resistance, and in the present case is a curve of sines of which the maximum ordinate LE is Wl. The angles being supposed reckoned in circular measure so that $AB = \pi$, the area of the diagram from A up to any point S will represent the work done. We can, however, in this example find this work otherwise, for g rises through the height NZ, and therefore if U be the work

Fig. 89b.



$$U = Wl(1 - \cos i).$$

By use of the integral calculus it can be verified that this is also the value of the area ASN.

It is not necessary that the axis of rotation should be fixed in estimating the work done during rotation, provided that the resistance be a couple, for then there is no pressure on the axis. An important example is that of a vessel floating in the water and steadily heeled over by the action of a couple M produced by external agency, or more frequently by shifting the weights on board in such a way that the displacement and trim remain constant. Then for each angle of heel this couple has a certain definite value which can be found either by calculation or by observation of the shift of the weights. The moment of resistance which is equal and opposite to M is called the Statical Stability of the vessel, and the curve of resistance drawn as above described is called the Curve of Stability.

According to the principles of this article the area ANS of the curve represents the work done in heeling the vessel over. This is called the Dynamical Stability, and as is shown elsewhere (see the chapter on Impact in Part IV.) represents the resistance to heeling over to that angle by a sudden gust.

For small angles of heel not exceeding 10°, or at most 15°, the statical stability (S) is given by the equation,

$$S = Wm \sin \theta,$$

where W is the displacement of the vessel and m is the "metacentric height," that is, the height of the "metacentre," through which the upward action of the buoyancy of the vessel passes at small angles of heel, above the centre of gravity. If this equation held good at large angles of heel the stability would increase to a maximum value Wm when $\theta = 90°$, and would not vanish until $\theta = 180°$. Such a curve is very exceptional, the maximum stability being in general much less than Wm, and occurring at a much smaller angle, while the vessel capsizes at an angle much less than 180°, known as the Angle of Vanishing Stability. An important typical case is when the actual curve is a reduced copy of a curve of sines given by the equation

$$S = W \cdot \frac{m}{k} \cdot \sin k\theta,$$

the maximum stability being now Wm/k, and the angle of vanishing stability π/k. The stability is then the same as that of a heavy pendulum of length m/k swinging through k times the actual angle of heel of the vessel, m being the metacentric height as before. The dynamic stability is evidently $1/k^{\text{th}}$ that of the pendulum, and consequently is given by

$$U = W \cdot \frac{m}{k^2}(1 - \cos k\theta),$$

a result which may also be reached by use of the integral calculus.

92. *Internal and External Work.*—In all that precedes, the position of a body has been changed by overcoming external resistances. All forces, however, arise from the mutual action between two bodies or between two parts of the same body, and every change of position must be with reference to some other body which is regarded as fixed. Work, then, consists in a change of relative position of two bodies notwithstanding a mutual action between the two which opposes the change. In raising weights the second body is the earth, but the pair of bodies may be such as occur in mechanism, and the mutual action between the two may be due to springs or an elastic fluid, or to the resistance of some body to separation into parts. In scissors, nut-crackers, bellows, and other similar instruments, the elements of the pair are exactly alike and their existence is recognized in popular language.

In reckoning the work done either body may be regarded as fixed, the result must be the same and will be unaffected by any movement of the pieces common to both; thus when air is compressed in a cylinder the work done depends on the pressure of the air and the amount of compression, not on the movements of the cylinder within which the air is contained. In other words, the motion to be considered is the motion of the pair as defined in Art. 46, p. 94, and the resistances consist exclusively of forces opposing this motion.

In every case where we have to do with a number of pieces connected in any way, we may distinguish between the resistances due to the mutual action between the pieces themselves and those due to the mutual action between the pieces and external bodies. The internal resistances require work to be done in changing the relative position of the pieces themselves, while the external resistances require work to be done in changing the position of each piece relatively to external bodies. These two kinds of work are called Internal Work and External Work respectively. In two cases we can at once foresee that the internal work will be zero, first when the pieces as disconnected, secondly when they are rigidly connected. Thus, for example, if a heavy mass of matter be raised, we need only consider the rise of the centre of gravity (Art. 89) if the mass be rigid; but if not, any change of form which occurs ought to be taken into account. In raising ordinary solid bodies and masses of earth the internal work may usually be disregarded.

93. *Energy. Principle of Work.*—Hitherto we have been speaking of the *resistance* which is being overcome during the process of doing work, let us now fix our attention on the *effort* which overcomes the resistance.

The forces arising from the mutual action between a pair of bodies, when not purely passive like the normal pressure between two surfaces in contact, are of two kinds. The first always oppose the motion of the pair; in other words, they are always resistances. Friction between two surfaces is the simplest example of this, and hence such actions are called Frictional Resistances. The second, on the other hand, promote or oppose the motion of the pair according to the direction in which the motion is taking place, so that a resistance becomes an effort when the direction of motion is reversed. Such actions are conveniently described as Reversible; and systems of bodies, in which they occur, possess, when the parts are suitably disposed, the power of doing work. This power is called ENERGY. As examples of bodies possessing energy may be taken a raised weight, a compressed spring, or steam of high pressure. Change of velocity in a moving body likewise gives rise to efforts and resistances, but this is a matter for subsequent consideration. For the present we suppose all bodies with which we have to do to be in a state of uniform motion, or to move so slowly and steadily that no sensible action of this kind can arise.

Energy is measured by the quantity of work which it is capable of doing, and the process called doing work may also be described as the exertion or expenditure of energy, so that we write

$$\text{Energy exerted} = \text{Work done.}$$

If the effort which is being exerted and the resistance which is being overcome be applied to the elements of the same lower pair, as when a weight is lifted vertically or a spring wound up, the effort and the resistance are equal, and the equation shows that the energy exerted by an effort is the product of the effort and the space through which it is exerted. Thus all the examples given above of the doing of work will also serve as examples of the exertion of energy simply by supposing the direction of motion reversed. In short, the exertion of energy and the doing of work are merely different aspects of the same process.

In this case the effort and the resistance may be regarded as applied at the same point, but the equation has a much wider application than this, for it is equally true if the points of application be different, provided only that they are rigidly connected. Thus, for example, if we dig in the ground, the energy we exert at the handle of the spade is--if the spade be perfectly rigid—exactly equal to the work done at the blade. This can be shown to be a necessary consequence of the forces we are considering being balanced, and the equation may be regarded as a concise statement of the conditions of equilibrium of forces applied to a rigid body. It is preferable, however, for our purposes to regard it as the simplest case of a fundamental mechanical principle continually verified by experience. This principle may be called the PRINCIPLE OF WORK.

We have now a means of transferring the power of doing work, that is to say, energy from one place to another; evidently we are not restricted to one piece, as in the case of the spade. We may make use of a series of pieces through which energy may be transferred from piece to piece in succession: and if there were no frictional resistances to the relative motion of the pieces, there would be no loss of energy in the process. Thus the principle of work is true when the points of application of the effort and the resistance are mechanically connected in any way. Frictional resistances however absorb a portion of the energy whenever any relative motion occurs which they tend to prevent, and therefore a certain loss always accompanies the transmission of energy. Nevertheless the principle of work still holds good if overcoming friction be reckoned as part of the work done.

It may here be remarked that though frictional resistances are never a source of energy, yet friction may, like normal pressure between surfaces, transmit energy, and hence, in cases where one only of the bodies between which it is exerted belong to the set of bodies we are considering, may be an effort by means of which work is done on the set. Thus, for example, in the case of a shaft driven by a belt, the whole power of the engine is transmitted by friction-closure between the belt and the pulleys; and if we consider the shaft alone apart from the rest of the mechanism, the friction may be regarded as the effort which drives the shaft. We cannot however in such cases properly speak of the friction as exerting energy; the source of energy is the steam, or other motive power, and the friction merely transmits it in the same way as the pressure between a connecting rod head and the crank pin transmits energy to the crank shaft. Nevertheless in both of these cases the phrase "energy exerted" may be used conveniently, though "energy transmitted" would be more precise.

If a piece of material through which energy is transmitted yield under stress applied to it, as in fact it always does, the energy exerted will not be equal to the work done. Either the change of relative position of the several parts of the piece will require work to be done in order to overcome the mutual actions between the parts which resist the change, or, conversely, those mutual actions exert energy during the change. In the first case the work is done at the expense of the energy transmitted; in the second the piece of material is a source of energy which increases the energy transmitted. In perfectly elastic material the mutual actions are reversible, and any energy exerted in overcoming them is stored up in the piece and recovered when the piece resumes its original form, as in the case of a watch spring. (Compare Art. 98.)

94. *Machines.*—A mechanism becomes a machine if we connect together two of its elements by a link capable of changing its form or dimensions, and so moving the mechanism, notwithstanding a resistance applied by a similar link connecting two other elements. In compound mechanisms some or all of the component simple mechanisms may be distinct machines, as will be seen farther on.

The elements connected may be called the "driving pair" and the "working pair," and these pairs often, though by no means always, have one element common, namely the frame-link of the mechanism. The driving-link is the source of energy. As examples, we may take steam which connects the piston and cylinder which form the driving pair in a steam engine, or gravity which, as in Art. 62, is to be conceived replaced by a link exerting the same effort. The working link is gravity in cranes and other hoisting machines, or a piece of material the deformation of which is the object of the machine, as in the case of machine tools.

In addition to the driving and the working links, the force of gravity acts on all the parts of the machine, and frictional resistances have to be overcome; but these are matters for subsequent consideration.

The driving and working pairs are very frequently kinematic pairs of the lower class. Let us suppose them in the first instance sliding pairs. Let the driving pair move through a space x, then the working pair will move through a space y, which is in a certain definite proportion to x depending on the nature of the mechanism. Let P be the driving effort, which, by taking x small enough, can be made as nearly uniform as we please; and let R be the resistance opposing the motion of the working pair, then

$$\text{Energy exerted} = Px; \quad \text{Work done} = Ry,$$

and these must be equal, therefore

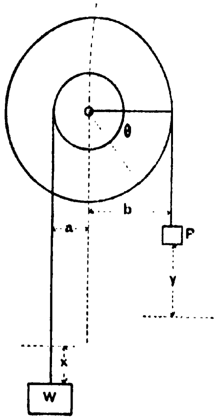
$$\frac{P}{R} = \frac{y}{x} = \frac{\text{Velocity of Working Pair}}{\text{Velocity of Driving Pair}};$$

from which it appears that the ratio of the effort to the resistance, or as we may briefly call it, the "force ratio," is the reciprocal of the velocity-ratio of the driving and working pairs. In works on mechanics this is also known as the Principle of Virtual Velocities.

If the pairs be turning instead of sliding pairs, then the effort and resistance are moments, and the velocities will be angular; and if one pair be sliding, the other turning, a suitable "radius of reference" must be selected (p. 94) to compare the motions and the forces, but the same principle holds good.

In the simplest machines, known frequently as the "mechanical powers," we have a 2 or 3-linked chain, so that the driving pair and working pair are identical or very closely connected. But they may belong to two or more distinct machines connected by a long train of mechanism and may have no common link. We are not then restricted to the consideration of the whole process of transmission; any intermediate pair upon which an effort or a resistance is being exerted, either directly or by transmission, may be regarded as a driving or a working pair in applying the principle. In all cases it must be carefully remembered that the effort and the resistance arise from the mutual action between the elements, each consisting of two equal and opposite forces, as will be *further described in a later chapter (Ch. XI.). Either of these as before measures the magnitude of the action opposing or promoting the motion of the pair.*

Fig. 90.



95. *Verification of the Principle of Work in Special Examples.*—We will now take some examples to illustrate and verify the principle of work, neglecting friction.

(1) Take the common wheel and axle. Suppose P to be just sufficient to lift the weight W, so that the two forces exactly balance one another. Now let P descend through the distance y (Fig. 90), and W rise through the corresponding distance x. As P falls it is said to exert energy. Energy exerted $= Py$. This is employed in overcoming the resistance to the rise of the weight W. Work done $= Wx$. The principle of work asserts that Energy exerted $=$ Work done, that is, $Py = Wx$.

Suppose the wheel and axle to turn through the angle θ, then $y = b\theta$ and $x = a\theta$. Then, in order that the weights P and W may statically balance one another, $Pb = Wa$; from which it follows that $Py = Wx$, verifying the principle of work.

Also, we may write,

$$\frac{P}{W} = \frac{a}{b} = \frac{x}{y} = \frac{V}{v},$$

where v, V are the velocities of P, W respectively, thus showing that the force-ratio is the reciprocal of the velocity-ratio.

In this simple example both the force-ratio and the velocity-ratio remain constant throughout the movement. In general this will not happen.

(2) Take the case of the mechanism of the steam engine for an example. Neglect friction and let the driving pressure on the piston be P. A thrust which we will call S will be produced along the connecting rod and transmitted to the crank pin as shown in Fig. 91. At the crank pin this force S may be resolved into two components,

one acting along the crank arm and the other, R, perpendicularly to it. The last alone will tend to turn the crank, the other component producing only a pressure on the shaft immediately balanced by the pressure of the bearings on the journals of the shaft.

This component R which tends to turn the shaft is called the *crank effort*. If the turning effort on the crank is perfectly balanced at all points of its revolution by some suitable resistance, then the resisting force which must be applied at the crank pin at right angles to the crank arm in order to balance perfectly the pressure of the steam on the piston must be equal and opposite to the crank effort. The force-ratio will be P/R. We have, with the notation employed in Chapter V., $S \cos \theta = P$ and $S \sin (\theta + \phi) = R$.

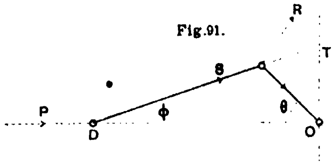


Fig. 91.

$$\text{Thus } \frac{R}{P} = \frac{\sin(\theta + \phi)}{\cos \phi} = \frac{\sin OBT}{\sin OTB} = \frac{OT}{OB}.$$

That is, the crank effort is to the steam pressure as the intercept OT is to the crank arm OB.

But we have previously shown (see p. 101) that this fraction expresses the velocity-ratio of piston to crank pin; hence we have again found in this case that the force-ratio is the reciprocal of the velocity ratio, and the curve which we previously drew to represent the varying velocity of the piston, the crank pin moving uniformly, will represent also the varying crank effort, the pressure of the steam on the piston being uniform throughout the stroke. It is therefore described as the Curve of Crank Effort.

(3) The same thing may be proved to be true for every mechanism, the forces acting on which balance one another. In some cases it may be easier to determine the force-ratio than the velocity-ratio or *vice versa*. In any case, either may be inferred by taking the reciprocal of the other. As an additional example take the case of two pieces driving one another by simple contact (Fig. 92). We have already found the velocity-ratio by a direct process (p. 152), but we may also determine it in the following way. When A presses on B there is a resistance R equal and opposite to the pressure, and normal to the portions of the surfaces in contact, if we suppose no friction to exist. Drop

perpendiculars p_A and p_B on the common normal. Then the moment of the driving pressure R which A exerts on B or the turning moment due to $A = M_A = Rp_A$. Similarly the moment of the resisting force which B exerts on A or the moment of resistance to turning which B opposes to $A = M_B = Rp_B$. Hence it appears that

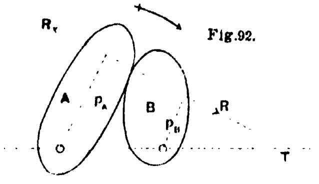


Fig.92.

$$\frac{\text{Driving moment}}{\text{Resisting moment}} = \frac{M_A}{M_B} = \frac{p_A}{p_B}.$$

But we have previously proved that this fraction is the angular velocity-ratio of the piece B to the piece A, and thus we show that the moment-ratio is the reciprocal of the angular velocity ratio.

96. *Periodic Motion of Machines.*—One of the most essential characteristics of a machine is the periodic character of its motion. Each part goes through a cycle of changes of position and velocity and returns periodically to its original place. When moving steadily the periods are equal and the velocity of each piece is the same at the beginning and end of each period. That this may be the case it is not necessary that the driving effort should balance the working resistance in every position: on the contrary, this seldom happens; it is sufficient if the mean effort be equivalent to the mean resistance, or as we may otherwise express it,

Energy exerted during a period = Work done in the period;

a condition which always governs the action of a machine in steady motion. In reckoning the energy and work the action of gravity on any piece of the machine may be omitted, for, if the piece rise through any height during one part of the period, it will fall through an equal height during another part. The work done consists partly of the work which the machine is designed to do, and partly of frictional resistance to the relative motion of the parts of the machine, or, in other words, of Useful Work and Waste Work. The ratio of the useful work to the energy exerted is called the Efficiency of the machine and its reciprocal the Counter-Efficiency. The efficiency of a machine depends partly on the kind of machine and partly on the speed, as will be explained in the chapter devoted to frictional resistances (Chap. X.). In estimating the power required to drive a machine a value is assumed for the efficiency derived from experience of machines of the same or nearly the same type. Examples will be given hereafter.

97. *Power. Sources of Energy.*—The sources of energy are—

(1) Living agents;

(2) Gravity acting usually by means of falling water;

(3) Springs and elastic fluids;

(4) Gunpowder and other explosive agents.

The energy thus derived may be traced further back to the action of heat and chemical affinity, and we may add to the list electric and magnetic forces, but the foregoing is a sufficient statement for our present purpose.

In general, the motion and effort which are proper to the source of energy, and to which the driving pair must be adapted, are entirely different from the motion and the resistance necessary in the working pair. Besides which the work will generally be required to be done at various places more or less distant from the source of energy. To connect the source and the work mechanism is therefore necessary, which (1) receives energy from the source and converts into a form suitable for transmission and distribution; and (2) receives the transmitted energy and adapts it to the work to be done. The same machine may serve both these purposes, especially when a living agent is the source of energy, as in a crane worked by hand, a sewing machine or a lathe driven by the foot. But in most cases distinct machines are employed, one of which receives energy directly from the source, and is described as a Prime Mover, or more briefly a Motor, while the rest receive energy from the motor, either directly or by a train of connecting mechanism, and adapt it to the work. A machine then effects something more than mere transmission of energy; it is directly connected with the source or the work, and converts the energy it receives into a form in which it can be utilized. Thus in a factory the engine is a machine which adapts the energy of the steam to the purpose of driving a shaft; the loom or the mule are machines which adapt the energy transmitted to them to the purposes of weaving or spinning, but the train of belt or wheel gearing distributing the energy through the factory is not a machine, for it is employed solely for transmission purposes. Theoretically the connection between the source of energy and the work might be effected by a single machine; the separation into distinct machines connected by a transmitting train is simply an augmentation (p. 139) adopted for constructive reasons. The variety of movements of which a living agent is capable renders the separation less necessary.

The rate at which energy is exerted is called Power; it is this which measures the value of a source of energy and the expense of the work which is being done. The ordinary unit of measurement is the conventional horse-power of 33,000 foot-pounds per minute, or 550 per second, a quantity greater than the working power of an ordinary draught horse on the average of a day's work, except under the most favourable conditions (see Appendix). The unit of power employed universally on the Continent is somewhat less, being 75 kilogrammetres per second or 32,550 foot-pounds per minute. In measurements of electrical power the "watt" is often used; 1000 watts, a quantity also known as a "kilowatt," being 1·34 horse-power. For small powers the watt is a convenient unit.

In prime movers the effort may generally be regarded as applied at a point which moves with a known mean velocity; then the horse-power is given by the equation

$$\text{H.P.} = \frac{PV}{33{,}000},$$

where P is the mean value of the effort in lbs. and V the mean velocity in feet per minute.

In machines driven from a prime mover the effort is generally a moment M which exerts the energy $M2\pi$ in every revolution of a driving shaft. We then have

$$\text{H.P.} = \frac{M \,.\, 2\pi n}{33{,}000},$$

where M is the mean moment and n the revolutions per minute.

98. *Reversibility. Conservation and Storage of Energy.*—The resistance overcome at the working point may be either frictional as in machine tools or reversible as in machines for raising weights. In the second case, if the machine were stopped and set in motion in the reverse direction it would, if friction could be neglected, work equally well, the driving effort and working resistance would be interchanged, and constructive modifications might be required, but otherwise the action is unaltered. This may be described by saying that the machine is Reversible. Many machines actually occur in both their direct and their reversed forms; thus a pump is a reversed hydraulic motor. Hence it appears that in reversible machines the power of doing work, that is to say, energy, is not lost after being exerted, for by reversing the machine it may be employed a second time. Thus it is that we describe the action of reversible machines as a transfer of energy, and are led to conceive of energy as indestructible, and speak of it as if it were independent of the bodies through which it is manifested. No machine, indeed, is completely reversible, for in all cases frictional resistances occur to a greater or less extent, while many machines are completely non-reversible; but we shall see as we proceed that even then energy is not lost but only converted into another form, so that we have in reversible machines the first and most simple example of the great natural law, called the Conservation of Energy. The importance of reversibility as a test of maximum efficiency will be seen more fully hereafter.

Again, we can store up energy and use it as required when it is inconvenient to resort to any of the usual sources. For example, by a few turns of the watch key we store energy in the mainspring which is supplied at a regular rate to the watch throughout the day. So the hydraulic accumulator (Part V.) receives energy from the pumping engines and supplies it at irregular intervals to the hydraulic machines which lift weights and move gates in a dockyard or work the guns in a ship of war.

A large part of what follows in the present work is merely a development of what has been said here: in the succeeding chapters of the present division we consider machines comprising solid elements only, while in a future division we shall consider the transmission and conversion of energy by means of fluids. The simpler machines are treated in much greater detail with numerous additional examples in a smaller treatise by the author of this work and Mr. J. H. Slade.*

EXAMPLES.

1. A waggon weighs 2 tons and its draught is $\frac{1}{50}$th of its weight. Find the work done in drawing it up a hill 1 in 20, half a mile long. Find also how long three horses will take to do it, supposing each horse to work at the rate of 16,000 foot-pounds per minute. Work done = 370 ft.-tons. Time occupied = 17′ 15″.

2. A force of 10 lbs. stretches a spiral spring 2″, find the work done in stretching it successively 1″, 2″, 3″, etc., up to 6″. *Ans.* 2½, 10, 22½, 40, 62½, and 90 inch-lbs.

3. Find the H.P. required to draw a train weighing 200 tons at the speed of 40 miles an hour on a level, the resistance being estimated at 20 lbs. per ton. Find also the speed of the train up a gradient of 1 in 100, the engine exerting the same power.

Ans. H.P. required = 426⅔. Speed up the incline = 18·87 miles per hour.

4. The resistance of H.M.S. "Iris" at 17 knots is estimated at 40,000 lbs., what will be the H.P. required simply to propel the ship? Find also in inch-tons the moment, on each of the twin screw shafts, equivalent to this power, the revolutions being 80 per minute.

Ans. H.P. required = 2088. Moment on each shaft = 367 inch-tons.

5. The curve of stability of a vessel is a common parabola, the angle of vanishing stability 70°, and the maximum moment of stability 4,000 ft.-tons. Find the statical and dynamical stabilities at 30°.

Ans. Statical stability = 3·918 ft.-tons. Dynamical stability = 1·283 ft.-tons.

6. Verify the principle of work, neglecting friction, in: (*a*) The differential pulley (Art. 59). (*b*) A pair of 3-sheaved blocks. (*c*) The hydraulic press (Art. 62).

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* *Lessons in Applied Mechanics.* Macmillan. 1891.

7. From the results in question 6, p. 103, deduce the crank efforts for the given positions of the piston and the mean crank effort, supposing the effective steam pressure on the piston 20 tons and neglecting friction.

Crank effort at quarter stroke in the { forward stroke = 18·4 tons. Mean = 12·74 tons.
{ backward ,, = 16·6 tons.

8. Show that the efficiency of a machine is equal to the velocity ratio multiplied by the force ratio.

SECTION II. UNBALANCED FORCES (KINETICS).

99. *Kinetic Energy of Translation. Sliding Pair.* We now proceed to consider the cases in which efforts or resistances arise from the changes of velocity of the parts of a system, which changes thus become a source of energy or require energy in order to produce them. The commonest observation is sufficient to show the importance of such cases: a cannon ball possesses a great power of doing work, and a railway train requires energy to be exerted by the steam to obtain the requisite speed, quite irrespectively of that necessary to maintain the speed when once produced.

First, suppose a weight under the action of gravity only. Unless it be supported by a vertical force exactly equal to the weight it will fall with a gradually increasing velocity. Let it be wholly unresisted by external bodies, let it start from rest and fall through a height h, then, whatever the material, we know that it will acquire a velocity v given by the formula

$$v^2 = 2gh,$$

where g is a number measuring the acceleration of the weight which, for velocities in feet per second, ranges from 32·1 at the equator to 32·25 at the pole, and having intermediate values at other points on the earth's surface according to the intensity of gravity at the point. The average value 32·2 is usually adopted for this important constant, and the height h is called the "height due to the velocity."

During the whole fall, the weight W of the body has been exerting an effort upon it which overcomes an equal resistance occasioned by the change of velocity which is taking place; thus an amount of energy has been exerted, and an amount of work done equal to Wh. Resistance of this kind is of the reversible kind, for if we imagine the weight, after reaching the ground, projected up again with the same velocity, it will, if not otherwise resisted, attain the height from which it originally fell. Hence we describe the weight as possessing energy, and the amount it possesses when moving with velocity v is

$$Wh = \frac{Wv^2}{2g}.$$

Energy due to motion is called Kinetic Energy, to distinguish it from that kind of energy considered previously, which is a consequence of the relative position of the parts of a system, and which is called Potential Energy. The kinetic energy of a body depends only on the velocity of each of the particles of which it is made up, not on the direction of its motion nor on the way in which its motion has been produced; and the energy exerted in changing the motion of a body is always represented by an exactly equivalent increase of kinetic energy, whether the effort be uniform or variable, or whether its direction coincide with the direction of motion or not.

The fall of a weight under the action of gravity is a particular case of the motion of a sliding piece under the action of a known force P in the direction of motion, the other element of the sliding pair being fixed. The piece here has a simple motion of translation, each particle traversing the same space with the same velocity. Let the velocity change from V to v as the piece moves through the space x, then equating the change of kinetic energy to the energy (Px) exerted by the force P

$$\frac{Wv^2}{2g} - \frac{WV^2}{2g} = Px,$$

an equation which is true whatever be the size, shape, or material of a sliding piece of weight W. The equation may be written

$$v^2 - V^2 = 2\frac{P}{W}gx,$$

showing that the piece moves with uniform acceleration as in the case of a falling weight, the magnitude of the acceleration being

$$f = \frac{P}{W}g.$$

If the sliding piece be under the action of a force S which is not in the direction of motion, then we know (p. 180) that the energy exerted by S is the same as if its resolved part P in the direction of motion existed alone. The acceleration of the sliding piece therefore is independent of the component of S perpendicular to the direction of motion. These results are, of course, in direct accordance with the laws of motion.

If P be a resistance instead of an effort, then work is done at the expense of the kinetic energy which is now diminished. If P be variable we must represent it graphically by a curve as in Art. 90, and it should be especially remarked that the ordinate of the curve of areas deduced in Art. 31 will, on affixing a suitable scale, and measuring the ordinates from a suitable base line, represent the height due to the velocity, or, as it may otherwise be described, the "height equivalent to the kinetic energy" of the body.

100. *Partially Unbalanced Forces. Principle of Work.*—Again, the effort which is changing the motion of the body may be partly balanced by an external resistance to which the body is subject. If this be the case we can imagine it separated into two parts, a part which is, and a part which is not, balanced. The energy exerted by the first is employed in overcoming the external resistance, while that exerted by the second is employed in increasing the kinetic energy of the body. Or the resistance may be greater than the effort, then the excess is overcome at the expense of the kinetic energy of the body, the velocity of which now diminishes.

In the present treatise we shall use the phrases "energy exerted" and "work done" only in reference to efforts and resistances other than those due to inertia, subject to which convention, we may state, the principle of work as applied to cases where the forces are partially unbalanced, as follows

Energy exerted = Work done + Change of Kinetic Energy.

In this statement the work done may be greater or less than the energy exerted. In the first case the change of kinetic energy is a decrease, in the second an increase.

Not only does this principle apply to a single body, but—subject to the observations of the preceding section—to a set of bodies mechanically connected in any way, provided that one of them be fixed to the earth; or, in other words, that a body of great mass like the earth be one of the set. A single body is in reality one of a set of two bodies, the other being the earth. When no one of the set predominates over the rest it is necessary to consider further how the kinetic energy should be reckoned: for the present, however, we shall suppose this condition satisfied.

A simple case is that of Atwood's machine. Let the descending weight P be greater than the rising one Q. Neglecting friction, the excess sets the two weights in motion. Let P descend through a distance y, then Q rises through the same distance, and therefore

$$\begin{aligned}\text{Energy exerted} &= Py.\\ \text{Work done} &= Qy.\end{aligned}$$

Let v be the velocity of the two weights; then supposing them to start from rest,

$$\text{Kinetic energy acquired} = (P+Q)\frac{v^2}{2g}.$$

From principle of work

$$Py = Qy + \frac{(P+Q)v^2}{2g} \; ; \quad \therefore \; v^2 = \frac{P-Q}{P+Q} 2gy.$$

The law of increase of velocity is, therefore, the same as that of a body falling freely, but the rate of increase is less. This formula is the same as that obtained by other methods, and we have therefore here a verification of the principle of work.

In applying the principle any pair of elements may be a driving or a working pair, whether or not one of them be the fixed link attached to the earth. Thus, for example, in a locomotive the steam exerts an amount of energy measured by its pressure and by the motion of the cylinder piston pair which it drives. This energy is employed in drawing the train while overcoming frictional and other resistances which oppose the motion of the various pairs making up the whole mechanism. Any excess or defect is represented by a change of kinetic energy in the whole train, inclusive of the mechanism of the locomotive estimated relatively to the earth as fixed. A rotating cylinder engine, in which the steam cylinders, instead of being fixed to the frame, are attached to a rotating fly-wheel, furnishes another instructive example.

101. *Kinetic Energy of Rotation. Turning Pair.*—Instead of a single body, every point of which moves with the same velocity, suppose we have a system of bodies, and we require to know the total kinetic energy of the system. The direct method is to find the energy of each separate particle of the system and add the results. In the particular case of a rotating rigid body we are able to express the result of the summation in a convenient and simple form. First consider a ring of small section rotating about an axis in the centre perpendicular to its plane. Every portion of the ring will move with the same velocity, v say, and the kinetic energy of the ring may, as before, be written $Wv^2/2g$.

We may express this another way, as follows:—If n be the revolutions per second, and a the radius, $v = 2\pi an$,

$$\therefore \; \frac{Wv^2}{2g} = W \cdot \frac{4\pi^2 n^2}{2g} a^2.$$

If the ring is not complete, but W is the weight of a portion which has the same centre of rotation, the expression will still hold.

Now, suppose we have a body consisting of a number of particles rigidly connected together, rotating about a centre O, at n revolutions per second.

Let the weights of the particles be w_1, w_2, w_3, w_4, etc.
rotating about O *at distances* y_1, y_2, y_3, y_4, etc.

By adding together the results for each particle, we obtain for the kinetic energy of the system,

$$\frac{4\pi^2n^2}{2g}(w_1y_1^2+w_2y_2^2+w_3y_3^2+\text{etc.}).$$

Now suppose a is such a radius that

$$a^2=\frac{w_1y_1^2+w_2y_2^2+w_3y_3^2+\text{etc.}}{w_1+w_2+w_3+\text{etc.}},$$

then substituting, we may write

$$\text{Kinetic energy}=\frac{4\pi^2n^2}{2g}(w_1+w_2+w_3+\text{etc.})a^2=\frac{4\pi^2n^2}{2g}Wa^2.$$

By this method we are always able to reduce any such system of rigidly connected particles to a ring sometimes called the *Equivalent Fly-Wheel*, and the radius a is called the *Radius of Gyration*. The quantity Wa^2/g is usually called the *Moment of Inertia*, and denoted by the symbol I. The quotient W/g measures the Inertia of the body, as will be explained hereafter (p. 263), but is commonly called the *Mass*.

However numerous the particles are, the expression obtained above will hold, and so will be true if they are sufficient in number to make up a solid body. In a continuous body, the separate weights w_1, w_2, w_3, etc., must be taken indefinitely small and close together to get accurate results, and the results of the summation may be most conveniently arrived at by the use of the calculus. The symbol I, but for the introduction of the mass as a factor would have the same meaning as in Chapter XII., and hence all the results there given may be used here for thin plates simply by multiplication by the mass of a unit of area. In addition, the following simple cases will be sufficient. The fourth is a particular case of the second.

1. Solid cylinder rotating about its axis. Radius $=r$. $\quad a^2=\dfrac{r^2}{2}$
2. Rectangular parallelopiped rotating about an axis. Diagonal of either end $=2d$. $\quad a^2=\dfrac{d^2}{3}$
3. Sphere rotating about a diameter. Radius $=r$. $\quad a^2=\dfrac{2r^2}{5}$
4. Rod rotating about an axis perpendicular to it through one end. Length $=l$. $\quad a^2=\dfrac{l^2}{3}$

In other cases such as occur in practice, the body is generally too irregular and complex in form to render mathematical formulæ useful;

we then apply the rule given in Chapter XII for plane areas, which by a similar process can readily be extended to solids. That is to say, if I be the moment of inertia of a body about any axis, I_0 that about a parallel axis through the centre of gravity at a distance h,

$$I = I_0 + mh^2,$$

where m is the mass of the body. In applying this rule the body is cut up into portions to which the values just given apply exactly or with sufficient approximation, just as in the chapter cited.

In estimating the kinetic energy of a fly-wheel, which consists of rim, arms, and boss, since the rim is by far the most important part for storing energy, it is generally sufficient to consider it alone. If it be desired to take the remaining parts into account, an addition of about one third the weight of the arms may be made to the weight of the rim. The combined effect of arms and boss is said to amount to an addition of, on the average, about 8 per cent. to the weight of the rim.

In any case of the motion of a rotating piece the other element of the turning pair being fixed, a change in the kinetic energy of the piece can only be produced by the action of forces which have a moment M about the axis of the pair. If M be constant, the energy exerted as the piece turns through an angle θ will be $M\theta$. Suppose the angular velocity at the same time to change from A_0 to A, then equating the change of kinetic energy to the energy exerted,

$$\frac{Wa^2A^2}{2g} - \frac{Wa^2A_0^2}{2g} = M\theta,$$

a formula exactly corresponding to that already given for a sliding piece (p. 197), and showing that the angular acceleration is uniform.

If M be variable we have only to represent it by a curve, as on page 184, and it should be observed that as before the ordinate of the curve of areas will represent the change of kinetic energy. The scale and base line of this curve may conveniently be so taken that the ordinate shall represent the height,

$$H = \frac{a^2A^2}{2g},$$

which may be described as the "height equivalent to the kinetic energy" of a rotating body.

Thus the motion of a rotating piece is governed by the same laws as the motion of a sliding piece, the same diagram applying to both cases. Examples will be given presently.

It is often convenient to write for brevity

$$h = \frac{g}{A^2} = \frac{2936}{N^2},$$

a quantity having a definite physical meaning, being the height in feet of a revolving pendulum rotating (Chap. XI.) with angular velocity A or at N revolutions per minute. It may be described as the "height due to the revolutions." The formula just given for the height equivalent to the energy of rotation becomes

$$H = \frac{a^2}{2k}.$$

102. *Kinetic Energy of the Moving Parts of a Machine.*—If the body have a motion of translation, combined with a motion of rotation about an axis through its centre of gravity, the two motions (p. 118) are equivalent to a rotation about a second axis parallel to the first. Applying the rule just given, it at once follows that the total kinetic energy is the sum of that due to the translation and the rotation taken separately, so that the whole can be found by preceding rules. As an example of the use of this principle, consider the case of a ball rolling down an inclined plane, the ball and plane being sufficiently rough that slipping does not take place between them; and suppose the resistance to rolling, called the rolling friction, is insensible. In this case the whole energy due to the descent of the ball is employed in generating kinetic energy in the ball, which will be stored in it by virtue of its two motions of translation and rotation. Let V be the velocity of translation, A the angular velocity, r the radius of sphere; then since no slipping occurs $V = Ar$.

Let the ball descend through a vertical height h, then the energy exerted is Wh, equating which to the kinetic energy stored we obtain

$$Wh = \frac{WV^2}{2g} + \frac{WA^2a^2}{2g},$$

where the ratio of gyration a is given by $a^2 = \frac{2}{5}r^2$.

$$\therefore\ Wh = \frac{WV^2}{2g} + \frac{WA^2}{2g} \cdot \frac{2}{5}r^2 = \frac{7}{5}W\frac{V^2}{2g}.$$

$$\therefore\ V^2 = \frac{5}{7}2gh.$$

Thus the velocity of the ball will be less than if it simply slid down the plane without rotating in the proportion $\sqrt{5} : \sqrt{7}$.

In a carriage on wheels, and in many other cases, the total kinetic energy may, as in the preceding example, be found by adding a suitable percentage to the energy of translation.

The total kinetic energy of the moving parts of a machine in any position may be found by drawing a diagram of velocity for that position in the manner explained in Chaps. V. and VI. Each part may be divided into a number of small portions, and the centre of each portion may be laid down on the diagram, as explained on page 120. If now the diagram be imagined to represent a set of particles rigidly connected, of masses equal to those of the particles in question, one-half the moment of inertia of those particles about the pole of the diagram must be the total kinetic energy required; the radius vector of each particle representing the velocity of the corresponding portion.

103. *Conservation of Energy.*—The principle of work may also be stated in another form, which, though not so convenient in practical applications, is much employed by scientific writers. It has already been explained that, when there are no frictional resistances, the power of doing work (energy) exerted in doing a given amount of work is not lost but merely transferred from one place to another (Art. 98), while it appears from the present section that any energy exerted in changing the motion of a body is represented by an exactly equivalent amount of kinetic energy stored up in the moving body; hence it follows that in any dynamical system, which receives no energy from without and supplies none to external bodies, the total amount of energy is always the same if there be no frictional resistances. We express this by the equation

$$\text{Kinetic Energy} + \text{Potential Energy} = \text{Total Energy} = \text{Constant},$$

and call it the principle of the Conservation of Energy. In all actual motions frictional resistances occur which gradually absorb the energy, but this process is accompanied by the generation of heat which is equivalent to the energy absorbed, a fact which leads us to conclude that heat is a form of energy, and that the principle still holds good.

A good illustration of the principle in this form is furnished by the motion of a body which oscillates to and fro about a mean position, a question which occurs in a great variety of forms.

(1) Let a body oscillate in a straight line under the action of a force P which varies as the distance x from a fixed point about which the oscillation takes place. For example, a weight suspended by a long elastic string when disturbed vertically from its position of equilibrium vibrates under the action of such a force, arising from the difference between the weight W downwards, and the tension T of the string upwards. This case will be fully considered in a later chapter (Chap. XVI.); for our present purpose it is sufficient to take

$$P = W - T = \mu W x,$$

where μ is a coefficient measuring the intensity of the force. The curve of effort is now a sloping straight line as shown in Fig. 99, p. 223 of the next chapter, C being the point about which the weight vibrates through the distance $AA = 2a$. Considering any position N, the work which must be done to move the weight from C through the space $CN = x$ will be represented by the area of the triangle on the base CN, and the potential energy reckoned from C is therefore $\frac{1}{2}W\mu x^2$.

If then the velocity be V we shall have

$$\text{Total Energy} = \frac{WV^2}{2g} + \tfrac{1}{2}\mu W x^2.$$

By the principle we are now considering this must be constant through the whole motion, which consists in a continual interchange between the kinetic and potential energies. It is of course supposed that the resistance of the air is neglected: this is a resistance of the frictional kind and continually absorbs energy from the weight which is thus at last reduced to rest unless it receives energy from without.

Since $V = 0$ when $x = a$ the equation may be written

$$V^2 = \mu g(a^2 - x^2).$$

To represent the velocity graphically, upon AA as diameter describe a semicircle AQA, draw the ordinate $QN = y$ and join CQ, then $y^2 = a^2 - x^2$, and therefore

$$V = \sqrt{\mu g}\,.\,y,$$

that is, the velocity of the weight is proportional to the ordinates of a semicircle. The curve of areas corresponding to the curve of effort which, as we have before found in a different problem (Ex. 5, p. 66), is a parabola, gives the kinetic energy, but it is not shown in the diagram, not being required for our present purpose. Let V_0 be the velocity with which Q moves as N returns with velocity V towards C, then since V_0 when resolved parallel to AA must be equal to V,

$$\frac{V_0}{a} = \frac{V}{y} = \sqrt{\mu g},$$

from which it appears that CQ rotates with uniform angular velocity, describing a complete circle in the

$$\text{Period} = \frac{2\pi}{\sqrt{\mu g}},$$

which gives the time of a complete oscillation to and fro.

As the formula shows, the period does not depend on the extent of the oscillation, but only on the intensity of the force as measured by the magnitude of the coefficient μ. If we call c the distance from the centre at which the force is equal to the weight of the vibrating mass, then $c = 1/\mu$ and the formula becomes

$$\text{Period} = 2\pi\sqrt{\frac{c}{g}},$$

being as will be seen presently the same as that of a pendulum of length c.

(2) Next take the case of a rotating piece vibrating backwards and forwards about a mean position under the action of a couple of magnitude proportional to the angle turned through. For example, the balance of a watch vibrating under the action of the balance spring which exerts a moment M, proportional to the angle θ turned through from the position of rest. Here the moment is given by the equation

$$M = \mu W\theta,$$

μ as before being a coefficient measuring the intensity of the moment. Writing now r for the radius of gyration of the wheel and referring to page 201 the equation of energy will be exactly as in the case just considered of a sliding piece,

$$\frac{Wr^2A^2}{2g} + \tfrac{1}{2}\mu W\theta^2 = \tfrac{1}{2}\mu W\theta_1^2.$$

The motion is now represented by the same diagram as before in which AA is now $2\theta_1$, the whole angle through which the wheel oscillates in the

$$\text{Period} = \frac{2\pi r}{\sqrt{\mu g}}.$$

In this as in the preceding case the time does not depend on the extent of the oscillation, and the oscillations are therefore described as "isochronous." In a wheel, however, the period also depends on the radius of gyration r: the coefficient μ is here a certain length, being the leverage at which W must act to balance the moment M at unit angle in circular measure, and the length of the corresponding pendulum is r^2/μ.

(3) In the two preceding cases the motion is of the kind called "harmonic," let us next consider a pendulum vibrating to and fro under the action of gravity. We have now a rotating piece, the radius of gyration of which is r (suppose), oscillating about a horizontal axis at a distance L from the centre of gravity g. Referring to page 184 it will be seen that in any position inclined at an angle θ to the vertical, the potential energy reckoned from the lowest position is

$$U = WL(1 - \cos\theta).$$

Applying once more the *principle of the conservation of energy* we have as the equation of energy

$$\frac{Wr^2A^2}{2g} + W.L(1 - \cos\theta) = W.L(1 - \cos\theta_1),$$

θ_1 being the extreme angle reached, that is to say, half the total angle of swing. The equation may be written

$$\frac{Wr^2A^2}{2g} + W.2L\sin^2\frac{\theta}{2} = W.2L.\sin^2\frac{\theta_1}{2}.$$

When the angle θ_1 is not too great $\sin\frac{\theta}{2}$ may be replaced by $\theta/2$ and the equation reduces to

$$\frac{Wr^2A^2}{2g} + \tfrac{1}{2}WL\theta^2 = \tfrac{1}{2}WL\theta_1^2.$$

This is the same equation as in the preceding case, and indicates that the vibrations are isochronous, an oscillation to and fro taking place in the

$$\text{Period} = 2\pi . \frac{r}{\sqrt{gL}}.$$

In a "simple" pendulum consisting of a heavy particle suspended by a string of length l from a fixed point, and vibrating in a vertical plane $r = L = l$, and

$$\text{Period} = 2\pi\sqrt{\frac{l}{g}}.$$

The length of such a simple pendulum is often adopted as a measure of the time of a vibration. In a so-called "compound" pendulum let the radius of gyration about a horizontal axis through the centre of gravity be r_0, then (p. 201)

$$r^2 = L^2 + r_0^2,$$

and consequently the length of the simple equivalent pendulum is

$$l = L + \frac{r_0^2}{L}.$$

This is least when $L = r_0$ and the quickest time of vibration of a body of radius r_0 is consequently that of a pendulum of length $2r_0$; but the period may be made as long as we please by taking the axis near the centre of gravity, as for example in the beam of a pair of scales which is balanced on knife edges slightly above the centre of gravity.

Returning to the original equation observe that $\sin\frac{\theta}{2}$ is always less than $\theta/2$ and that therefore the potential energy is always less than if the motion were harmonic. The difference is greater the greater the value of θ, it is therefore greater for θ_1 than for θ, and the kinetic energy is consequently always less than in harmonic motion. When θ_1 is not small the diminution is perceptible and the vibrations are then not isochronous, but the period is less the greater the angle of swing. If T_0 be the time of small oscillations and T the actual time for the half angle of swing θ_1, then it is shown in treatises on the kinetics of a particle that

$$T = T_0\left\{1 + \frac{\theta_1^2}{16}\right\} \text{ approximately.}$$

(4) When a vessel rolls in still water a part of her kinetic energy corresponds to the movement of her centre of gravity: this, however, is usually a small fraction of the whole and may be neglected. If we also

neglect the resistance to rolling due to friction and disturbance of the water the equation of energy will be

$$\frac{W \cdot r^2 A^2}{2g} + U = U_1,$$

where r is the radius of gyration about a horizontal longitudinal axis through the centre of gravity, U the potential energy at the angle of heel θ, and U_1 the value of U at the extreme angle through which she rolls. The potential energy is here the same quantity as that already described as the "dynamical stability" and in the typical case considered on page 185 is given by the equation

$$U = W\frac{m}{k^2}(1 - \cos k\theta),$$

hence by substitution and multiplication by k^2

$$\frac{W \cdot r^2 \cdot k^2 A^2}{2g} + Wm(1 - \cos k\theta) = Wm(1 - \cos k\theta_1).$$

Referring now to the equation of energy of a simple pendulum just obtained, suppose it to swing through k times the angle of heel of the vessel it will be seen that the angular velocity of the pendulum will be kA, and that therefore the motion of the rolling vessel will follow the motion of such a pendulum if

$$l = \frac{r^2}{m}.$$

Hence the period of the small isochronous oscillations of a vessel when unresisted is

$$\text{Period} = 2\pi \frac{r}{\sqrt{mg}},$$

where r is the radius of gyration and m the metacentric height. Being independent of k the formula applies to any case whatever the stability curve so long as the oscillations are small, not exceeding 15° probably on each side of the vertical. For larger oscillations the deviation from isochronism is much greater than in a simple pendulum swinging through the same angle, being proportional to k^2 in the case just considered.

It should be observed that throughout this article the periods given refer to a complete oscillation to and fro. By many writers the time of a single oscillation is described as the period. In the case of a pendulum the "time of vibration" generally means the time of a single oscillation. The number of vibrations per second is known as the "frequency."

EXAMPLES.

1. The energy of 1 lb. of pebble powder is 70 foot-tons. Find the weight of charge necessary to produce an initial velocity of 1300 feet per second in a projectile weighing 700 lbs., neglecting the recoil of the gun and the rotation of the shot.

Wt. of powder required = 117 lbs.

2. In Example 1 suppose the gun fired at an elevation of 30°, and resistance of the atmosphere neglected, find the kinetic and potential energies of the shot at its greatest elevation. Also deduce the greatest elevation.

Horizontal velocity = velocity at highest point $= 1300\frac{\sqrt{3}}{2}$.

Kinetic energy at highest point = 6150 ft.-tons,

Potential „ „ = 2050 „

$$\frac{\text{Potential energy}}{\text{Wt. of shot}} = 6560 \cdot 6 \text{ feet} = \text{maximum elevation.}$$

3. A train is running at 40 miles an hour, find the resistance in pounds per ton necessary to stop the train in 1000 yards on a level. Also find the distance in which the train would be brought up by the same brake power on a gradient of 1 in 100, both when going up and when going down.

Resistance = 39·9 lbs. per ton.

Distance required to bring up the train when ascending the gradient ... ... = 640 yards.

When descending ... ... = 2280 „

4. The reciprocating parts of an engine running at 75 revolutions per minute weigh 25 tons, of which parts weighing 20 tons have a stroke of 4 feet, and parts weighing 5 tons a stroke of 2 feet. Find the energy stored in the parts, assuming a pair of cranks, OP, OQ at right angles and neglecting obliquity of connecting rod.

Here if V is the velocity of the crank pin and PN, QM are perpendiculars on the line of centres,

Velocity of parts attached to crank $P = PN\frac{V}{OP}$.

„ „ „ $Q = QM\frac{V}{OP}$.

Further assuming weights attached to these cranks each equal W.

Energy stored in these weights $= \frac{WV^2}{2g}(PN^2 + QM^2)\frac{1}{OP^2} = \frac{WV^2}{2g}$.

In example, total kinetic energy = 40·7 ft.-tons.

5. One weight draws up another by means of a common wheel and axle. The force-ratio is 1 to 8 and the velocity-ratio is 9 to 1. Find the revolutions per minute after 10 complete revolutions have been performed, neglecting frictional resistances and the inertia of the wheel and axle. Diameter of axle 6 inches.

Revolutions per second = 2·14.

6. In Ex. 1 suppose the gun rifled so that the projectile makes 1 turn in 40 diameters, find the additional powder charge required to provide for the rotation of the shot, the diameter of shot being 12 inches and the radius of gyration 4½ inches.

Additional powder required = ·407 lb.

7. A disc of iron rolls along a horizontal plane with velocity 15 feet per second, and comes to an incline of 1 in 40 on to which it passes without shock. Find how far it will ascend the incline, neglecting friction.

Distance along incline it will run = 209·6 feet.

8. In Ex. 5 suppose the weight of wheel = weight of axle, and the two together = sum of weights, obtain the result, taking account of the inertia of the wheel and axle.

After 10 revs. it will rotate at 1·22 revs. per second.

9. A fly-wheel, the radius of gyration of which may be taken as 8 feet, rotates at 40 revolutions per minute; find the height due to the revolutions and also the height equivalent to the energy of rotation. *Ans.* $h = 1·835$; $H = 17·45$.

10. The beam of a pair of scales is 2 feet long, radius of gyration 6 inches; the scale, pans, and weights are equivalent to a weight of 3 lbs. placed at each end of the beam,

which itself weighs 3 lbs. If the beam rest on knife edges placed ¼ inch above the centre of gravity, find the time of vibration. *Ans.* 3·3 seconds.

11. The centre of gravity of a connecting rod 5 feet long has been found by the method of suspension to be 3 feet from the crosshead end. To determine the radius of gyration it is made to oscillate as a pendulum on knife edges fixed at the crosshead end. It is then found that 53 vibrations are made in a minute; find the radius. *Ans.* 3 feet 6 inches.

12. From a curve of "tons per inch immersion" it is found that a vessel sinks one inch in the water by the addition to the weight on board of a small fraction c of her original displacement; show that the period of small unresisted dipping oscillations is

$$\text{Period} = \frac{2\pi}{\sqrt{386c}}.$$

REFERENCES.

Numerous elementary examples on the application of the Principle of Work will be found in Cotterill and Slade's *Lessons on Applied Mechanics*.

CHAPTER

DYNAMICS OF THE STEAM ENGINE.

104. *Construction of Polar Curves of Crank Effort.*—One of the most common and important applications of the principles of the preceding chapter is to the working of steam engines, and we shall investigate this question, chiefly with reference to fluctuations of stress energy and speed. Throughout, frictional resistances are neglected.

In Chapter V. a curve was constructed which shows the velocity-ratio of piston and crank pin, and it has been proved (p. 191) that this curve must also give the ratio of the effort tending to turn the crank to the pressure of the steam on the piston, so that it may also be called a Curve of Crank Effort. If there are two or more cranks, the crank effort can be obtained by suitably combining the results for each taken separately, and a curve may then be drawn representing the combination. There are two kinds of such curves, the Polar and the Linear. First suppose two cranks at right angles, steam pressure uniform, and the same on both pistons. Let us commence with the polar curve.

Suppose $OT_1'B_1$, $OT_2'B_2$ (Fig. 93) to represent the polar curve of crank effort for an engine constructed as in Art. 49, and let the two cranks be in the positions OQ_1, OQ_2, each pointing towards the cylinder. Add together the corresponding crank efforts OT_1', OT_2', which are given by the curve, and set off their sum along OQ_1, we thus obtain a radius OT'', which represents the total crank effort for the two engines taken together. It may also be considered as the leverage at which the pressure on one piston must act to produce the same turning moment. Performing this construction for a number of positions of the cranks, we obtain a polar curve showing the crank effort in every position.

If the connecting rod is indefinitely long the single curve of crank effort consists of the pair of circles on OB_1, OB_2, shown dotted in the diagram. If we add together radii of these circles, the combined curve of crank effort will consist of four portions of circles passing the points

$A_1B_1A_2B_2$; each of the circular arcs if produced would pass through the point O. These arcs are also dotted in the diagram. When the crank is in a quadrant lying towards the engine, the actual crank effort

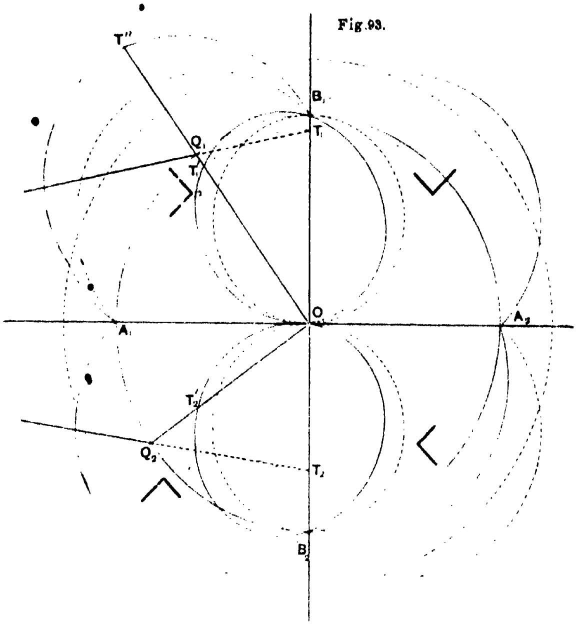


Fig. 93.

is in excess of that due to a long connecting rod. So for the positions OQ_1, OQ_2, shown, for each the crank effort is in excess, and thus the curve of combined effort will for the quadrant A_1B_1 lie outside the circular arc. When the cranks are in the two upper quadrants the effort for the leading crank is less than when the connecting rod is long, whereas for the following crank it is greater; and the diminution of one is very approximately equal to the excess of the other; that is, the sum is the same as that obtained by neglecting the shortness of the rod. The true combined effort is then for the quadrant B_1A_2 represented by the circle. In the next quadrant both are diminished; and the true curve will lie inside the circle A_2B_2, while for the fourth quadrant it will again coincide with the circular arc.

We may, if we please, lay off the sum of the radii on the following crank instead of the leading; the same series of curves would be obtained, but would be turned backwards through an angle of 90°.

To add to this the circle of mean crank effort we equate the work done on the two pistons in the double strokes to the work due to the mean effort R_m exerted through a complete revolution.

$$P \times 2 \times 4a = R_m \times 2\pi a.$$

$$\therefore\ R_m = \frac{4}{\pi}P.$$

In these curves the steam pressure P is represented by the radius of the crank-pin circle, so the mean crank effort will be represented on the diagram by drawing a circle, shown dotted, with centre O and radius $= 4OQ/\pi$.

If there are three or more cranks inclined at any angles, the combined crank effort diagram can be constructed by adding together three or more radii vectores of the curve of single crank effort, and laying the sum off on either of the cranks.

105. *Construction of Linear Curves of Crank Effort.*—The linear curve of crank effort, which is more useful for most purposes, is constructed as follows:

Take a base line, A_1A_2 = semi-circumference of the crank-pin circle, and let the circle and this base line be divided into the same number of equal parts, and at the points of division of the base line set off ordinates such as SN, VM both above and below the base equal to lengths of the common ordinates of the single crank effort diagram such as OT_1', OT_2', and so we construct the linear crank effort diagram for a single crank. Neglecting the obliquity of the connecting rod, the diagram will consist of two curves of sines shown dotted, one above, the other below (Fig. 94). To get the combined crank effort diagram we have only to add together proper ordinates according to the angle between the cranks, just as we did in drawing the polar diagram. When the cranks are at right angles it will be seen that when the leading crank is, for example, at Q_1 or N the following crank is at Q_2 or M; and if the ordinate MV is laid off on the top of ordinate NS we obtain a point W on the curve of combined crank effort. If the same process be followed throughout we obtain the diagram, shown in Fig. 94, consisting of four curves. If the connecting rod be taken as indefinitely long, and ordinates of the dotted curve be added together the combined diagram will consist of four curves, also curves of sines shown dotted in the diagram, all alike and all of the same height. But taking proper account of the shortness of the rod, we observe that for one quadrant of the revolution when both cranks lie towards the cylinder, each ordinate added is in excess of that, neglecting obliquity, and then we obtain the highest curve. In the next quadrant the height of the curve is less and is the same as if we neglected the shortness of the rod. In the next quadrant when both cranks are away from the cylinder the shortness of the rod makes the crank effort for each engine less, and we get a very low curve for the combination. This is followed in the last quadrant by a curve like the second.

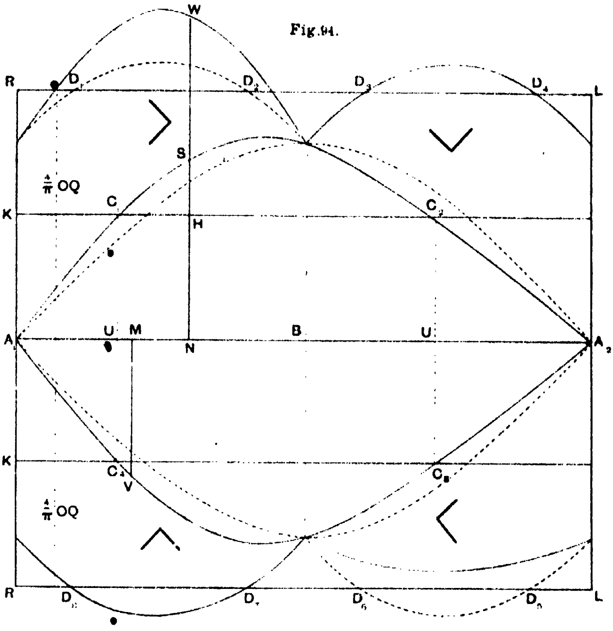


Fig. 94.

The mean crank effort will be represented by a horizontal line at a height $4OQ/\pi$, as before. Setting off this line we observe that unless the connecting rod is longer than is usual in ordinary practice, the actual crank effort will be less than the mean throughout the whole of one of the quadrants.

At the points where the straight line *RL* cuts the curves the actual crank effort is equal to the mean.

106. *Ratio of Maximum and Minimum Crank Effort to Mean.*—One of the principal objects in the construction of curves of crank effort is the determination of the ratio which the maximum and the minimum values of that quantity bear to its mean value as determined from the power of the engine. It is on these quantities that the strength required for the shaft depends, besides which, too great an inequality in the turning moment on the shaft is frequently injurious to the machine which is being driven by the engine, or to the work which the machine is doing.

Approximate mathematical formulæ, analogous to those given on page 102 for piston velocity, may be used in simple cases, but in general it is preferable to construct a diagram. The annexed table gives some numerical results.

FLUCTUATION OF CRANK EFFORT WITH UNIFORM STEAM PRESSURE.

Ratio to Mean } for	One Crank.	Two Cranks at right angles.	Three Cylinders at 120°, driving the same Crank.	Connecting Rod.
Maximum.	1·57	1·112	1·047	Indefinitely long.
Minimum.	0	·785	·907	
Maximum.	1·62	1·31	1·077	Four Cranks.
Minimum.	0	·785	·794	

The great influence which the length of the connecting rod has on the results should be especially noticed; we shall return to this hereafter, but now go on to consider the motion of the engine under the action of the varying crank effort.

107. *Fluctuation of Energy.*—We have already referred to the periodic character of the motion of a machine, and explained that when the mean motion is uniform we have for a complete period

Energy exerted = Work done.

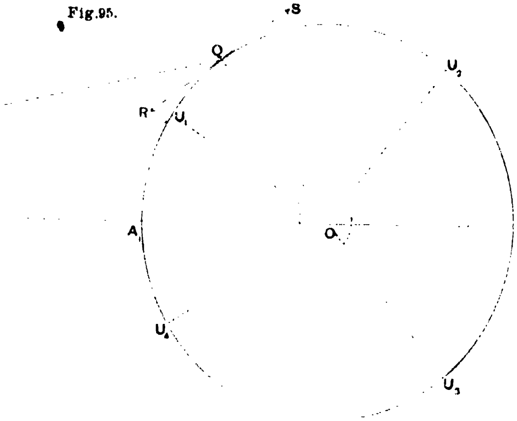
It will seldom happen however that this equation holds good for a portion of the period. In general, during some part of the period the work done will be greater, and in some part less, than the energy exerted.

In the first case some part of the kinetic energy of the moving parts is absorbed in doing a part of the work, and the speed of the machine diminishes; while in the second, a part of the energy exerted is employed in increasing the kinetic energy of the moving parts and the speed of the machine increases. Thus the kinetic energy of the moving parts alternately increases and diminishes, the increase exactly balancing the decrease. At some instant in its motion, the energy of the moving parts will be a minimum, and at some other point a maximum. The difference between the maximum and minimum energies is described as a "fluctuation" of energy of the machine. In general a number of these fluctuations occur in the course of a period, and the greatest of them is called the Fluctuation of Energy. It is most conveniently expressed as a fraction of the whole energy exerted during a complete period of the machine, and this fraction is called the Coefficient of Fluctuation of Energy.

All this will apply to any machine taken as a whole, or to any part of that machine; for every piece of the machine has a driving point and a working point, and the equation of energy may be applied to it.

Take now the case of the mechanism of a direct-acting engine. Suppose the pressure P on the piston to be uniform. This through the connecting rod will produce a crank effort S, the magnitude of which for each position of the crank may be found as just now shown. To the crank and shaft S is the driving force and furnishes the energy exerted. At every point of the revolution of the shaft a certain resistance will be overcome, which resistance will tend to prevent the shaft from turning; it will not depend on the steam pressure, but on the sort of work that is being done. As the most simple ordinary case we will suppose the resistance overcome to be uniform, and we will neglect the inertia of the reciprocating parts (Art. 110). We may represent this constant resistance by a constant force R applied to the crank pin Q (Fig. 95), at right angles to the crank arm, resisting its motion. The magnitude of R is immediately determined by the application of the principle of work to a complete period, say one revolution. We have

Fig. 95.



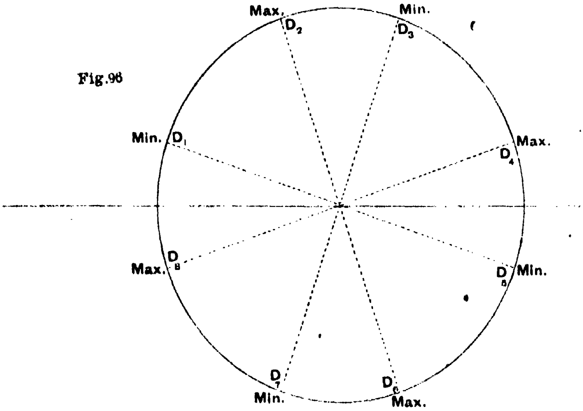
$$P4a = R \times 2\pi a. \quad \therefore R = \frac{2}{\pi}P.$$

This constant resisting force is the same as the mean crank effort.

Then so long as $S>R$ the speed of the crank shaft will increase, and when $S<R$ it will diminish.

Referring to the linear curve of crank effort (Fig. 94, p. 213) let $A_1N=$ the arc A_1Q (Fig 95), then $NS=$ crank effort S for this position of the crank. If an ordinate A_1K be set up to represent the constant resistance or mean crank effort, and a horizontal line parallel to base line be drawn, then NH being the representation of R the resistance overcome, the effort S will be greater for this position of the crank, and the difference HS will be employed in accelerating the motion of the machine. From the commencement of the revolution up to this position, the energy exerted is represented by the area A_1NS, whereas the work done is represented by the area A_1KHN. As the crank revolves from the position A_1 the crank effort increases until when at U_1 it is equal to the resistance. Up to this point the speed of rotation will have been diminishing. After passing the point U_1 the effort will be greater than the resistance and the speed of the engine will increase.

Fig. 96



Thus U_1 is a point of minimum speed at which the kinetic energy is a minimum. When the crank reaches the position U_2, the effort will again be equal to the resistance; and, since from U_1 to U_2 the effort has been greater than the resistance, during the whole of which time the engine has been increasing its speed, it follows that at the point U_2 the speed and the kinetic energy will have reached a maximum. The energy stored during this interval will be equal to the area C_1SC_2, and this will be the fluctuation of energy. During all the movement from U_2 to U_3 the speed of the engine will diminish, so that U_3 is another point of minimum kinetic energy. The kinetic energy stored from U_2

to U_3 is negative, and represented by $C_2A_2C_3$, which quantity also is the fluctuation of energy. Again at U_4 the kinetic energy is a maximum. If the resistance had not been uniform, but its varying magnitude represented by the ordinates of some curve of resistance, then where the curve of resistance intersected the curve of crank effort would be the points where the kinetic energies would be maximum and minimum, as just explained. By the graphical construction of such a curve of resistance the fluctuation of energy may be estimated by measuring the area of the crank effort curve cut off above or below the curve of resistance, which area will lie between consecutive points of maximum and minimum energies. If the energy be E, the fluctuation of energy may properly be denoted by ΔE. It is convenient to express this as a fraction of the total energy $4Pa$ exerted in a revolution We have then for the co-efficient of fluctuation of energy $\dfrac{\Delta E}{4Pa} = k$.

The value of k does not depend on the size of the engine, but only on the length of the connecting rod and the way in which the steam pressure and resistance vary. If the connecting rod is indefinitely long, steam pressure and resistance uniform, $k = \cdot 1052$. The shorter the connecting rod the greater will be the value of k.

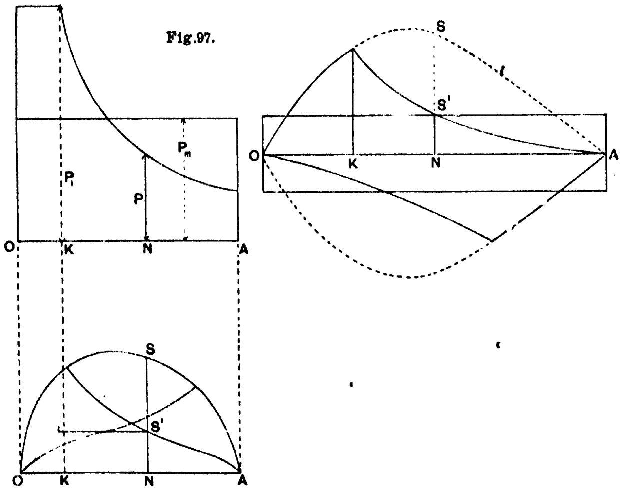
FLUCTUATION OF ENERGY.

Values of k supposing.			Length of Rod.
One Crank.	Two Cranks at right angles.	Three Cylinders at 120°, driving the same Crank.	
·1052	·01055	·00325	Infinite.
·1245	·0314	·0084	Six Cranks.
·1358	·0418	·0115	Four Cranks.

An equally important case is that of two cranks at right angles also shown in Fig. 94. Neglecting the shortness of the connecting rod, then the line of resistance cuts each of the four curves in two points, the first of which is a point of minimum energy as shown in Fig. 96, on the preceding page. For this case $k = \cdot 01055$ or one-tenth of its value for a single crank: eight fluctuations of equal magnitude occur in each revolution. When the connecting rod is short the curves of crank effort are not the same in each quadrant (see Fig. 94), and one of them lies wholly below the line of resistance. There are then six fluctuations in each revolution: four of these are nearly the same as before, but the other two are much greater, the values of k being ·037 and ·042, with a connecting rod of four cranks. The table on the preceding page gives the maximum value of k for various cases, supposing steam pressure uniform and resistance uniform.

As before, the great influence of length of connecting rod on the results should be noticed. Frictional resistances, which are here neglected, *generally* increase the value of k.

In general the pressure of the steam in the cylinder of an engine varies throughout the stroke, and the construction of the curve of crank effort previously described must be modified on account of this. Suppose, instead of the steam being admitted throughout the stroke, it is cut off at a certain point and expanded so that the expansion curve is hyperbolic. For simplicity neglect the back pressure. At the point N in the stroke (Fig. 97) the pressure will have fallen to P, such that $\frac{P}{P_1}=\frac{OK}{ON}$. If we draw an ordinate P_m



such that the area of the rectangle enclosed is equal to the area of indicator diagram, then $P_m=P_1\frac{1+\log_\epsilon r}{r}$ where $r=\frac{OA}{OK}$. Up to the point K the crank effort diagram will be the same as previously described, but after that point the crank effort will be less than that due to a uniform steam pressure. At the point N in the stroke, for example, the crank effort instead of being NS will be NS', found by drawing OS in the lower figure, to cut a vertical through the point K of cut-off and making $NS'=KL$, for the ratio NS'/NS is then equal to P/P_1. In the expanded diagram, the base of which is taken equal to the circumference of the crank-pin circle, ordinates must be taken equal to NS', and a diagram so constructed, from which the fluctuation of energy may be calculated. Assuming the resistance to be uniform, it will have a value R such that

$$R\pi a = P_m 2a = 2aP_1 \frac{1+\log_\epsilon r}{r},$$

$$R = \frac{2}{\pi} P_1 \frac{1+\log_\epsilon r}{r};$$

and drawing a horizontal line above the base at a height to represent R, it will cut off an area above it which will be the fluctuation of energy. The diagram for the return stroke is shown below. It is not exactly the same as that for the forward stroke, because the effect of obliquity is different. A general method of procedure applicable with any given indicator diagram is explained at the end of this chapter.

108. *Fluctuation of Speed. Fly-Wheels.*—Fluctuation of energy in an engine or any other machine is necessarily always accompanied by a fluctuation of speed; but the heavier the moving parts the less will be the fluctuation of speed. In most cases it is necessary that the fluctuation of speed should not exceed certain limits, as it would be injurious to the working parts of the machine and would sometimes impair the character of the work done; so it is a question of some importance to inquire as to what the weight of the moving parts must be to confine the fluctuation of speed within a given limit.

Consider the steam engine, and, first, take the case of a single crank. We have already for this case determined the points of the revolution at which the energy of the moving parts is a maximum and minimum, and also the fluctuation of energy. The energy of the moving parts consists of the energy of the rotating crank shaft and all its connections, as well as that of the reciprocating parts. If we imagine a case in which the shaft and all the parts which rotate with it are comparatively very light, then the points determined will be the points at which the piston and reciprocating parts move fastest and slowest, the motion would be very irregular, and, in fact, the engine would not get over the dead points. To avoid this the weight of the rotating parts is made considerable as compared with that of the reciprocating parts, and the heavier they are the more uniform the motion of the engine will be. To increase the uniformity, the weight must generally be artificially increased by the addition of a heavy fly-wheel to the shaft, and the inertia of this is predominant over that of the other moving parts of the engine. For the present we may neglect the inertia of the reciprocating parts and consider the fly-wheel alone.

On this supposition the energy and speed of the fly-wheel will be greatest and least at the points previously described, viz., where the curve of crank effort is cut by the line of uniform resistance. Let W be the weight, V the velocity of rim of fly-wheel; then

$$\frac{WV^2}{2g} = \text{Energy of Rim.}$$

The energy of the arms and boss may be estimated by the addition of a percentage to the weight of the rim, or be considered as furnishing a margin in favour of uniformity. On account of the danger of fracture the speed of periphery V should not exceed 80 feet per second. This is the limit of speed commonly stated, but the liability to fracture depends very much on the straining action of the arms of the wheel due to inequality between the crank effort and the resistance, and not merely on centrifugal forces. (See Ch. XI.) In large wheels the rim is in segments, and the speed should not be more than from 40 to 50 feet per second.

Let V_1 and V_2 be the greatest and least speed of periphery due to the fluctuation of speed, then $\frac{W}{g2}(V_1^2 - V_2^2)$ is the fluctuation of energy of the wheel. By the graphical process previously described, we have been able to determine the fluctuation of energy in terms of the total energy E_0 expended in one revolution.

Equating these two we have

$$\frac{W}{2g}(V_1^2 - V_2^2) = kE_0,$$

where k is the co-efficient previously found.

Suppose now that it is required that the fluctuation of speed should not exceed a certain amount, then we may write

$$V_1 - V_2 = q \cdot V_0,$$

where V_0 is the mean speed and q is a co-efficient depending on the degree of uniformity which is considered desirable. In some cases q must not exceed ·02 or even less, whilst in others ·05 or even more may be sufficient. In driving dynamos great steadiness is necessary and q should not be greater than ·007.

We may generally assume with sufficient accuracy that

$$V_0 = \frac{V_1 + V_2}{2}$$

(see next article), then we find by substitution that, at the mean speed,

$$\text{Energy of Wheel} = \frac{k}{2q} \cdot E_0.$$

In a single crank non-expansive engine the value of k ranges, as we have seen, from ·1 to ·14 when the resistance is uniform. In expansive engines k may be ·25 even with a uniform resistance, and when an engine is doing very irregular work k may be unity.

If we have a pair of cranks at right angles, the kinetic energy of the reciprocating parts is the same, at the same speed, for all positions of the cranks. (Ex. 4, p. 208.) Consequently these parts may be considered as so much added to the weight of the fly-wheel. Besides this the value of k is much less, seldom reaching ·1 if the resistance is approximately uniform. Hence a lighter fly-wheel may be used. The difference however is not so great as it might appear, for in estimating the weight of wheel required, it is important to consider not merely the change of speed, but also the time in which the change takes place. A small change taking place rapidly may be as injurious as a much greater change taking place slowly. The values of the acceleration and retardation at any instant are proportional to the difference between the crank effort and resistance at that instant, which can be found from tables such as that on page 214, and some regard should be paid to these numbers in considering what value of q should be employed.

In any case then we may write

$$\text{Energy of Wheel} = K \, . \, E_0,$$

where K is a co-efficient, which will vary within narrower limits than the two co-efficients of speed and energy on which it depends. In general, in the very cases in which the resistance is most irregular a greater variation in speed is admissible.

An old rule for fly-wheels, said to date from the time of Watt, was equivalent to taking the energy of the wheel as 3·75 times the energy exerted per stroke. This corresponds to $K = 1{\cdot}875$, and would be satisfied by $k = 1$, $q = {\cdot}267$, or by $k = {\cdot}125$, $q = \frac{1}{30}^{\text{th}}$. The first of these cases would be a very irregular resistance with a great variation in speed, and the second a moderately uniform resistance with a uniformity of speed which would be sufficient for most purposes. Heavier wheels are common in modern practice, and it may be here remarked that the minimum weight necessary may depend partly on the rigidity of the shafting.

There is another method of obtaining the fluctuation of energy which, though not practically so convenient, is for some purposes advantageous. A curve representing the energy exerted may be constructed in this way: Suppose the steam pressure P constant, then in the movement of the crank pin from A to Q the piston moves from A to N and the energy exerted $= P \times AN$, which will be proportional to AN. Now in Fig. 98 take a base line AA' equal to the semi-circumference, and at the various points such as Q, set up ordinates $QK = AN$, $A'A'' = AA'$, and so on; a curve $AKLA''$ will be obtained, which will represent by its ordinates the energy which has been exerted from the commencement up to the various points in the stroke. At the same time, the resistance being uniform, the work done will be proportional to the length of the arc AQ, since work done $= R \times AQ$. If from the base line AA' we set up ordinates to represent the work done. a straight sloping line will be obtained. If the work done = energy exerted in the complete stroke, they will both be represented by the same ordinate $A'A''$, and so the sloping line will meet the curve at the point A''. The intercept between the curve and line AA'' measured on the vertical ordinate will at any point be the difference between the energy exerted and the work done reckoned from the commencement of the stroke up to that point, and what we have called the fluctuation of energy will be the vertical intercept between two tangents to the curve $AKLA''$ drawn parallel to AA''.

From this we can derive a curve which will represent the varying angular velocity of the crank; but, in order to simplify the measurement and description, let the vertical intercepts of the curve just described be laid off from a horizontal base line, as shown below.

For suppose we know the moment of inertia of the equivalent fly-wheel of the engine and the angular velocity of the crank in some one position: the ordinate of the curve ALA'' at this point measured from a properly taken base line must represent the energy of the moving parts. Thus, if the base line be drawn in proper position, all ordinates measured from it will represent the square of the velocity of revolution of the crank shaft. If the speed of the machine is great, the base line will be some distance below the curve. On the other hand, if the speed is small, the base line will be close to the curve. There is manifestly a minimum speed at which the machine can be kept revolving; it is that which corresponds to the case in which the base line touches the curve. At one instant of the period of the machine the energy will then be zero.

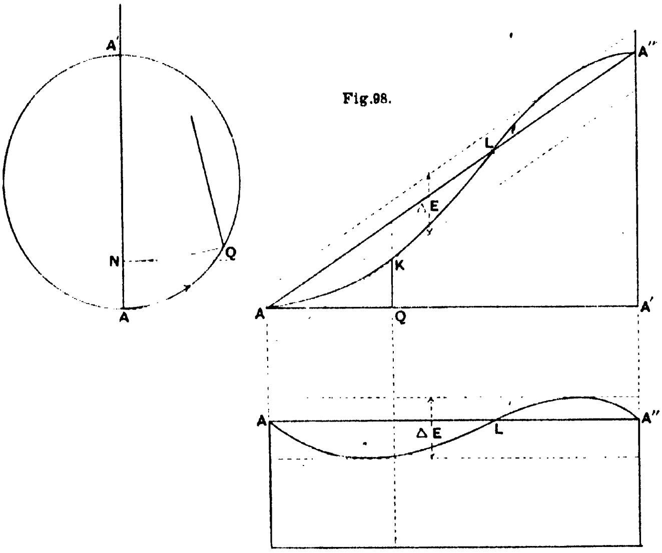


Fig.98.

Drawing such a base line all the ordinates measured from it will represent the square of the angular velocity, and we can from this deduce a curve of angular velocity. It will be noticed that half the sum of the greatest and least angular velocities is not exactly, but only approximately, the mean angular velocity. The true mean may be determined by means of the curve of angular velocity, the construction of which has just been described.

A curve of fly-wheel velocity has been constructed by M. Dwelvshauser-Déry * from data derived from indicator diagrams taken from an experimental engine belonging to the Mechanical Laboratory of the University of Liége. The mean velocity in this case

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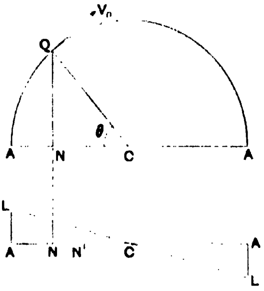
* "On the Application of Governors and Fly-wheels to Steam Engines." *Proceedings of the Institution of Civil Engineers*, vol. civ., p. 196.

was found to approximate closely to the arithmetic mean between the maximum and minimum velocities in accordance with the usual assumption.

An attempt has been made by Mr. H. B. Ransom * to measure the actual velocity of the fly-wheel at different points of its revolution by means of a tuning fork making a known number (about 500) of vibrations per 1″, a point on one leg of the fork marking a continuous series of waves on a drum rotating with the wheel. The length of a set of 10 waves gave the angle turned through by the wheel in one-fiftieth of a second.

109. *Correction of Indicator Diagram for Inertia of Reciprocating Parts.*—All that has been said respecting the fluctuation of energy and speed of a machine as a whole, applies to each of the several parts of which it is constructed. The energy supplied by the driving power is transmitted through each piece in succession from the driving pair to the working pair. For each piece the energy exerted is equal to the work done for the whole period; but for a part of the period the two are unequal, so that the kinetic energy of the piece varies. If the motion of the piece be known, the variation of its energy can be used to determine the difference between the driving force on the piece considered and on the piece immediately following it. Of this calculation an important example is the change in the crank effort caused by the inertia of the reciprocating parts of an engine.

Fig. 99.



In this calculation we neglect, in the first instance, the obliquity of the connecting rod, and suppose the crank to rotate uniformly. Let Q (Fig. 99) be the centre of the crank pin describing a circle AQA with velocity V_0, then the position of the piston is represented by N, and its velocity is

$$V = V_0 \cdot \sin\theta,$$

from which it follows that the kinetic energy of the reciprocating parts must be given by

$$\text{Kinetic Energy} = \frac{WV_0^2 \sin^2\theta}{2g} = \frac{WV_0^2}{2g}\left(1 - \frac{x^2}{a^2}\right),$$

where W is the weight of the piston, piston rod, and other reciprocating parts, and x is the distance of the piston from the centre of its stroke.

Take now two positions N, N' at distances x_1, x_2 from the centre and find by this formula the change of kinetic energy as the piston moves from N to N'. Evidently we shall have

$$\text{Change of Kinetic Energy} = \frac{WV_0^2}{2g} \cdot \frac{x_1^2 - x_2^2}{a^2}.$$

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* "The Cyclical Velocity Variations of Steam Engines." *Proceedings of the Institution of Civil Engineers*, vol. xcviii., p. 357.

Now this energy must have been obtained from the steam pressure which drives the piston and accelerates its motion. Let P be the mean value of that part of the whole steam pressure which is employed in this way between N and N', then $P\,.\,NN'$ is the energy exerted in this way, so that

$$P(x_1 - x_2) = \frac{WV_0^2}{2g} \cdot \frac{x_1^2 - x_2^2}{a^2};$$

or dividing by $x_1 - x_2$,

$$P = \frac{WV_0^2}{2g} \cdot \frac{x_1 + x_2}{a^2}.$$

This formula gives the mean value of the pressure in question between any two points N, N', and therefore, if we take the points near enough, we shall obtain the actual pressure at any point of the stroke. Putting $x_1 = x_2 = x$ we get

$$P = \frac{WV_0^2}{ga} \cdot \frac{x}{a}.$$

It is convenient to express our result as a pressure in lbs. per square inch by dividing by the area of the piston in square inches, then

$$p = p_0 \cdot \frac{V_0^2}{ga} \cdot \frac{x}{a} = p_0 \cdot \frac{x}{h}$$

where p_0 is the weight of the reciprocating parts divided by the area of the piston, or, as we may call it, the "pressure equivalent to the weight of the reciprocating parts," and h the "height due to the revolutions," as on page 202.

When $x = a$ we get the pressure at the commencement of the stroke required to start the piston: here the pressure is greatest, and elsewhere varies as the distance from the centre. At the centre the pressure is zero: the piston then for the moment moves with uniform velocity and requires no force to change its motion. When past the centre the pressure is so much addition to the steam pressure because the piston is at every instant being stopped: this is shown by the formula, since x is then negative. All this is shown graphically by drawing a straight line LCL through C such that

$$AL = p_0 \cdot \frac{V_0^2}{ga} = p_0 \cdot \frac{a}{h},$$

the ordinate of that straight line represents the pressure due to inertia for each position of the piston. After subtracting this from the actual steam pressure the effective pressure is found, which is transmitted to the crank pin, and furnishes the crank effort.

The value of p_0, the pressure equivalent to the weight of the reciprocating parts, varies considerably according to the size and type of engine, but in ordinary cases ranges from 1½ to 3 lbs. per square inch.

In return connecting rod engines, and in some other types where the reciprocating parts are exceptionally heavy, p_0 may reach 4½ or 5 lbs. per square inch. This being given, the pressure due to inertia will vary inversely as the stroke and directly as the square of the speed; in the high-speed engines common in the present day, the correction for inertia is very considerable. It is hardly necessary to say that it is only the value of the crank effort at particular points of the stroke which is affected. The mean value must remain unaltered, for any energy employed in overcoming inertia at one part of the stroke must be given out again at another part, so that the total energy exerted by the steam remains the same. Further, when there are a pair of cranks at right angles the total crank effort is little altered. The effect is best seen by correcting an indicator diagram for the inertia of reciprocating parts in the following way. Consider, for simplicity, a theoretical indicator diagram (Fig. 100) $SQZA$, in which BB is the back pressure line, QZ the expansion curve, then, but for inertia, the ordinates reckoned from BB of SQZ give the effective pressure of the steam. Set up BL equal to the pressure necessary to start the piston found above and draw the straight line LCL, then the actual effective pressure will be obtained by measuring the ordinates to the sloping base LCL instead of the original base BB. It will be seen that the general effect is to equalize the steam pressure throughout the stroke.

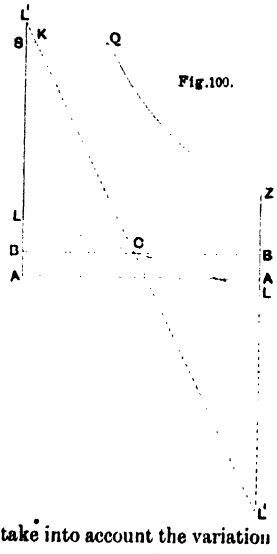


Fig. 100.

In a high-speed engine the effect of inertia is so great that BL is sometimes greater than BS as shown by the dotted line $L'CL'$. This case will be considered in a later chapter.

The question here considered is evidently the converse of Case I., Art. 103 of the last chapter; the motion now being given instead of the force. In the case of a piston it is usually more complicated (1) because the crank does not rotate uniformly, (2) from the effect of obliquity of connecting rod. To take into account the variation in the velocity of the crank it would be necessary to draw a curve representing that velocity by an approximate method already described and to deduce from it a curve representing the real piston velocity in any position. In general however the inertia of the rotating parts is *sufficient to reduce the variation in speed within narrow limits and the error caused by disregarding it may be neglected. The effect of* obliquity is of more importance. *In the first place the motion of the* piston is not exactly harmonic, and in the second place the connecting rod does not simply reciprocate but has, in addition, an angular motion. These two points will now be considered separately and in order.

To find the effect of the deviation from harmonic motion three methods may be adopted:—

(1) A linear curve of piston velocity being drawn as in Fig. 48, page 100, we may derive from it, by an easy graphical process, a curve of kinetic energy. Divide the stroke into a convenient number of equal parts and draw the corresponding ordinates; the differences of these ordinates show the changes of kinetic energy and consequently the mean pressures necessary to produce them.

(2) If f be the acceleration of the piston, P the inertia-pressure, W the weight, then (p. 197)

$$P = W \cdot \frac{f}{g}.$$

In Exs. 9 and 10, p. 103, two constructions are given, by either of which the acceleration of the piston may be found graphically. A third (generally preferable) will be found in the Appendix. An acceleration curve can thus be drawn which will also be the curve of inertia required, for reciprocating parts of given weight.

(3) An acceleration curve may be constructed by graphical differentiation of the velocity curve, a method which it may be worth while to illustrate in detail.

Divide the crank-pin circle into a number of equal parts, and supposing the connecting rods drawn, let them cut the vertical through 0 in the points 1′, 2′, 3′, in Fig. 101. Also find and mark off the corresponding positions of the piston 1″, 2″, 3″, etc. Now, since the lengths, 01′, 02′, 03′, etc., represent the velocities of the piston and reciprocating parts when in positions 1″, 2″, 3″, etc., the difference between any two consecutive lengths, for

Fig. 101.

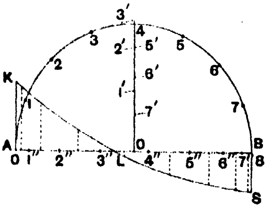
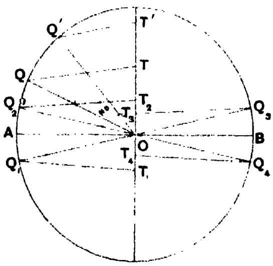


Fig. 102.



example 1', 2', will represent the change of velocity that has taken place in the corresponding movement of the piston 1", 2". If we suppose the crank pin to revolve uniformly and divide the circle into equal parts, equal times will be occupied in the motions from point to point, and therefore equal times in the motions between consecutive positions 1", 2", 3", 4", etc., of the piston. Accordingly the differences 01', 1'2', 2'3', etc., will represent the force required to change the velocity of the reciprocating parts; and if we set them up as ordinates between the corresponding positions of the piston, we shall obtain the curve expressing the effect of inertia. The ordinate should be erected from the position of the piston when the crank pin is at the middle of the intervals 1, 2, 3, etc.

It will be seen that the greater the number of parts into which we divide the crank-pin circle the less will be the ordinates representing the effect of inertia, though in all the curves the same character will be preserved. Accordingly it is possible to determine the number of parts into which the crank circle should be divided, or to determine the angle between consecutive radii, 01, 02, etc., so that the ordinates of the inertia curve may be of a length proper to represent the pressure per square inch of piston area required for inertia on the same scale that the indicator diagram is drawn. The ordinates of the resulting inertia curve may then be directly employed to correct the indicator diagram.

Let N be the number of revolutions per minute; Q, Q' consecutive points on the crank-pin circle; and let $QOQ' = n°$ be the required angle. Further suppose that the crank pin circle is drawn on a scale of x inches to the foot. Then the change of velocity of piston Δv, in feet per second, is evidently

$$\therefore \Delta v = \frac{2\pi N}{60} \cdot \frac{TT'}{x}, \text{ where } TT' \text{ is to be measured in inches.}$$

Now this change of velocity takes place in the time Δt occupied by the movement through $n°$, that is in $n/6 \cdot N$ seconds.

$$\therefore \text{ Force due to inertia} = \frac{W}{g}\frac{\Delta v}{\Delta t} = \frac{W}{g}\frac{2}{10}N^2\frac{(TT' \text{ in inches})}{xn°}.$$

And if the indicator diagram is drawn on the scale of y lbs. to the inch it will be found that

$$n = \cdot 0195 p_a \cdot \frac{N^2}{xy}.$$

The curve will cross the base line at the point L where the piston has it maximum velocity, that is approximately when the crank is at right angles to the connecting rod, hence

$$OL = \sqrt{(\text{con. rod})^2 + (\text{crank})^2} - \text{connecting rod}.$$

It can seldom be necessary to apply any of these methods, for the error (Δf) in the acceleration due to obliquity is given by the simple approximate formula (p. 102)

$$\Delta f = \frac{V_0^2}{na} \cdot \cos 2\theta,$$

while the corresponding error in the position of the piston is found by direct construction or by the formula on page 98. When obliquity is neglected the curve of inertia is a sloping straight line LCL (Fig. 99, p. 223), the extreme ordinate AL of which corresponds to the acceleration at the ends of the stroke. The deviation of the actual curve of inertia from the straight line must therefore be

$$\text{Deviation} = \frac{AL}{n} \cdot \cos 2\theta.$$

This vanishes when the crank stands at 45° from either dead point and is equal to AL/n at the ends of the stroke. When the crank is upright *the deviation is* $-AL/n$, *and at* 30° *from either dead point* $AL/2n$. Finally the deviation at 60° from either dead point is $-AL/2n$, and thus corrected ordinates are determined for each of nine corresponding positions of the piston. In addition, if necessary, the point L where the curve crosses the axis may be found by the simple rule given above (p. 227), giving altogether ten points on the curve through which it can be readily traced. If n is not less than 4, the form of the curve is similar to that shown in Fig. 101, but with a shorter rod the formula here used for the acceleration

$$f=\frac{V_0^2}{a}\left\{\cos\theta+\frac{1}{n}\cos 2\theta\right\}$$

shows that the negative ordinates reach a maximum value at a point given by

$$\cos\theta=-\frac{n}{4}.$$

The curve therefore becomes horizontal, and slopes upwards before reaching the point S at the end of the stroke.

To ascertain the effect of the angular motion of a connecting rod of length l, consider a small portion of weight w at a distance z from the cross-head end: then with the same notation as on page 223 the vertical velocity of that portion is

$$\text{Vertical Velocity}=V_0\cdot\frac{z}{l}\cdot\cos\theta.$$

The horizontal velocity when the rod is very long is sensibly the same as that of the piston (V), and therefore if W be the whole weight of the rod

$$\text{Kinetic Energy}=\frac{WV^2}{2g}+\frac{\Sigma wz^2}{l^2}\cdot\frac{V_0^2}{2g}\cdot\cos^2\theta.$$

Now Σwz^2 is equal to βWl^2, where β is a fraction, which for a uniform rod would be one-third, and which in any case can be calculated by summation, or determined experimentally as in Ex. 11, page 209. When the rod is very long we have also $V=V_0\sin\theta$, and hence by substitution

$$\text{Kinetic Energy}=(1-\beta)\frac{WV^2}{2g}+\beta\frac{WV_0^2}{2g}.$$

Thus it appears that the investigation already given for the pressure P necessary to accelerate the piston will apply when the inertia of the rod is taken into account if we suppose the fraction $1-\beta$ of its weight to be added to the other reciprocating parts.

The formula here given for the kinetic energy of a connecting rod is, however, closely approximate only for very long rods, unless the crank be at dead points or at 90° with the line of centres; it is therefore desirable to study the question more thoroughly in order that the nature of the inertia-curve may be better understood.

For brevity let us take

$$l - z = \mu l = \mu n a\,;$$

then μ is a fraction which ranges from zero to unity in proportion to the distance of the point considered from the crank-pin, and with the usual notation (p. 102) the horizontal distance of the point from the crank centre is $\mu na\,.\cos\phi + a\,.\cos\theta$. By differentiation and substitution, remembering the relation between θ and ϕ (compare Ex. 9, p. 103),

$$\text{Horizontal Velocity} = V_0\{\sin\theta + \mu\,.\tan\phi\,.\cos\theta\}.$$

This is less than the velocity of the piston by $V_0(1-\mu)\tan\phi\,.\cos\theta$, a quantity which is positive, zero, or negative according as θ is less, equal, or greater than 90°. We also see that the effect of obliquity on the horizontal velocity of points on the rod is diminished in proportion to the distance from the cross-head end till at the crank-pin it vanishes. The exact value of the kinetic energy (U) of a weight w placed at the point in question is now found to be

$$U = \frac{wV_0^2}{2g}\left\{(\sin\theta + \mu\,.\tan\phi\,.\cos\theta)^2 + (1-\mu)^2\cos^2\theta\right\}$$

$$= \frac{wV_0^2}{2g}\left\{1 + \mu^2\,.\frac{\cos^2\theta}{\cos^2\phi} - 2\mu\,.\cos(\theta+\phi)\frac{\cos\theta}{\cos\phi}\right\}.$$

Next let r be the crank effort which as the crank turns through a small angle $\Delta\theta$ would produce the corresponding change of kinetic energy ΔU; then

$$r\mu\Delta\theta = \Delta U,$$

and if S be the corresponding pressure on the piston, we have, as on p. 191,

$$S = \frac{r\,.\cos\phi}{\sin(\theta+\phi)} = \frac{1}{a}\,.\frac{\cos\phi}{\sin(\theta+\phi)}\,.\frac{dU}{d\theta}.$$

By differentiation and reduction, always remembering the relation between θ and ϕ, we now find

$$S = \frac{wV_0^2}{ga}\left\{\mu\,.\cos\theta + \frac{\mu-\mu^2}{\cos^2\phi}\,.\cos\theta - \frac{\mu}{n}\,.\frac{\sin^2\theta}{\cos\phi} + \frac{\mu^2\cos^2\theta}{n\cos^3\phi}\right\}.$$

If $\mu = 1$ we get for the case where w is at the cross head end of the rod, that is to say, virtually attached to the piston,

$$S = \frac{wV_0^2}{ga}\left\{\cos\theta - \frac{1}{n}\,\frac{\sin^2\theta}{\cos\phi} + \frac{1}{n}\,.\frac{\cos^2\theta}{\cos^3\phi}\right\}.$$

As the quantity in the bracket is easily shown to be the acceleration-ratio of piston and crank-pin, this agrees with previous results. (See Appendix, note to p. 102.)

The nature of the inertia-curve is best seen by putting $\cos\phi = 1$, a supposition which for the present purpose is quite a sufficient approximation, and expressing $\sin^2\theta$, $\cos^2\theta$ in terms of $\cos 2\theta$, then

$$S = \frac{wV_0^2}{ga}\,.\left\{(2\mu-\mu^2)\cos\theta - \frac{\mu-\mu^2}{2n} + \frac{\mu+\mu^2}{2n}\,.\cos 2\theta\right\}.$$

The value for the whole rod is now found by summation, to express the result of which we may conveniently write

$$\Sigma w\mu l = \sigma Wl\,;\quad \Sigma w\mu^2 l^2 = \gamma Wl^2,$$

where σl is the distance of the centre of gravity of the rod from the crank pin, and γl^2 the square of the radius of gyration of the rod about the crank-pin end, quantities connected with the fraction β previously used by the equation

$$2\sigma - \gamma = 1 - \beta.$$

Thus finally the inertia-curve for a connecting rod is given very approximately by the formula

$$P = \frac{WV_0^2}{ga}\left\{(1-\beta)\cos\theta - \frac{\sigma-\gamma}{2n} + \frac{\sigma+\gamma}{2n}\,.\cos 2\theta\right\}.$$

For a uniform rod $\beta = \gamma = \frac{1}{3}$, $\sigma = \frac{1}{2}$. On comparing this result with that for the piston, which may be deduced from it by putting $\sigma=\gamma=1$, when the value already used (p. 228) is obtained, it is clear that the inertia-curve of a connecting rod is the same as that of a weight $(1-\beta)W$ attached to the piston, except that (1) the whole curve is shifted downwards from the line of centres through the small space $AL(\sigma+\gamma)/2n(1-\beta)$, and (2) that the correction for obliquity is less, the length of the rod being virtually increased in the ratio $2(1-\beta)/(\sigma+\gamma)$. In a uniform rod this ratio is 1·6. The curve may evidently be plotted by the method already explained.

In the present article the effect of the inertia of the reciprocating parts of an engine on the crank effort at different points of the revolution has been exclusively considered. The disturbing forces called into play on the engine as a whole will be described in a later chapter (see Arts. 144, 144A, Ch. XI.), to which the question properly belongs.

110. *Construction of Curves of Crank Effort for any given Indicator Diagram.*—If the varying magnitude of the steam pressure is given by the actual indicator diagram of the engine we may deduce the true crank effort as follows: Let Fig. 1, Plate V., be a pair of indicator diagrams. The examples chosen are from the low pressure cylinder of H.M.S. "Nelson."* Before proceeding to make use of them they should be corrected for inertia, and, where the engines are vertical, for the weight of the reciprocating parts. The curve of pressure due to inertia is KLS in Fig. 1, which has been drawn, as just described, to the same scale as the indicator diagram. If we draw a line MN parallel to the base line of the inertia curve to represent p_0, the pressure due to the weight of the reciprocating parts, then the intercept between MN and KLS will be the necessary correction for inertia and weight combined. In applying the correction, the forward pressure in one of the pair of diagrams should be taken in conjunction with the back pressure of the other, for it is the difference between these which gives the true effective pressure on the piston. Let the dotted lower curves be the result of the correction, so that the virtual pressure which is transmitted to the crank-pin is to be measured by the vertical intercept between the upper steam curve and the dotted curve, such as BC for example. Immediately below the diagram draw a crank-pin circle with diameter equal to the length of the indicator diagrams. Divide the crank-pin circle into, say 20, equal parts, and suppose the crank-pin to be successively at these points of division; determine the corresponding positions of the piston in its stroke. Whilst doing this, mark the directions in which the connecting rod lies when the crank-pin is in these several positions. Let the positions of the piston in the line of stroke be set off along the diameter 0, 10. Through

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* The author is indebted to Mr. (now Prof.) Hearson for the example here given, and for the method of drawing the curve of inertia described in small type on page 226.

Plate. V.

Scale 1 inch 12 lbs

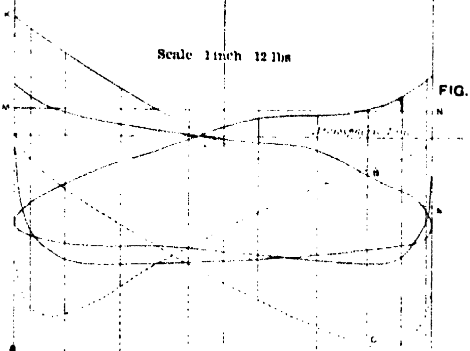


FIG. 1.

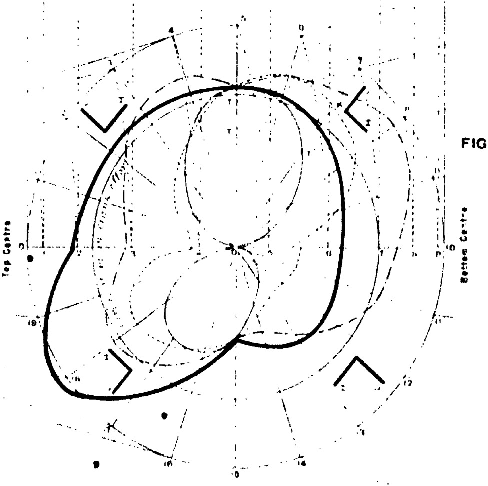


FIG. 2.

Co-efficients—

Of Crank Effort, 1·72. Of Fluctuation of Energy, ·108.

To face page 280.

these points draw verticals to intersect the indicator diagrams. The intercepts of these verticals will give us the virtual steam pressure at each of the points of the stroke and corresponding to each position of the crank in its revolution. Next, having in Fig. 2 drawn a number of radii through the points 1, 2, 3, etc., lay off from the centre O along each, the respective intercepts of the indicator diagram which represent the virtual pressures of the steam when the cranks are in those positions. We thus draw what we may call a polar curve of virtual steam pressure. We have for example taken OK equal to BC in the figure, and similarly for all other radii.

Now, referring to page 191, we observe that if the connecting rod in any position be drawn to cut the vertical through O, in a point T, as for example in Fig. 2 when the crank is at 7, then the length OT will represent the crank effort on the same scale that the length of the crank arm $O7$ represents the magnitude of the steam pressure. If now through K we draw KT'' parallel to $7T$, then by similar triangles $\frac{OT''}{OK}=\frac{OT}{O7}$, and thus on the same scale that OK represents the steam pressure OT'' will represent the crank effort. Now along the crank $O7$ set off a length $OT''' = OT''$, and perform a similar operation for each of the positions of the crank. If through the points so obtained we draw a continuous curve it will be the polar curve of crank effort which we require, for it will represent by its radii in any position the actual crank effort when the crank is in that position; and we see that, in the construction, account is taken not only of the angular position of the crank, but also of the steam pressure which is available for turning the crank. Taking both indicator diagrams we thus draw the curve for the complete revolution of the engine. By transfer of the radii of the polar curve to the crank circle unrolled, we can construct a linear curve (Art. 105), and thus determine the fluctuation of energy.

In Fig. 2 the thick curve has been drawn to show the crank effort due to the high and low pressure cylinders combined, by adding to the radii of the original curve the corresponding radii of the high pressure curve (not shown in the figure) after correction for difference of scale. In this engine the high pressure crank is 90° in advance of the low: if it had been 90° behind the low the fluctuation of crank effort would have been less. This is shown by the large dotted curve in the figure. The circle of mean crank effort is added to facilitate comparison.

111. *Pumping Engines.*—We have hitherto supposed the engine employed to turn a shaft against a resistance represented by an approximately uniform force applied at the crank-pin. In cranes and other hoisting machines, though the resistance to be overcome is linear, the reciprocating movement of the piston cannot conveniently be converted into a continuous lift without the use of a rotating shaft driven by the engine, while for transmitting and distributing power to other machines shafting is generally necessary. In pumping water, however, a reciprocating movement is required in the pump piston, which is converted into an intermittent upward movement of the water by the ratchet action (p. 156) of the valves. The pump piston is therefore connected with the steam piston either directly, so as to form one element of the same pair, or by the intervention of a vibrating beam introduced either for constructive reasons or for the purpose of altering the stroke and speed of the pump.

The external resistance to the motion of the steam piston is now approximately constant instead of being indefinitely great at dead points and least at the beginning of the stroke; but in order that the engine may work economically the steam must be used expansively, that is, its pressure must be great at the commencement of the stroke and gradually diminish to the ends. There is therefore, as before, a difference between effort and resistance which gives rise to a fluctuation of energy of the moving parts. Referring to Fig. 97 (p. 218) it will be seen that at the beginning of the stroke the pressure P_1 of the steam is greater than the mean pressure P_m which represents the approximately uniform resistance to be overcome in lifting the water; the excess sets in motion the reciprocating parts, which, unless regulated in a proper way, will move with gradually increasing velocity until the point is reached where the mean pressure line crosses the expansion line. At this point the pressure of the steam has by expansion fallen to its mean value and storage of energy ceases. The part of the indicator diagram lying above the mean pressure line represents in this case the Fluctuation of Energy, which can be calculated readily in any particular case. It is stored in the moving parts and completely restored when they are brought to a standstill at the end of the stroke.

The work of pumping requires a slow and steady movement of the pump piston, combined with a pause at the ends to allow time for the opening and closing of the valves, otherwise dangerous shocks will occur and much loss by hydraulic resistance. Moreover in each stroke the mean pressure of the steam must exactly correspond to the work done in raising the water. Hence some method of controlling and regulating the motion of the reciprocating parts and of the water moving with them is generally necessary. Three ways of doing this will now be briefly noticed.

(1) A crank and fly wheel may be employed as in the common form of small steam pump shown in Plate II., or in a modified form in the Stannah pump (Plate I.) referred to in Chapter V. Many pumping engines of the largest size are constructed in this way for waterworks, the drainage of a town, or other similar purpose. They are sometimes direct acting but more often beam engines, a constructive difference which need not be here considered; in either case the crank and fly, wheel is simply an appendage provided for the purpose of defining the stroke and regulating the motion. The greater part of the fluctuation of energy is in this case accounted for by the kinetic energy of the reciprocating parts, inclusive of the water being pumped: the weight of these parts being generally very considerable. The excess to be provided for by the fly-wheel can be found by methods already explained.

(2) In the Worthington direct acting horizontal engine, a regulator of a different kind has been recently introduced.

A cylinder containing compressed air enclosed behind a piston is mounted on trunnions and oscillates to and fro in a vertical plane. The steam and pump pistons are attached to opposite ends of the same rod, to the middle of which the piston rod of the air cylinder is connected by a pin. The air cylinder is placed above the rod so as to be vertical at mid-stroke of the steam piston, and in consequence the air piston is then pushed furthest in and the pressure of the air is greatest. At the ends of the stroke of the steam piston the air cylinder is inclined to the vertical, its piston is furthest out and the pressure of the air least. Hence at the beginning of the stroke the thrust of the air piston acts against the steam pressure and at the end in favour of it. When the pressure of the enclosed air is properly adjusted by forcing fresh air in or allowing it to escape, the compensation for varying steam pressure is nearly perfect and the engine works with great regularity. The air plays the part of a fly-wheel, storing energy in the first half of the stroke and restoring it in the second.

(3) In the Cornish engine, which is usually but not necessarily of the beam type, a single acting plunger pump at the bottom of a mine is operated from the surface by long and heavy "spear rods." The work of raising the water is done on the down stroke of the pump by the weight of the rods, which has to be properly adjusted for the purpose. The steam cylinder, also single acting, is employed solely for the purpose of raising the rods to the top of the stroke of the pump. In this case no special regulator is necessary, the fluctuation of energy being wholly stored in the heavy spear rods with pump plunger and beam (if any). The upward movement continues in the second half of the stroke until the energy thus stored is exhausted, the whole energy

exerted by the steam being now represented by the potential energy of the elevated weights, which is made use of for pumping in the down stroke. The stroke in this type of engine depends on the exact adjustment of the mean steam pressure to the weight lifted, and is accordingly subject to variation instead of being precisely defined as when a crank and fly-wheel is used. The alternate starting and stoppage of the heavy moving parts represents a large fluctuation of energy, for which a suitable amount of expansion is necessary, and it is believed that expansion was first employed for this reason and not from motives of economy, the type of engine in question being of very early date.

In all pumping engines the valves and valve gear form a most important subject for consideration, but this is beside the purpose of the present chapter.

111A. *Periodic Motion of Machines in General.*—The motion of a steam engine, which we have been describing in detail in this chapter, may be taken as a typical example of the transmission of energy by any machine whatever. Neglecting frictional resistances the energy is transmitted without alteration from a driving pair to a working pair—when the complete period of the machine is considered; but the rate of transmission varies from instant to instant during the period. The alternate excess and deficiency of energy is provided for by the moving parts of the machine, which serve as a store of energy or "kinetic accumulator," which can be drawn upon at pleasure.

It has been supposed that the mean resistance at the working pair is exactly equivalent to the mean effort at the driving pair. If this be not the case the machine will rapidly alter its mean speed till the balance is restored by alteration of the effort or the resistance or both. The balance seldom exists for long, and some means of controlling the machine is therefore generally indispensable. We have also to consider the straining actions due to the motion of the machine, especially at high speeds. These, however, are matters for subsequent consideration.

EXAMPLES.

1. In the case of a pair of cranks at right angles, draw the polar diagram of crank effort when the connecting rod is indefinitely long, and find the ratio of maximum crank effort to mean. Find also the position of the cranks when the actual crank effort is equal to the mean. *Ans.* Maximum crank effort = 1·11 mean.

2. Draw the diagram and obtain the results as in the last question, when the length of connecting rod is equal to 4 cranks.

Maximum crank effort = 1·307 mean.

3. Draw the linear diagram of crank effort, assuming two cranks at right angles and connecting rod = 4 cranks.

4. What is the maximum length of connecting rod for which the crank effort is less than the mean throughout one quadrant?

Connecting rod = 7·1 cranks.

5. From the diagram of crank effort constructed in question 3, determine the co-efficient of fluctuation of energy, 1st. When the connecting rods are indefinitely long; 2nd. When the length equals 4 cranks.

Connecting rod indefinitely long. Co-efficient of fluctuation of energy = ·011.

Connecting rod = 4 cranks. Co-efficients are ·011, ·042, ·011, ·009, ·038, ·009.

6. A pair of engines of 500 H.P., working on cranks at right angles with connecting rods = 4 cranks, are running at 70 revolutions per minute. Find the maximum and minimum moments of crank effort, and the fluctuation of energy in ft.-lbs.; assuming the steam pressure and resistance uniform.

Maximum moment of crank effort = 49,125 ft.-lbs.
Minimum moment of crank effort = 29,465 ft.-lbs.
Mean moment of crank effort = 37,500 ft.-lbs.
Fluctuation of energy = 9,900 ft.-lbs. Co-efficient ·042.

7. In the case of a single crank the steam is cut off at one-fourth of the stroke. Neglecting back pressure and inertia, find the ratio of maximum to mean crank effort, and also the ratio of the fluctuation of energy to the energy of one revolution. Connecting rod = 4 cranks.

Maximum = 2·45 mean crank effort. Fluctuation of energy = ¼ energy of one revolution.

8. Construct a diagram of crank effort for three cranks at angles of 120°. The lines of stroke of the three pistons are parallel, the steam pressure constant, and the resistance uniform. Find the ratio of maximum to mean crank effort, and the co-efficient of fluctuation of energy for a connecting rod of 4 cranks.

Ans. Maximum = 1·077 mean crank effort. *k* = ·0115.

9. In engines with a pair of cranks at right angles, connecting rod 4 cranks long, the reciprocating parts attached to each crank have a stroke of 4 feet and weigh 20 tons. The steam pressure is uniform and equal to 50 tons on each piston, and the resistance moment is uniform. Find the least number of revolutions the engines can make without the aid of a fly-wheel and draw a curve of angular velocity ratio for this case.

Ans. At the point of maximum speed the least number of revolutions will be 50 per 1′. To obtain the curve and the least number of *complete* revolutions, see page 222.

10. The pressure equivalent to the weight of the reciprocating parts of an engine is 4 lbs. per square inch, the stroke is 4 feet. Neglecting obliquity find the pressure necessary to start the piston, when the engines are making 75 revolutions per minute. If the steam pressure be initially at 30 lbs above the atmosphere, and the cut off at ¼th the stroke, find the effective pressure at each eighth of the stroke, taking account of the inertia of the piston, and assuming a constant back pressure of 3 lbs.

Pressure equivalent to inertia at commencement of stroke = 15·3 lbs. per sq. in.

Effective pressure at commencement = 26·4
" " 1st eighth = 30·3
" " 2nd " = 34·0
" " 3rd " = 23·0
" " 4th " = 19·4
" " 5th " = 18·7
" " 6th " = 19·5
" " 7th " = 21·2
" " 8th " = 23·5

11. In the last question, assuming the connecting rod to be 4 times the crank, plot a curve of inertia and obtain corrected values of the effective pressure.

12. *In question 11 construct a curve showing the kinetic energy of the piston at each point of the stroke, and deduce a curve showing the pressure due to inertia of the piston.*

Take the curve of piston velocity previously constructed, and PN being any ordinate of it, the kinetic energy of the piston will be proportional to the square of PN, so we have only to draw a curve whose ordinates vary as $(PN)^2$.

Having drawn the curve of kinetic energy, take the difference between consecutive equidistant ordinates of that curve and set them up as ordinates in the same way as on page 226.

13. In the connecting rod of question 11, page 209, find what fraction of the weight of the rod should be added to the other reciprocating parts when calculating the effect of inertia on the crank effort. *Ans.* ·5.

14. If n be the revolutions per minute of a fly-wheel and d its diameter; show that the weight of wheel necessary for a given regularity in an engine of given indicated power is

$$W = C \cdot \frac{IHP}{n^3 d^2},$$

where C is constant.

NOTE. The diameter is generally about 3½ times the stroke (S), and according to a well-known empirical rule for piston speed (V) employed in calculating nominal horse-power $V^3 \propto S$. If this be assumed $n^3 d^2$ is constant and the weight of wheel is then proportional to the indicated horse-power, a rule sometimes employed, 100 lbs. being allowed for each horse-power.

15. The fluctuation of energy of an engine of 150 *I.H.P.* is 13 per cent. of the energy exerted in one revolution. The revolutions are 35 per minute, find the weight of a fly-wheel 20 feet in diameter, that the fluctuation in speed may not exceed one-fortieth. *Ans.* 8 tons.

16. Find the "inertia pressure" in the case of an oscillating engine.

Here if z be the distance of the centre of the crank pin from the centre of motion of the oscillating cylinder, the angular velocity-ratio of cylinder and crank (p. 106) is readily shown to be

$$\text{Angular velocity-ratio} = \tfrac{1}{2}\left(\frac{z_0^2}{z^2} - 1\right),$$

z_0 being the value of z for the extreme position of the cylinder. Hence the kinetic energy (U) of the cylinder is known in terms of z. But z differs from the space traversed by the piston only by a constant;

$$\therefore P = \frac{dU}{dz}$$

is the value of the inertia-pressure.

The effect of the inertia of an oscillating cylinder is the reverse of that of the piston and connecting rod of the ordinary type, the crank effort being increased at the beginning and reduced at the end of the stroke.

Proceeding as on p. 229, the kinetic energy of the piston and rod may be found in terms of z, and the inertia-pressure then deduced as above.

CHAPTER X.

FRICTIONAL RESISTANCES.

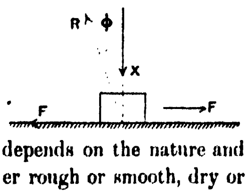
112. *Preliminary Remarks.*—The action of a machine consists, as we have seen, in a transmission of energy from a driving pair to a working pair, through a number of intermediate pairs, which change in a given way the motions proper to the source of energy. In the absence of friction, the energy transmitted from piece to piece in a complete period would be the same for all the pairs, but, in consequence of frictional resistances, a certain part of the energy is lost at each transmission. These frictional resistances are of two kinds, one due to the relative motion of the elements of the pairs one upon another, the other to the changes of form which the flexible parts of the machine undergo, for example to the bending of ropes and belts. It is to the first kind that the word "friction" is specially appropriated, although it is not essentially different from the second kind, which in some cases is also called "stiffness."

We commence with the case of linkwork mechanisms in which the friction is due simply to the sliding of one surface upon another. The pairing is in this case of the lower class.

SECTION I.—EFFICIENCY OF LOWER PAIRING.

113. *Ordinary Laws of Sliding Friction.*—If one body rests on another (Fig. 102) and is pressed against it with a force X, a mutual action takes place between the two which resists sliding. The magnitude of this mutual action or tangential stress (Ch. XII.) is measured by the force F which is necessary to produce sliding, and the ratio F/X is called the co-efficient of friction and will be denoted by f. The value of f depends on the nature and condition of the surfaces in contact, whether rough or smooth, dry or lubricated. Under certain circumstances and within certain limits it is independent of the area of the surfaces in contact and of the velocity

Fig. 102.



of sliding. These statements may be called the "ordinary" laws of friction. The evidence on which they rest and the limitations to their truth will be considered hereafter; for the present we assume them as applicable to all the cases we consider.

The work done in overcoming friction may be estimated just as in the case of any other resistance. If the body move through a space x the work done is Fx or $f.Xx$ if X be uniform, and if it be not, a curve is constructed giving X at every point, then the area under that curve multiplied by the co-efficient f is the work done (See Ex. 2). If R be the reaction of the surface upon which the body we are considering rests, ϕ the angle its direction makes with the normal to the plane,

$$R.\cos\phi = X : R.\sin\phi = F;$$

$$\therefore \tan\phi = f,$$

an equation which shows that the total mutual action between two plane surfaces, which slide over one another, makes an angle with the normal to the plane, the tangent of which is the co-efficient of friction The magnitude of this angle then is fixed, but its direction varies according to the direction of the sliding. It may therefore be called the "friction angle," but it is also often called the "angle of repose," because it is the greatest inclination of a plane on which the body can rest under the action of gravity without slipping. In the solution of questions respecting friction, graphically or otherwise, it is often convenient to suppose it known.

114. *Friction of Bearings.*—Next suppose the surfaces in contact cylindrical. In Fig. 103 ABA represents a cylinder pressed down into a semi-circular bearing by a force S, the direction of which passes through the point O, which is the intersection of the axis of the cylinder with the plane of the paper. We may take this to represent the ordinary case of a shaft and its bearings from which the cap has been removed, S being the resultant of all the forces acting on the

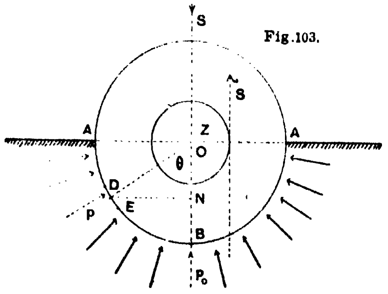


Fig. 103.

shaft which, for the moment, are supposed to have no tendency to turn the shaft. The force S is balanced by the reaction of the bearing which, when the bearing is in good condition, consists of a pressure distributed over the whole semi-cylindrical surface. Let DE be a small element of the surface, p the pressure, θ the angle the radius of DE makes with the direction of S, then we must have

$$\Sigma p DE \cos\theta = S.$$

If now we knew the law according to which p varies from point to point, we could by use of this equation find the actual value of p and also find the total amount of the distributed pressure, that is to say, $\Sigma p \,.\, DE$ which we will call X. Evidently then we shall have

$$X = k \,.\, S,$$

where k is a co-efficient depending on the law of distribution and therefore to some extent uncertain. When a bearing is well worn and imperfectly lubricated it is probable that (see Art. 115) if p_0 be the pressure at B

$$p = p_0 \,.\, \cos\theta,$$

that is, that the intensity of the pressure at any point varies as ON the distance of the point below the centre. This is the same law as that which the pressure of a heavy fluid follows, supposed occupying the semi-cylinder ABA, and it is shown in books on hydrostatics that

$$\frac{\text{Total pressure}}{\text{Resultant pressure}} = \frac{4}{\pi} = k.$$

Next suppose the shaft to be turned by the action of a couple M applied to it, then if a be the radius

$$M = \Sigma f \,.\, p \,.\, DE \,.\, a = f \,.\, Xa = fk \,.\, Sa.$$

In this formula we have some doubt as to the value of k, and we are not sure that the co-efficient f would be the same for a curve as for a plane surface; we therefore replace fk by f', where f' is a special co-efficient of axle friction determined by experiment. If there is a cap on the bearing, which is screwed down, the value of S is increased by an amount about equal to the tension of the bolts.

The loss of energy per revolution in overcoming axle friction is evidently $M \,.\, 2\pi$, or, if d be the diameter,

$$\text{Work lost} = \pi f' S d.$$

The reaction of the bearing surface on the shaft is partly normal and partly tangential. The normal part balances S and the tangential part balances M, hence the two parts may be combined into a single force opposite and parallel to S at such a distance z from O that

$$Sz = M, \text{ or } 2z = fd,$$

that is to say, the line of action of the mutual action between the shaft and its bearing always touches a circle, the diameter of which is f' times the diameter of the shaft. This circle is called the Friction Circle of the shaft or pin considered. When the bearing has a cap on, the force S must be increased by the tension of the bolts in calculating M, but not for any other purpose, and the diameter of the friction circle is consequently increased, it may be very considerably. The utility of this rule will be seen presently.

The real pressure between a shaft and its bearing varies from point to point, as we have seen. What is conventionally called the "pressure on the bearing" is something different. Let l be the length of the bearing, then ld is the area of the diametral section, and

$$p=\frac{S}{ld}$$

is the quantity in question. It is a sort of mean value of the actual pressure, and will bear some definite relation to it depending on the law of pressure. For the particular law of pressure given above

$$p=p_0\cdot\frac{\pi}{4}.$$

The work lost by friction per square inch of bearing surface per 1′ is evidently proportional to pv, where v is the rubbing velocity in feet per minute. An equivalent amount of heat is generated as we shall see hereafter, and it is upon the rate at which this heat can be abstracted by the cooling influences to which the bearing is exposed that the amount of bearing surface required depends. We shall return to this hereafter (p. 249); for the present, it is sufficient to say that the pressure on a bearing may sometimes be as much as 1000 lbs. per sq. inch when the load is alternating, as in the case of a crank-pin, but is limited to 300 lbs. or less when the load is always in one direction.

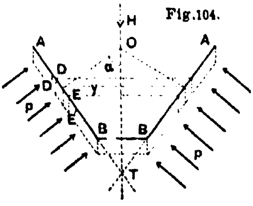


Fig.104.

115. *Friction of Pivots.*—In pivots and other examples in which the revolving shaft is subject to an endways force the surfaces in contact are frequently conical. In Fig. 104 a conical surface AB is pressed against a corresponding conical seating by a force H, and revolves at a given rate. If the surface be divided into rings, one of which is seen in section at DE, the pressure on those rings may be resolved vertically upwards and must then balance H. Hence if p be the pressure on DE a ring the radius of which is y,

$$\Sigma p\,.\,DE\,.\,2\pi y\cos a=H,$$

where α is the angle a normal to the conical surface makes with the axis.

When the bearing is somewhat worn the conical surface will have descended through a certain space, and it may be assumed that all points such as DE will descend through an equal space, so that the wear of the surface measured normal to itself is proportional to $\cos\alpha$.

But if v be the velocity of rubbing of the ring DE, the wear will be proportional to pv, that is to py: hence

$$py \propto \cos\alpha.$$

This principle determines the most probable distribution of the pressure on worn surfaces in any case, and has already been used above for the case of a journal. In the present case α is constant, and we have

$$py = \text{constant} = p_1y_1 = p_2y_2,$$

where the suffixes 1 and 2 refer to the upper and lower edge; hence, by substitution, if l be the length AB of the conical surface,

$$py \,.\, 2\pi l \,.\, \cos\alpha = H,$$

a formula which determines the pressure at every point. The moment of friction is evidently

$$M = f\Sigma pDE2\pi y^2$$
$$= f \,.\, py \,.\, 2\pi \,.\, \Sigma y \,.\, DE = \frac{fpy2\pi\Sigma y\Delta y}{\cos\alpha},$$

where Δy is written for the projection of DE on the transverse plane. By use of the integral calculus this is readily seen to be

$$M = fpy2\pi\frac{y_1^2 - y_2^2}{2\cos\alpha} = fpy2\pi l \,.\, \frac{y_1 + y_2}{2},$$

$$\text{or } M = f \,.\, H \,.\, \frac{y_1 + y_2}{2\cos\alpha},$$

a formula which shows that the friction is the same as that of a ring of small breadth, of diameter equal to the mean of the greatest and least diameters of the portion of a cone considered. In the case of a simple flat-ended pivot the equivalent ring is half the diameter of the pivot. If the pressure were uniform throughout, the diameter of the equivalent ring would be $\frac{2}{3}$ instead of $\frac{1}{2}$ the diameter of the pivot, and the actual diameter in practice will probably vary between these limits.

Pivots are sometimes used in which the surfaces in contact are not cones, but are curved, so that in wearing the pressure and wear are the same throughout (Schiele's pivots). That this may be the case we must have, since p is constant,

$$y \propto \cos\alpha,$$

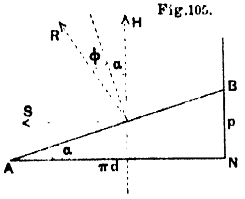
that is to say, if we draw a tangent DET to meet the axis in T, ET must be constant. The curve which possesses this geometric property is called the "tractrix." It is traced readily by stepping from point to point, keeping the tangent always of the same length. Pivots of this kind are very suitable for high speeds, as the wear is very smooth.

116. *Friction and Efficiency of Screws.*—In any case of a machine in steady motion the principle of work takes the form (Art. 96)

$$\left.\begin{array}{c}\text{Energy exerted}\\ \text{in a period}\end{array}\right\} = \left\{\begin{array}{l}\text{Useful work done + Work wasted}\\ \text{in overcoming frictional resistance.}\end{array}\right.$$

The simplest case is that of a screw which we will suppose to be square threaded and applied to a press, or to some similar purpose. The pressure between the nut and the thread is distributed uniformly along the thread, if the screw be accurately constructed and slightly worn. As shown in the last article in the similar case of a pivot, the friction may be regarded as concentrated on a spiral traced on a cylinder the diameter of which may be expected to be about the mean of the external and internal diameter of the screw. Fig. 105 shows one convolution of this spiral unrolled. AB is the thread, BN parallel to the axis of the screw is the pitch p, and AN is the circumference πd. H is the thrust of the screw, being the force which the screw is overcoming by means of a couple applied to turn it about its axis. R is the action of the screw thread which (Art. 113) makes an angle ϕ with the normal, where ϕ is the angle of repose. The normal itself makes an angle α with the axis of the screw, where α is the pitch angle given by the formula

Fig. 105.



$$\tan\alpha = \frac{p}{\pi d}.$$

This force R arises from the turning forces applied to the screw, and must have the same moment M about the axis of the screw; its vertical component therefore must be H and its transverse component a force S such that

$$S\cdot\frac{d}{2}=M.$$

Hence the equations

$$M=\frac{Rd}{2}\cdot\sin(\alpha+\phi),$$

$$H=R\,.\cos(\alpha+\phi).$$

Also considering a complete revolution of the screw,

$$\text{Energy exerted} = M\,.\,2\pi = R\pi d\,.\sin(\alpha+\phi),$$

$$\text{Useful work done} = H\,.\,p = Rp\,.\cos(\alpha+\phi),$$

from which it follows that the efficiency of the screw is

$$\text{Efficiency} = \frac{\tan \alpha}{\tan (\alpha + \phi)}.$$

It is not difficult to show that this fraction is greatest when $\alpha = 45^\circ - \frac{1}{2}\phi$, and its value is then

$$\text{Maximum efficiency} = \left(\frac{1 - \frac{1}{2}f}{1 + \frac{1}{2}f}\right)^2 \text{ approximately.}$$

For ordinary values of f then, the best pitch angle is approximately 45° and the efficiency is considerable.

In practice, however, the pitch angle is much smaller, its value in bolts and the screws used in presses ranging from ·035 in large screws to ·07 in smaller ones; the efficiency is then less, often much less, than one-third, the object aimed at being not efficiency but a great mechanical advantage.

If the pitch be sufficiently coarse, it will be possible to reverse the action, the driving force being then H and the resistance a moment opposing the rotation of the screw. In a well known kind of hand drill and a few other cases this occurs in practice; the force R is then inclined on the other side of the normal, and the efficiency is in the same way as before found to be

$$\text{Efficiency} = \frac{\tan (\alpha - \phi)}{\tan \alpha}.$$

In most cases, however, α is less than ϕ, and the screw is then incapable of being reversed. Non-reversibility is often a most valuable property in practical applications, the friction then serving to hold together parts which require to be united or to lock a machine in any given position.

In estimating the efficiency of screw mechanisms the friction of the end of the screw acting like a pivot or of the nut upon its seat must be included; in screw bolts this item is generally as great as the friction of the threads. The friction due to lateral pressure of the screw on its nut may usually be neglected, but when necessary it may be estimated by the same formula as is used for shafts. The above investigation, strictly speaking, applies only to square-threaded screws; it has, however, been shown that the efficiency is only slightly diminished by the triangular or other form of thread usually adopted for the sake of strength.* The formulæ here given for screws may be applied to any case of a sliding pair in which the driving effort is at right angles to the useful resistance. A simpler case is that in which the driving effort is parallel to the direction of sliding. This is given in Example 1, page 260. In all cases observe that the efficiency diminishes rapidly when the velocity-ratio is increased. This, which is common to most mechanisms, limits the mechanical advantage practically attainable. The hydraulic press is an exception, as will be seen hereafter.

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* *Cours de Mécanique Appliquée aux Machines*, par J. V. Poncelet, p. 386. Paris, 1874.

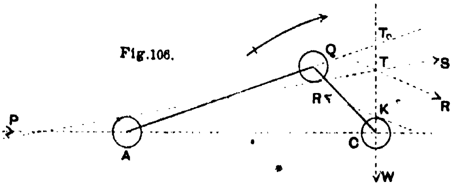
117. *Efficiency of Mechanism by Exact Method.*—In the preceding cases the efficiency is the same for any motion of the mechanism whether large or small. Generally, however, it will be different in each position of the mechanism, and by the "efficiency of the mechanism" is then to be understood the ratio of the useful work done in a period to the energy exerted in the period.

The exact calculation of the loss of work by frictional resistances in mechanism is generally very complicated, so that it is best to proceed by approximations the nature of which will be understood on considering an example with some degree of thoroughness. The case we select is that of the mechanism of the direct-acting vertical steam engine such as is represented in Plate I., page 108.

The losses by friction are (1) the loss by piston friction, (2) friction of guide bars, (3) friction of crosshead pin, (4) friction of crank pin, (5) friction of crank shaft-bearings. Of these, the first two are considered separately (Ex. 2, p. 261), and for the present neglected, whilst the last three are treated by a graphical method as follows.

In Fig. 106 *CQA* are the friction circles of the three parts in question,

Fig.106.



which for the sake of clearness are drawn on a very exaggerated scale while the bearings themselves are omitted. We will neglect the weight of the connecting rod and its inertia; of these the first is generally relatively inconsiderable, but in high speed engines the last is often very large and makes the friction very different at high speeds and low speeds (see Chap. XI.). The weight of the crank shaft and all the parts connected with it is supposed to act through the centre of the shaft; for simplicity we will call it *W* The pressure on the piston after correction for piston and guide-bar friction is denoted by *P*. Then, in the absence of friction, the line of action of the thrust on the connecting rod is the line joining the centres of the friction circles, and the moment of crank effort is $P . CT_0$, where T_0 is the intersection of that line with the vertical through C. But the line of action in question must now touch the friction circles (Art. 114), and the true moment of crank effort on the same principle must be $P . CT$, where T is the intersection of this common tangent with the vertical CT. Thus $P . TT_0$ is the correction for friction of the crosshead and crank pins. Next observe that the forces acting on the crank shaft are W the weight and S the thrust of the connecting rod; these may be compounded into one force R passing through T as shown in the diagram. The reaction of the crank-shaft bearing is an equal and opposite force R which must touch the friction circle and cut CT in a certain point K. Now the horizontal component of R is the same as that of S, namely P; therefore the true moment of crank effort after allowing for friction is $P . TK$.

By performing this construction for a number of positions, as in the last chapter, we obtain a diagram of crank effort corrected for friction. The area of this curve will give us the useful work done in a revolution, the ratio of which to the energy exerted is the efficiency of the mechanism: and its intersections with the line of mean resistance will give the points of maximum and minimum energy and the fluctuation of energy as corrected for friction. When the crank makes a certain angle with the line of centres TK vanishes. Within this angle no steam pressure, however great, will move the crank, as is well known in practice. It may be called the "dead angle," all points within it being dead points.

118. *Efficiency of Mechanism by Approximate Method.*—The process just described is not too complicated for actual use in the foregoing example, but in many cases it would be otherwise, and it may therefore be frequently replaced with advantage by a calculation of the efficiency of each of the several pairs of which the mechanism is made up taken by itself.

Each pair consists of two elements, one of which transmits energy to the other, with a certain deduction caused by the friction between the elements. The ratio of the energy transmitted to the energy received may be called the efficiency of the pair. If $e_1, e_2, e_3, \ldots$ be the efficiencies of all the pairs in the mechanism it is evident from the definition that the efficiency of the whole mechanism must be

$$e = e_1 . e_2 . e_3 \ldots .$$

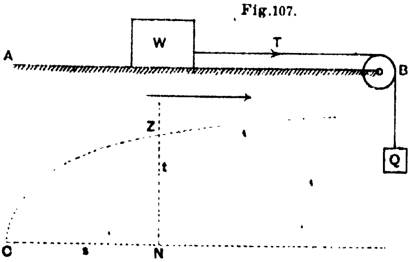
In some cases the efficiency of each pair will be independent of the frictional resistances of all the other pairs, and may be found separately. In general this is approximately, but not exactly, true, a point which will be best understood by a consideration of the foregoing diagram.

For example, the friction of the guide bars is diminished in consequence of the friction of the crank pin, because the obliquity of the connecting rod is virtually diminished. The supposition is, however, often sufficiently nearly true to enable a rough estimate to be made of the efficiency of the mechanism by finding the efficiencies of the several pairs taken alone, all the others being supposed smooth. In doing this mean values are taken for variable forces, if the amount of variation be not considerable. The uncertainty and variability of the co-efficients on which frictional efficiency depends are such as to render refined calculations of little practical value.

119. *Experiments on Sliding Friction (Morin).*—The ordinary laws of friction, which may be comprised in the single statement that the co-efficient of friction depends on the nature of the surfaces alone, and not on the intensity of the pressure or on the velocity of rubbing, were originally given by Coulomb in a memoir published in 1785, although some facts of a similar kind were previously known. They are therefore often called Coulomb's laws. Yet Coulomb's experiments were scarcely sufficient to establish them, and the subject was reinvestigated by others, especially by the late General Morin, whose memoirs were presented to the French Academy in 1831-4. Morin's experiments were so elaborate and exact that they may be considered as conclusively proving the truth of Coulomb's laws within certain limits of pressure and velocity, and under the circumstances in which they were made: it will therefore be advisable to explain them briefly.

A sledge loaded with a given weight was caused to slide along a horizontal bed *AB* more than 12 feet long (Fig. 107), the rubbing

Fig.107.



surfaces being formed of the materials to be experimented on. The necessary force was supplied by a cord passing over a pulley at *B* to a descending weight *Q*. The tension of the cord *T* was measured by a spring dynamometer, and could likewise be inferred from the magnitude of the weight after correction for the stiffness of the cord and the friction of the pulley. In one form of experiment the weights were so arranged that the sledge moved nearly uniformly: the corresponding friction was measured and found to be constant. In a second form, the times occupied by the sledge in reaching given points were automatically measured and compared with the spaces traversed, by setting them up as ordinates of the curve *CZ* shown below. The curve proved to be a parabola, showing that the space varied as the square of the time, from which it was inferred that the acceleration of the sledge was constant.

From both methods it appeared that the co efficient of friction was exactly the same, whatever the pressure and whatever the velocity, provided the nature and condition of the surfaces were the same. A few important results are given in the annexed table; they are taken from Morin's latest memoir,* containing, besides many new experiments, tables of the results of the whole series. The limits to their application will be considered presently.

NATURE OF SURFACES.	CONDITION OF SURFACES.	CO EFFICIENT OF FRICTION.
Wood on Wood,	Perfectly dry and clean,	·25 to ·5
Metal or Wood on Metal or Wood,	Slightly oily,	·15
Do., do.,	Well lubricated,	·07 to ·08
Do., do.,	Lubricant constantly renewed,	·05

Full tables of Morin's results will be found in Moseley's work cited on page 257. The friction between surfaces at rest is often greater than when they are in motion, especially when the surfaces have been some time in contact: the excess, however, cannot be relied on, as it is liable to be overcome by any slight vibration. In the case of metal on metal whether dry or oily, or in wood on wood when dry and clean, Morin, however, found that there is no such difference.

120. *Limits of the Ordinary Laws.*—From the exactitude with which Coulomb's laws were verified by Morin's experiments the inference was naturally drawn that they were universally true, but this is probably erroneous, even for dry surfaces without sensible adhesion. Although no complete and thorough investigation has been made, it can hardly now be a matter of doubt that there are cases in which the laws of friction are widely different. The known cases of exception for plane surfaces may be grouped as follows:—

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* Nouvelles Experiences . . . faites à Metz en 1834. Page 99.

(1) At low pressures the co-efficient of friction increases when the pressure diminishes. This has been shown by various experimentalists, as, for example, by Dr. Ball.* The lowest pressure employed by Morin was about three-fourths of a lb. per square inch, and this is about the pressure at which the deviation noticed by Ball becomes insensible. This effect may be ascribed to a slight adhesion between the surfaces independent of friction proper.

(2) At high pressures, according to certain experiments by Rennie,† the co-efficient increases greatly with the pressure. The upper limit of pressure in Morin's experiments was from 114 to 128 lbs. per square inch. At 32·5 lbs. per square inch Rennie found for metallic surfaces at rest ·14 to ·17, nearly agreeing with Morin; but on increasing the pressure the co-efficient became gradually greater, ranging from ·35 to ·4 at pressures exceeding 500 lbs. per square inch. The metals tried were wrought iron on wrought and cast iron, and steel on cast iron. Tin on cast iron showed only a slight increase in the co-efficient. This increased friction at high pressures may be ascribed to abrasion of the surfaces.

(3) At high velocities the co-efficient of friction, instead of being independent of the velocity, diminishes greatly as the velocity increases. This was shown by M. Bochet in 1858. Similar results have been obtained by others, especially by Captain (now Sir Douglas) Galton in some important experiments on railway brakes.‡ The limit of velocity in Morin's experiments was 10 feet per 1″, and at somewhat greater velocities than this the diminution becomes perceptible. Morin's results have been shown to be applicable at the very lowest velocities by the late Professor F. Jenkin and Mr. (now Professor) Ewing,§ the friction increasing at excessively low velocities in those cases only in which there is a difference between the friction of rest and the friction of motion.

It appears difficult to explain the diminution at high speeds merely by a change in the condition of the surfaces; it should, probably, be regarded as part of the law of friction. Professor Franke in the *Civil Ingenieur* for May, 1882, has proposed the formula

$$f = f_0 \cdot \epsilon^{-av},$$

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* *Experimental Mechanics*, by R. S. Ball, p. 78. Macmillan, 1871.

† *Phil. Trans.* for 1829.

‡ *Engineering*, vol. 25, pages 469-472.

§ *Phil. Trans.*, vol. 167, part II.

where f_0 is about ·29, and a (for velocities in metres per 1″) ranges, from ·02 to ·04, according to the nature and state of the surfaces.

121. *Axle Friction.*—It has already been pointed out that the co-efficient of axle friction is not necessarily the same as that for plane surfaces sliding on one another, and, besides, the continuous contact of a shaft and its bearing is very different from the brief contact occurring in sledge experiments. Morin, however, made special experiments on the friction of axles, and showed that the co-efficients were constant and nearly the same in the two cases. The diameters employed, however, were 4 inches and under, while the revolutions did not exceed 30 per minute, so that the rubbing velocity was not more than 30 feet per minute. The pressures were not great, the value of pv not exceeding 5,000.

Much greater values of pv than this are common in modern practice, and then it is certain that the value of the co-efficient is much less and diminishes with the pressure. Already in 1855 M. Hirn had made a long series of experiments on friction, especially of lubricated surfaces. The following summary of his results is given by M. Kretz, editor of the third edition of the *Mécanique Industrielle*.*

(*a*) That a lubricant may give a regular and minimum value to the friction it must be "triturated" for some time between the rubbing surfaces.

(*b*) The friction of lubricated surfaces diminishes when the temperature is raised, other things being equal.

(*c*) With abundant lubrication and uniform temperature friction varies directly as the velocity. When the temperature is not maintained uniform, the relation between friction and velocity depends on the law of cooling of the special machine considered. In ordinary machinery friction varies as the square root of velocity.

(*d*) The friction of lubricated surfaces is nearly proportional to the square root of the area and the pressure.

Experiments made in 1883-4 by Mr. Tower,† and subsequently by others, have however conclusively shown that, with thoroughly efficient lubrication, the friction of a bearing under pressures exceeding 100 lbs. per sq. inch is independent of the pressure. This is the well-known law of friction between a fluid and a surface in contact with it, and indicates that the bearing surfaces are not in actual contact but are separated by a thin film of lubricant, a fact which has also been proved by direct experiment. The co-efficient of friction in such cases varies inversely as the pressure and under heavy loads may be less than one-tenth of the value (·05) given by Morin. Thus Hirn's law of pressure, though verified by other experiments, is only true in special cases; in other respects his conclusions appear on the whole correct. With thorough lubrication very heavy pressures and high speeds may be employed, the value of pr reaching, or even exceeding, 100,000, but any inaccuracy of fit in consequence of which parts of the bearing surfaces come into actual contact will at once cause heating. On the causes which produce heating the reader is referred to a paper by Professor Denton.*

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* *Introduction à la Mécanique Industrielle*, par J. V. Poncelet. Troisième édition, Paris, 1870. Page 516.

† *Proceedings of the Institution of Mechanical Engineers.*

SECTION II.—EFFICIENCY OF HIGHER PAIRING.

122. *Rolling Friction.*—We now proceed to consider higher pairing, commencing with the case of rolling contact. The friction is then described as "rolling friction."

When a wheel rolls on soft ground the resistance to rolling is due to the fact that the wheel makes a rut and depresses the ground as it advances over it. Thus the resistance to motion is proportioned to the product of the weight moved into the depth of the depression. The depth of the rut depends on the radius as well as the breadth of the wheel. It is found that the resistance may be expressed by

$$R = \frac{bW}{r},$$

where W = weight, r = radius of wheel, and b is approximately a constant length. This might have been anticipated, since the depth of the rut is of the versed sine of the arc of contact, and therefore for a given small arc is inversely as the radius. If the wheel roll on hard ground over a succession of obstacles of small height the law of resistance will be expressed by the same formula.

When the surface rolled over is elastic, and the pressure on it is not sufficient to produce a permanent rut, the resistance to rolling is not so easily explained. If we consider an extreme case, as for instance a heavy roller rolling on india-rubber, we shall be able to see to what action the resistance is due. The wheel will sink into the rubber, which will close up around it both in advance and behind, as shown in Fig. 108. At C the rubber will be most compressed. As the wheel advances and commences to crush the rubber in advance of it the rubber moves away to avoid the compression, heaping itself up continually in advance of the wheel. In this movement it rubs itself over the surface *Ca* of the wheel, exerting on it a frictional force in the direction shown by the arrow *F*, which opposes the onward motion of the wheel. Again, the rubber in the rear is continually tending to recover its normal position and form of flatness, and in doing so rubs itself over the surface *bC* of the wheel in the direction shown by the arrow *F'*, which also tends to oppose the onward motion of the wheel. The effect of this creeping action of the rubber over the surface of the wheel is to cause the onward advance of the centre of the wheel to be different from that due to the circumference rolled out.* Moreover the vertical component of the reaction of the surface no longer passes through the centre of the wheel as it must do in the absence of friction, but is in advance by a small quantity *b* such that *Wb* is the moment of resistance to rolling.

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* "*Special Experiments on Lubricants,*" by Prof. J. E. Denton. American Society of Mechanical Engineers. November, 1890.

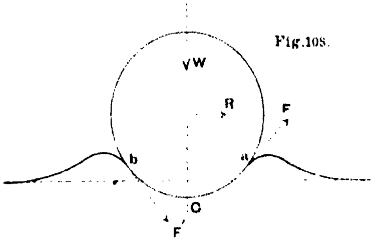

Fig. 108.

Experiments on rolling resistance present considerable discrepancies, but within the limits of dimension of rollers which have been tried it appears that *b* is independent of the radius; this leads to a formula of the same form as before for the force necessary to draw the roller, namely

$$R = \frac{Wb}{r},$$

where *b* is a constant which for dimensions in inches is from ·02 to ·09 according to the nature of the surfaces. With very hard and smooth surfaces of wood or metal, the lower value ·02 may be employed. Rolling friction is not sensibly diminished by lubricants, but depends mainly on smoothness and hardness of the surfaces. It is probably influenced by the speed of rolling, but this does not appear to have been proved by experiment unless in cases where the resistance of the atmosphere and other causes make the question more complicated.

In many cases of rolling the surfaces are partly elastic and partly soft, so that the resistance to rolling is partly due to surface friction and partly to permanent deformation. The value of the constant b is then much increased. For wagon wheels on macadamized roads in good condition the value of b is about ·5", and on soft ground four to six times greater. The draught of carts is said to be increased by the absence of springs.

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* See a Paper by Professor Osborne Reynolds, *Phil. Trans.*, vol. 166, to whom the true explanation of resistance to rolling in perfectly elastic bodies is due.

123. *Friction of Ropes and Belts.*—Frictional resistances are also produced by the changes of form and dimension of the parts of a machine occasioned either by the stresses necessarily accompanying transmission of energy or by shock. In the present chapter we consider tension elements only, that is to say, chiefly ropes and belts.

In Fig. 109 AB is a pulley, the centre of which is O, over which a rope passes embracing the arc AKB and acted on by forces T_1T_2 at its ends. If their be sufficient difference between T_1 and T_2 the rope will slip over the pulley notwithstanding the friction which tends to prevent it. Let the rope be just on the point of slipping, then its tension will gradually diminish from T_1 at A to T_2 at B. Let T, T' be the tensions at the intermediate points K, L, then the portion KL of the rope is kept in equilibrium by the forces T, T' at its ends, and a third force S due to the reaction of the pulley, the three forces meeting in a point E. On OL set off to Ol to represent T, and draw lk perpendicular to S to meet OK in k, then the sides and the triangle Okl will be proportioned by the three forces, so that Ok represents T' and lk S. The angle S makes with the radius will be the same for all arcs of the same length, and if KL be taken small enough will be the angle of friction (Art. 113).

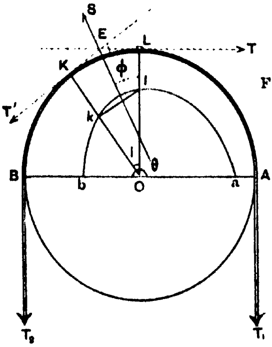


Fig. 109.

This construction can, if we please, be commenced at A and repeated for a number of small portions of the rope till we arrive at B; we shall obtain a spiral curve $alkb$, the last radius Ob of which represents T_2 on the same scale as the first Oa represents T_1. It is convenient, however, to have an algebraical formula to calculate T_2. Let the angle KOL be i and the angle S makes the radius ϕ, then

$$\frac{T}{T'} = \frac{Ol}{Ok} = \frac{\sin Okl}{\sin Olk} = \frac{\cos(i-\phi)}{\cos\phi} = \cos i + \sin i \tan\phi.$$

If now the angle i be diminished indefinitely we may write $\cos i = 1$ and $\sin i = i$, so that

$$\frac{T - T'}{T'} = i \,.\, \tan \phi.$$

Replacing i by $\Delta\theta$, $T - T'$ by ΔT, and proceeding to the limit

$$\frac{1}{T}\frac{dT}{d\theta} = \tan \phi = f,$$

which being integrated gives

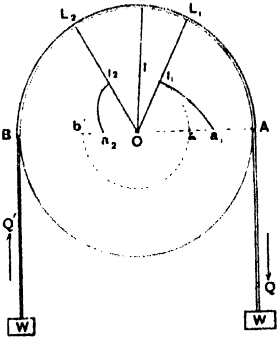
$$\frac{T_1}{T_2} = \epsilon^{f\theta},$$

where f is the co-efficient of friction, θ the angle subtended by the part of the pulleys embraced by the rope, and ϵ the number 2·718 being the base of the Naperian system of logarithms. The formula is applicable even if the pulley be not circular. For a circular pulley the spiral curve, representing graphically the tension at every point, is the equiangular or logarithmic spiral of which the formula may be regarded as the equation. In constructing it graphically, the value of ϕ, for a small yet finite angle i, is found by replacing $T:T$ by ϵ^{fi} and expanding the exponential: we thus get approximately

$$1 + fi = \cos i + \sin i \,.\, \tan \phi = 1 - \tfrac{1}{2}i^2 + i \,.\, \tan \phi,$$
$$\therefore\ \tan \phi = f + \tfrac{1}{2}i.$$

With small values of the co-efficient $2f$ may be a sufficiently small angular interval, but in general it will be advisable to take the angular interval equal to the angle of friction. then the value of ϕ is $1\frac{1}{2}$ times that angle. The construction being one in which errors accumulate. the formula is preferable when great accuracy is desired.

Fig. 110.



124. *Driving Belts.*—When a belt is stretched over a pulley by equal weights, the tension of the belts is not necessarily the same everywhere in the first instance; but if the pulley move steadily and the stiffness of the belt be disregarded, it must be so. Assuming this, let one of the weights be increased by a certain quantity Q and the pulley be held fast, then the tension of that side of the belt will be increased by an amount equal to Q at A, but diminishing to zero at L_1, a point determined by the intersection of the friction spiral a_1, l_1 (Fig. 110) with the circle alb, the radius of which represents the weight W.

Similarly if the other weight be diminished by Q', the tension will be diminished by an amount equal to Q' at B, but diminishing to zero at L_2. The portion L_1L_2 will remain at the original tension W. If QQ' be increased sufficiently, L_1, L_2 will coincide in one point L, the position of which will depend on the proportion between Q and Q'. While these changes take place in tension, corresponding changes of length must occur in the parts of the belt exposed to them, AL_1 increases and BL_2 diminishes in length. Hence both these parts slip over the pulley and work is lost by friction, while L_1L_2 remains fixed. If now, instead of altering the weights W, we imagine these weights held fast, and the pulley forcibly rotated so as to increase A's tension by Q, and diminish B's tension by Q', L_1L_2 will rotate with the pulley, and the total increase of length of the one side must be equal to the total diminution on the other, from which consideration it is possible to calculate the ratio Q bears to Q'. In practical cases, however, the difference between Q and Q' is so small that it may be neglected without sensible error, and therefore, in all questions relating to the working of belts, it may be assumed that the mean tension of the two sides of the belt is independent of the power which is being transmitted. The difference of tensions, however, is directly proportional to the power, and may at once be calculated if the speed be known, while the ratio of tensions may be determined, so that the belt shall just not slip, by means of the formula above obtained. The value of the co-efficient of friction of leather on iron ranges from ·15 to ·46 according to the degree of lubrication: under ordinary circumstances ·25 may be considered an average value. This, however, is often greatly exceeded in practice, and one reason why large values are admissible is said by some to be the effect of atmospheric pressure. The sectional area of belts is fixed by consideration of strengths, and as their thickness varies little, this is equivalent to saying that a certain breadth of belt is required for each horse-power transmitted. (See Ex. 11, p. 261.)

125. *Slip of Belts.*—When a belt is stretched over a pair of pulleys, one of which drives the other, notwithstanding a resistance not so great as to cause slipping of the belt as a whole, it appears from what has been said that a certain arc exists on each pulley on which the belt does not slip. The length of these arcs has already been found, but in the present cases the movement of the pulleys causes them to place themselves where the belt winds on to the pulleys, so that the driving pulley has the speed of the tight side of the belt and the driven pulley that of the slack side. The two sides have different speeds, because the same weight of belt must pass a given point in a unit of time, wherever that point be situated, and therefore the speed must be greater the greater the elongation, that is to say, the greater the tension. Hence the driving pulley moves quicker than the driven pulley by an amount which can be calculated when the tensions and the elasticity of the leather are known, and the "slip" measures the loss of work due to the creeping of the belt over the pulleys described above. In ordinary belting the slip does not exceed 2 per cent., and is believed to be often insensible. The length of belts, however, must not be too great, or its extensibility will be inconvenient, especially if the motion of the machine be not sufficiently uniform.* Within moderate limits extensibility is favourable to smooth working.

126. *Stiffness of Ropes.*—When a rope is bent it is found that a certain moment is required to do it depending on the dimensions of the rope and, besides, on its tension. The reason of this is best understood by referring to the corresponding case in a chain with flat links united by pin joints. If d be the diameter of the pin, T the tension of the chain, there will be a certain moment of friction resisting bending which, if the pin be any easy fit, will be simply $\frac{1}{2}fTd$, but if it be tight will be

$$M = \frac{1}{2}fTd + \frac{1}{2}fT_0d,$$

where T_0 is a constant depending on the tightness. If the chain pass over a rotating pulley without slipping, this frictional moment has to be overcome both when bending on and when bending off the pulley. The effect shows itself by a shaft outwards on the advancing and inwards on the retiring side of the chain, so as to increase the leverage of the resistance and diminish that of the effort. In the present case the two shifts are equal, being each given by the formula

$$x = \frac{1}{2}fd\left\{1 + \frac{T_0}{T}\right\}.$$

The case of a rope differs from this only in being more complex: in the act of bending, the fibres move over each other, and the relative motion is resisted by friction due to pressures which are partly constant and partly proportional to the tension. The shift of the centre line of the rope is visible on the side of the resistance, but hardly perceptible on the side of the hauling force, showing that most of the loss of work is due to the bending on the pulley. The magnitude of the shift varies so much according to the mode of manufacture and the condition of the rope that it is useless to attempt more than a very rough estimate.

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* See a footnote by M. Kretz, *Cours de Mécanique Appliquée aux Machines*, par Poncelet, p. 264.

According to a formula given by Eytelwein, if d be the diameter of the rope,

$$x = c \, . \, d^2,$$

where c is a constant, which for dimensions in inches is taken as ·47 for hemp ropes; but this value is too large, except for light loads, and small diameters of pulley. The loss of work per revolution is $T . 2\pi x$, and if D be the effective diameter of the pulley,

$$\text{Efficiency} = \frac{D}{D + 2x}.$$

There is a loss of work by the stiffness of belts of a similar kind, but of uncertain amount. By most authorities it is considered so small as to be negligible.

The shift of the line of action of the tension of a rope due to its stiffness has the effect of diminishing its strength.

127. *Friction of Toothed Wheels and Cams.*—The friction of toothed wheels is partly rolling and partly sliding, but the first is relatively small and may be neglected. To determine the sliding friction, let $PT = z$ (see Fig. 71, page 119), then (page 153) the velocity of rubbing is given by the formula,

$$v = (A + A')z,$$

which may be written, if V be the speed of periphery of the pitch circles, R, R' the radii,

$$v = z\left\{\frac{1}{R} + \frac{1}{R'}\right\}V.$$

If, therefore, the wheels be supposed to turn through a small space δx measured on the pitch circles, the pair of teeth will slide on one another through the small space δy, given by the formula

$$\delta y = \left(\frac{1}{R} + \frac{1}{R'}\right) z \delta x.$$

This enables us to find the work done in overcoming friction, for if P be the pressure between the pairs of teeth,

$$\text{Work done} = \int f \, . \, P dy = \left(\frac{1}{R} + \frac{1}{R'}\right) \int f \, . \, P z dx.$$

The pressure between the teeth will vary as the wheels turn according to some unknown law, depending on the way the teeth wear and the co-efficient f probably varies. Assuming fP constant, and further, supposing that the chord PT (Fig. 71) is equal to the arc PT, and therefore to x the arc turned through by the wheels after the teeth pass the line of centres,

$$\text{Work done} = f \, . \, P \, . \left(\frac{1}{R} + \frac{1}{R'}\right)\frac{x^2}{2}.$$

The same formula applies before the line of centres, and if we assume the arcs of approach and recess each equal to the pitch p, we shall have for the whole work lost by the friction of a pair of teeth,

$$\text{Whole Work lost} = fP\left(\frac{1}{R} + \frac{1}{R'}\right)p^2.$$

The energy transmitted during the action of a pair of teeth is $2Pp$, therefore the counter efficiency is

$$1 + e = 1 + f\left(\frac{1}{R} + \frac{1}{R'}\right)\frac{p}{2} = 1 + f\pi\left(\frac{1}{n} + \frac{1}{n'}\right),$$

where n, n' are the numbers of teeth in the wheels. A smaller arc of action is sometimes employed in practice, and the friction will then be less. This is also the case in bevel gear. The formula shows that the friction is diminished by increasing the number of teeth.

A more exact solution of this question* can be obtained on the assumption that P varies as it would do if there were only one pair of teeth; but as this is uncertain it is not practically useful.

In all cam and wheel mechanisms the efficiency for a small movement in any position can be determined exactly by a graphical or other process. For the velocity-ratio can be found, as shown in Part II., and the force-ratio is determinate by the principles of statics, therefore the quotient which gives the efficiency can also be found. In the case of toothed wheels this method shows at once† that the friction of the teeth before the line of centres is greater than the friction after the line of centres. The difference appears insufficient to account for the injurious effects generally ascribed to friction before the line of centres, which however may be due to other causes. In cam mechanisms the efficiency in one position is little guide to the efficiency in a complete period, which can only be found by a process too intricate to be useful, or by making some supposition as the mean value of the pressure between the rubbing surfaces.

The counter efficiency of a train of m equal pairs of wheels is

$$1 + e = 1 + mf\pi\left(\frac{1}{n} + \frac{1}{n'}\right).$$

Assume now that a given velocity-ratio is to be provided by the train, and that the number of teeth in one wheel is given, then it is possible to find the value of m that the friction may be least. The solution of this problem is the same as that of finding the least possible number of teeth, and it was shown by Young that, for this, we ought to take m, so that the velocity-ratio for each pair of wheels is, as nearly as possible,

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* See Moseley's *Mechanical Principles of Engineering*.

† Ibid., page 286.

3·59. For example, if the train is to give a total velocity-ratio of 46, there should be three pairs of wheels. The gain over a single pair in this case is one-third, but will be much greater for higher velocity-ratios. The solution (first given by Mr. Gilbert) takes no account of axle friction, a circumstance which would greatly modify the result.

Some experiments on the friction of toothed and worm gearing have been made by Mr. W. Lewis, a brief account of which, with a table of results, will be found in a treatise on the *Mechanics of Machinery* by Prof. A. B. W. Kennedy. The table shows that the efficiency of spur gearing increases from ·94 at a speed of periphery of 10 feet per minute to ·986 at a speed of 200 feet per minute, but is nearly independent of the pressure. In a worm wheel there is a similar increase with the speed, but the efficiency is much smaller, diminishing with the angle of the worm. The experiments by Sir Douglas Galton on railway brakes already referred to (p. 248), show that the friction of a wheel when "skidding," that is when sliding on the rail without rotation, is much less than the friction of the brake blocks on a rotating wheel. It diminishes rapidly as the speed increases. In cases of higher pairing by contact we may therefore probably say generally that the co-efficient of friction is relatively small except at the lowest speeds, the difference being greater the higher the speed.

SECTION III.—FRICTIONAL RESISTANCES IN GENERAL.

128. *Efficiency of Mechanism in General.* It appears from what has been said that an exact calculation of the frictional resistances is impracticable, partly because the process is too complex to be useful, but chiefly because the co-efficients to be employed are variable according to circumstances, and within limits which are not precisely known. Hence when possible the efficiency of a machine is estimated, not by considering each particular element, but by direct experiment on the machine as a whole, and we conclude this chapter with some general principles which bear on this question.

The effort employed to drive a machine may be greater or less, according to the resistance which is being overcome, and therefore the stress between each element will also vary according to this effort. As, however, these stresses depend also on other forces, such as weight and elasticity, which have no connection with the effort, but are always the same, they will not increase so fast, and the frictional resistances will accordingly be proportionally less the greater the effort. Some resistances are absolutely constant, for example, the friction of bearings, the load on which is simply the weight of a fly-wheel or other moving part:

or the friction of a piston rod in its stuffing box. Others are sensibly proportional to the driving effort or the useful resistance, in which case, when the ordinary laws of friction apply, the loss of work increases in direct proportion to these quantities. The greater number depend on both variable and constant forces, but these may be in great measure separated into two parts, one of which is approximately constant and the other approximately proportional either to the driving effort or to the useful resistance. Hence, if U be the useful work done and E the energy exerted in a period of the machine,

$$E = U + kU + k' . E + B,$$

where k, k' are numerical co-efficients and B the work done in overcoming the constant resistances. In hydraulic and other machines, where fluid resistances occur, terms depending on the speed of the machine must be added, indeed this is so in all machines when driven at a high speed; because forces due to inertia increase the friction, and besides, shocks and the resistance of the atmosphere have to be considered. Such cases, however, are not considered here.

If we transfer the term $k'E$ to the other side of the equation and divide by $1 - k'$, we get

$$E = (1 + e)U + E_0,$$

where e, E_0 are two new constants derived from the former ones, of which E_0 is the work done in driving the machine when unloaded, and $1 + e$ the counter-efficiency when the load is very great.

The same formula may also be written in a way which is sometimes more convenient. Let P be the mean value of the driving effort and R that of the useful resistance during a complete period, r the mean value of the velocity-ratio of the working and driving pairs, then

$$P = (1 + e)Rr + P_0,$$

where P_0 is now the effort required to drive the machine when unloaded. In hoisting machines R is the weight lifted and P the hauling force usually called the power, R/P is the mechanical advantage or purchase.

In the steam engine, if p_m be the actual mean effective pressure, p'_m the part of that pressure employed in overcoming the useful resistance, p_0 the pressure necessary to drive the engine when unloaded,

$$p_m = (1 + e)p'_m + p_0.$$

The value of e may be taken as ·15 or in large engines somewhat less. The constant p_0, often called the "friction pressure," is from 1 to 1½ lbs. or in marine engines 2 lbs. or more per square inch. At high speeds and pressures the ordinary laws of friction fail and e is diminished, the constant friction is then relatively of more importance.

Experiments recently made by Professor Thurston on the friction of a horizontal engine driving a crank shaft and fly-wheel showed that the loss of work by friction was nearly independent of the power transmitted, $47\frac{1}{2}$ per cent. of the whole being due to friction of the crank shaft. The question is one on which little is definitely known, but it seems clear that the "constant" friction must be the most important element, and that it must be to a great extent uncertain, varying from time to time even in the same engine.

If the direction of motion of the machine be reversed so that the original resistance becomes the driving effort and the effort the resistance. the same general formula is approximately true, but the constants k, k' are interchanged. Unless under special conditions the efficiency is not the same in the two cases, and in fact is generally very different. Let us suppose that in a machine working against a known reversible resistance, the driving effort is gradually diminished until the machine reverses, and let E' be the work done when reversing, we have the equations

$$E = U + kU + k'E + B,$$
$$U = E' + k'E' + kU + B,$$

from which by subtraction and dividing by U we find

$$\frac{E'}{U} = \frac{2}{1+k'} - \frac{1-k'}{1+k'} \cdot \frac{E}{U},$$

a formula which gives the efficiency when reversing. If the original efficiency be less than $\frac{1}{2}(1-k')$, the machine will not reverse even when the driving force is entirely removed. In most forms of hoisting machines k' is small enough to be neglected, and we have the important principle that a machine will not reverse if its efficiency is less than ·5. It will not reverse under any circumstances if $k > 1$. As previously explained in the case of a screw, non-reversibility is a property so valuable in practical applications as to be worth obtaining at the sacrifice of efficiency. The differential pulley block is a common example.

Frictional resistances, though a source of waste of energy, are usefully employed in machines for various purposes. In screws and driving belts we have already found them used for the purpose of locking a pair or closing a kinematic chain, and many instances of the same kind might be referred to. Another application of equal importance will be considered in the next chapter.

EXAMPLES.

1. A weight is moved up a plane inclined at 1 vertical to n horizontal by an effort parallel to the plane: show that the counter-efficiency is $1 + nf$, where f is the co-efficient of friction. Find the value of n for a mechanical advantage of 10 : 1 and a co-efficient ·05. *Ans.* $n = 20$.

2. Show that the pressure on the guide bars of a direct-acting engine is approximately proportional to the ordinates of an ellipse, and deduce the work lost per stroke.

Referring to Fig. 91 let X be that pressure, then

$$X = S \sin\phi = P \,.\, \tan\phi = \frac{P}{n}\sin\theta \text{ approximately}$$

If the radius of the crank circle represent P, and an ellipse be drawn with the same major axis, and minor axis $= P/n$, X will be the ordinate of the ellipse at a point representing position of piston.

$$\text{Loss of work per stroke} = f \times \text{Area of semi-ellipse}$$

$$= \tfrac{1}{2} f \,.\, \pi \,.\, a \frac{P}{n} = \frac{\pi f s P}{4n},$$

where s is the stroke and f the co-efficient of friction.

3. A bearing 16" diameter is acted on by a horizontal force of 50 tons and a vertical force of 10 tons. Find the work lost by friction per revolution, using a co-efficient of one-eighteenth. Find also the horse-power lost by friction at 70 revolutions per minute. *Ans.* Loss of work = 11·87 foot tons. H.P. = 56·4.

4. The thrust of a screw propeller is 20 tons, the pitch 28 feet. The thrust block is 18" diameter at the centre of the rings. Find the efficiency with a co-efficient of friction of ·06. *Ans.* Efficiency = ·986.

5. Find the efficiency of a common screw and nut with pitch angle 45° and co-efficient ·16. *Ans.* Efficiency = ·72.

6. A screw bolt is ½" diameter outside and ·393" at the base of the thread. The effective diameter of the nut is 3", the pitch angle ·07, and the co-efficient of friction ·16; supposing it screwed up by a spanner two feet long, find the mechanical advantage.

Tension of bolt = 218 × pull on the spanner.

7. Find the efficiency of a pair of wheels, the number of teeth being 10 and 75, and the co-efficient of friction ·15. *Ans.* ·951.

8. The stroke of a direct-acting engine is 4 feet, piston load 58 tons, load on crank-shaft bearings 10 tons, connecting rod 4 cranks; trace the curve of crank effort when friction is taken into account, assuming all bearings 16" diameter and co-efficient one-eighteenth. Find the "dead angle."

9. In the last question, if the engine drive the screw propeller of question 4, find the efficiency of the mechanism, including thrust block, by the approximate method. The connecting rod may be supposed indefinitely long except for the purpose of estimating the efficiency of the guide bars.

$$\text{Efficiency} = \cdot 989 \times (\cdot 97)^2 \times \cdot 986 = \cdot 92.$$

10. A rope is wound thrice round a post, and one end is held tight by a force not exceeding 10 lbs. What pull at the other end would be necessary to make the rope slip, the co-efficient of friction being supposed ·366? *Ans.* 10,000 lbs.

11. Find the necessary width of belt three-sixteenths inch thick to transmit 1 H.P., the belt embracing 40 per cent. of the circumference of the smaller pulley and running at 300 feet per 1'. Co-efficient = ·25. Strength 285 lbs. per sq. inch. *Ans.* Breadth = 4½".

12. In question 10 construct the friction spiral showing the tension of the rope at every point.

13. The axles of a tramway car are 2½" diameter, and the wheel 2' 6": find the resistance, being given, that the co-efficient of axle friction is ·08 and that for rolling ·09. *Ans.* Resistance = 28½ lbs per ton.

14. Find the efficiency of a pulley 6" diameter, over which a rope ½" diameter passes, the axis of the pulley being ½" diameter, and the load on it twice the tension of the rope. Co-efficient of axle friction ·08. Co-efficient for stiffness of rope ·47. *Ans.* Efficiency = 94 per cent.

15. From the result of the preceding question deduce the efficiency of a pair of three-sheaved blocks. *Ans.* Efficiency = 71 per cent.

16. A wheel weighing 20 lbs., radius of gyration 1′, is revolving at 1 revolution per second on axles 1″ diameter. It is observed to make 40 revolutions before stopping: find the co-efficient of axle friction. *Ans.* Co-efficient = ·059.

17. In a pair of three-sheaved blocks it is found by experiment that a weight of 40 lbs. can be raised by a force of 10 lbs., and a weight of 200 lbs. by a force of 40 lbs. Find the general relation between P and W, and the efficiency when raising 100 lbs.

$P = \frac{3}{16} W + \frac{5}{2}$. Efficiency = ·784 when raising 100 lbs. $e = \frac{1}{8}$.

18. Find the distance to which power can be transmitted by shafting of uniform diameter, with a loss by friction due to its weight of n per cent., assuming that the angle of torsion is immaterial, and co-efficient for strength 9,000 lbs. per square inch.

If f be co-efficient of friction, then the length of shafting is $13\frac{1}{2} \cdot \frac{n}{f}$ in feet.

References.

For further information on subjects connected with the present chapter the reader is referred to

Kennedy, *Mechanics of Machinery*. Macmillan.

CHAPTER XI.

MACHINES IN GENERAL.

129. *Preliminary Remarks.*—In the motion of a machine the relative movements of the several parts are completely defined by the nature of the machine, and the principal action consists in a transmission and conversion of energy. Hence it is that the principle of work is of such importance in all mechanical operations that it is desirable to consider it as an independent fundamental law verified by daily experience. Even in applied mechanics, however, we have sometimes to do with sets of bodies, the relative movements of which are not completely defined by the constraint to which they are subject, but partly depend on given mutual actions between them. When this is the case, the principle of work, though still of great importance, is not by itself sufficient to determine the motions.

Again, if we wish to study the forces which arise when the direction of a body's motion is changed, the principle of work does not help us, for no work is done by such forces. For example, the position of the arms of a governor, revolving at a given speed, cannot be found, except, perhaps indirectly, by the methods hereto employed. We then resort to the ordinary laws connecting matter and motion, which form the base of the science of mechanics, and of which the principle of work itself is often considered as simply a consequence.

The present chapter will be devoted in the first place to a brief summary of elementary dynamical principles, and afterwards to various questions relating to machines and the forces to which they are subject.

SECTION I.—ELEMENTARY PRINCIPLES OF DYNAMICS.*

130. *Quantity of Matter. Mass.*—The effect of an unbalanced force P, acting during a certain time t, on a piece of matter, is to generate a velocity v, which is proportional to P and t directly and the quantity of

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* The brief statement here made of principles assumed in subsequent articles of the treatise is not intended as a substitute for a treatise on elementary dynamics.

matter inversely. When the force P is equal to the weight W, as in the case of a body falling freely, the velocity generated in 1″ is known to be g, where g is a number which varies slightly for different positions on the earth's surface (Art. 99), but is precisely the same for all sorts of matter. We may express this by the equation

$$Pt = \frac{W}{g}v.$$

Since we use gravitation measures exclusively, the symbol W in this formula must be understood to mean the weight of the piece of matter as compared with that of a standard piece at some definite place, as, for example, Greenwich Observatory. The weight W then varies according to the actual position of the piece of matter upon, above, or below the earth's surface; but these variations are in exact proportion to corresponding changes in the value of g, so that the quotient W/g, commonly known (p. 200) as the Mass, furnishes a definite measure of the inertia and therefore of the quantity of matter in the piece.

The quotient thus described as the "mass," however, is not numerically equal to the quantity of matter because the unit of measurement is different. The unit of mass is here derived from the unit of force, being necessarily a quantity of matter such that W/g is unity, that is, the weight of the unit mass must be g units of force. But the weight of the standard piece at Greenwich is one unit of force, and therefore the unit mass is the quantity of matter in the standard piece multiplied by g_0, the value of g at Greenwich. Now quantities of matter are practically determined by the process of weighing them against the same standard pieces as are employed in measuring forces; the quantity of matter in the standard piece is therefore the unit of measurement. So much is this the case that in ordinary language the terms "pound" or "kilogramme" are used indiscriminately for force, or the matter on which force acts.

The unit of mass then in gravitation measure, as usually defined, is the unit quantity of matter multiplied by g_0, and therefore, if μ be the quantity of matter, and m the mass,

$$m = \frac{\mu}{g_0}.$$

It is obvious that the value of μ is an absolute measure of the quantity of matter, being independent of time, space, and the place where the weighing takes place; it is numerically the same as W_0, the force with which the quantity of matter μ is drawn to the ground at Greenwich, for which reason the term "weight" in ordinary language is used in the sense of quantity quite as often as in that of force. On the other hand, the value of m is a measure which is only relative to the numerical value of g_0. Hence in gravitation measure, the word "mass" means the quantity of matter measured on a special scale, dependent on the units of time, space, and force adopted.

Some remarks on the "absolute" system of measurement employed by physicists, in which the mass and the quantity of matter are identical, will be found in the Appendix, but as this system has not as yet been introduced into practice, either at home or abroad, it is unnecessary for the purposes of this work to dwell on the subject here.

131. *Equation of Momentum. Centrifugal Force.* Denoting, then, the mass by m, the equation connecting P, t, and v, becomes

$$Pt = mv.$$

The products Pt, mv are called IMPULSE and MOMENTUM respectively, and the equation may be written

Impulse exerted = Momentum generated.

A unit of impulse is unit force exerted for unit time, usually 1 lb. for 1", a quantity for which the expression "second-pound" may conveniently be used. If P be variable, then impulse is calculated in the same way as the energy exerted by a variable force (Art. 90), the abscissæ of the diagram now representing time instead of space.

The body we are considering may have a velocity at the commencement of the time t, and the force may be partially balanced; if so, v must be understood to be the *change* of velocity, and P the unbalanced, part of the force.

So far, the equation of momentum is analogous to the equation of work, impulse representing the time effect of force as energy represents its space-effect. There are, however, two important differences, which we consider in the present and next succeeding article.

Change of kinetic energy arises from a change in the magnitude of the velocity irrespectively of direction, whereas change of momentum must be estimated in the direction of the force producing it, and includes change of direction. Hence the equation is applicable when the direction of the force is perpendicular to the direction of motion, so that the only effect produced is change of direction. The rate of change of velocity taken in the most general sense, is called Acceleration, and the equation of momentum may also be written

$$P = mf,$$

where f is the acceleration estimated in the direction of the force. By taking the force perpendicular to the direction of motion we get the equation which connects the curvature of the path of a moving body with the force R, which compels it to deviate from the straight line, namely,

$$R=\frac{mv^2}{r},$$

where v is the velocity and r the radius of the circle in which it is moving at the instant considered. Since v/r is the angular velocity of the line in which the body is moving the formula shows that the deviating force is equal to the product of the momentum and the rate of deviation.

Like other forces this arises from the mutual action between two bodies: one of these is the moving body; the other, the fixed body which furnishes the necessary constraint. If we are thinking of the fixed body instead of the moving body, we call the force R the Centrifugal Force, being the equal and opposite force with which the moving body acts on the body which constrains it. The two forces together constitute what we have already called a Stress (Art. 1). To determine a stress of this kind it is necessary to refer the direction of motion to some body which we know may be regarded as fixed, and we are not at liberty to choose any body we please for this purpose, as in kinematical questions. What constitutes a fixed body is a question of abstract dynamics, into which we need not enter. For practical purposes the earth is taken as fixed.

If a body rotate about a fixed axis the centrifugal forces, arising from the motion of each particle, will not balance one another unless the axis be one of three lines, passing through the centre of gravity, which are called the "principal axes of inertia" at that point. In most cases occurring in practical applications the position of these lines can be at once foreseen as being axes of symmetry. This is the case, for example, in homogeneous ellipsoids and parallelopipeds. In the common case of a homogeneous solid of revolution, the axis of revolution, and any line at right angles to it through the centre of gravity, are principal axes. If the axes of rotation be parallel to one of these axes, but does not pass through the centre of gravity, the centrifugal forces reduce to a single force, which is the same as if the whole mass were concentrated at the centre of gravity. In all other cases there is a couple depending on the direction of the axis of rotation, as well as the force just mentioned. (Ex. 15, p. 291.)

132. *Principle of Momentum.*—Again, every force arises from the mutual action between two bodies, consisting in an action on one accompanied by an equal and opposite reaction on the other. Hence, if we understand by the total momentum of two bodies in any direction, the sum or the difference of the momenta of each, according as the bodies move in the same or in the opposite direction, it appears that the total momentum will not be affected by the mutual action between the two. And more generally, if there be any number of bodies we shall have

Total impulse exerted = Change of total momentum,

where, in reckoning the impulse, we are to take into account external forces alone, and not the internal forces arising from the mutual action of the parts of the set of bodies we are considering. This equation expresses one form of what we may call the Principle of Momentum: other forms will be explained hereafter in connection with questions relating to fluid motion (Part V.).

The total momentum of a number of bodies may be reckoned by direct summation, with due regard to sign, but it may also be expressed in terms of the velocity of the centre of gravity: for, let m be the mass of any particle of the system, the ordinate of which, reckoned from a given origin parallel to a given line, is x; also, let Σmx denote the sum of all the separate products mx, for all the particles of the system, and let M be the total mass, then we know that the ordinate of the centre of gravity * is given by the formula

$$x = \frac{\Sigma mx}{M}.$$

Let the velocity of a particle parallel to the given line be u, then if x_1, x_2, be the ordinates at the beginning and end of 1" we shall have

$$u = x_2 - x_1.$$

Hence if u be the velocity of the centre of gravity parallel to the same line,

$$u = x_2 - x_1 = \frac{\Sigma m(x_2 - x_1)}{M} = \frac{\Sigma mu}{M},$$

which equation may be written

$$Mu = \Sigma mu,$$

showing that the total momentum of the system is the same as if its total mass were concentrated in its centre of gravity. We conclude from this that the motion of the centre of gravity can only be influenced by external forces and not by any action between the parts of the system.

133. *Internal and External Kinetic Energy.*—If we multiply the equation just obtained by $2u$ and remember that u being constant may be placed within the sign of summation, we obtain

$$2Mu^2 = \Sigma m \,.\, 2uu,$$

---

* Called more correctly by Young "the centre of inertia" and by modern writers on mechanics the "centre of mass," or more briefly the "centroid."

which, adding Σmu^2 to each side and re-arranging the terms, may be written

$$Mu^2 + \Sigma m(u - u)^2 = \Sigma mu^2.$$

This is true in whatever direction the velocities are estimated, and we can therefore write down two similar equations for the velocities in two directions at right angles to the first. Now the resultant of three velocities at right angles is the square root of the sum of the squares of the components, also $u - u$ is the velocity parallel to x relatively to the centre of gravity; hence if U be the resultant velocity of the centre of gravity, v, r the velocities of any particle relatively to the body regarded as fixed and relatively to the centre of gravity respectively, we have, adding the three equations together, and dividing by 2,

$$\tfrac{1}{2}MU^2 + \Sigma\tfrac{1}{2}mr^2 = \Sigma\tfrac{1}{2}mv^2.$$

The first term on the left-hand side of this equation is what the energy would be, if the whole mass were concentrated at its centre of gravity, a quantity which may be described as the External Energy, or otherwise as the Energy of Translation of the system. The second term is the energy relatively to the centre of gravity considered as fixed, which may be called the Internal Energy. The right-hand side is the total energy of motion, and we see therefore that this is the sum of the internal and external energies. In the case of a single rigid body the motion relatively to the centre of gravity is always a rotation about some axis, and therefore

Energy of Motion = Energy of Translation + Energy of Rotation,

a principle already employed in a preceding chapter (p. 202).

In the case of a set of rigid bodies the internal energy is the sum of the energies of rotation of each together with the internal energy of a set of particles of the same mass occupying the centres of gravity of the bodies and moving in the same way.

134. *Examples of Incomplete Constraint.*—In the cases which occur in applications to machines and structures we usually have to consider two bodies moving in straight lines without rotation.

CASE I. *Recoil of a Gun.*—When a cannon is fired the shot is projected and the cannon recoils with velocities dependent on the relative weights of the shot, the cannon, and the charge of powder.

Here, the motion is due to the pressure of the gases generated by the combustion of the powder one way on the shot, the other way on the cannon. If the inertia of these gases could be neglected these pressures would be exactly equal at each instant and would cease as soon as the shot left the bore. The impulse exerted on shot and cannon would then be equal. In fact, the inertia of the powder gases causes the pressure to be greater and to last longer on the cannon than on the shot, so that the impulses on the two are not nearly equal. For the present we shall neglect this, and shall further suppose that the material of both shot and gun is sensibly rigid.

In general, recoil is checked by an apparatus called a "compressor," which supplies a gradually increasing resistance to the backward movement of the gun, while friction and the resistance to rotation of the shot resist the forward movement of the shot. In the first instance suppose there are no such resistances, let V be the velocity of recoil and M the mass of the gun, v the velocity and m the mass of the shot; then, since the impulse exerted is the same for both,

$$MV = mv.$$

Further, if the weight of the charge and the amount of work 1 lb. of it is capable of doing be known, the explosion will develop a definite amount of energy (E) which will be all spent in giving motion to the shot and the cannon.

$$\text{Energy of Explosion} = \tfrac{1}{2}MV^2 + \tfrac{1}{2}mv^2.$$

Here E is the sum of two parts—

$$\text{Energy of Shot} = \frac{M}{M+m}E,$$

$$\text{Energy of Recoil} = \frac{m}{M+m}E.$$

The energy of recoil has to be absorbed by the compressor, usually an hydraulic brake, which will be considered hereafter (see Part V.).

CASE II. *Collision of Vessels.*—When two vessels come into collision an amount of damage is done depending on the size and velocities of the vessels.

Here we may suppose the vessels moving in given directions with given velocities; let the velocities parallel to a given line be u_1, u_2, and the masses m_1, m_2, then, as in Art. 133, the velocity of the centre of gravity parallel to the same line is

$$u = \frac{m_1u_1 + m_2u_2}{m_1 + m_2},$$

and therefore the velocities of the vessels relatively to their common centre of gravity must be

$$u_1 - u = \frac{m_2(u_1 - u_2)}{m_1 + m_2}; \quad u_2 - u = \frac{m_1(u_2 - u_1)}{m_1 + m_2}.$$

Two similar equations may be written down for the velocities in a direction at right angles to the first. Square and add corresponding equations, multiply by $\frac{1}{2}m_1$, $\frac{1}{2}m_2$, and add the pair of products, then (Art. 133)

$$\text{Internal Energy} = \frac{1}{2} \cdot \frac{m_1 m_2}{m_1 + m_2} V^2,$$

where V is the velocity of either vessel relatively to the other, a quantity found immediately from the given velocities of the vessels by means of a triangle of velocities.

The total kinetic energy of the vessels is found by adding the energy of translation. As, however, this quantity cannot be altered by the collision, it is clear that the amount of work done must depend on the internal energy alone: we may properly call it therefore the "energy of collision." If the displacements in tons of the vessels be W_1, W_2, we shall have, in foot-tons,

$$\text{Energy of Collision} = \frac{W_1 W_2}{W_1 + W_2} \cdot \frac{V^2}{2g}.$$

It is not, however, to be supposed that the whole of this is necessarily expended in damage to the vessels; if the circumstances of the collision be such that the vessels, even though completely devoid of elasticity, would have a motion of rotation or a velocity of separation of their centres of gravity, then the corresponding internal energy must be deducted. Also the influence of the water surrounding the vessels has been left out of account; this somewhat augments the effect by increasing the virtual mass of the vessels.

The same formula may be used for other cases of impact, but the effects of impact depend so much on the strength and stiffness of the colliding bodies that the subject cannot be further considered here (Ch. XVI.).

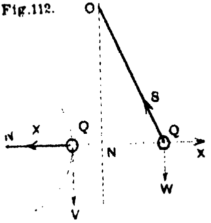
CASE III. *Free Rotation.*—If the axis of rotation of a solid be free to move, it will shift its position as already stated unless the axis be one of the principal axes of inertia: but if it be a principal axis it will remain fixed in direction unless external forces act upon it. When the solid rotates rapidly it offers a considerable resistance to any change of direction of its axis which can only be overcome by the action of forces which have a moment about an axis inclined to the axis of rotation. In consequence a body in rapid rotation may possess considerable stability in circumstances where in the absence of rotation equilibrium would be impossible. The principle is important and has many applications, the well-known gyroscope being a common example. The question, however, requires a considerable amount of explanation to render it intelligible, and the limits of this work render it impossible to do more than mention it here. The theory of the gyroscope is given in a clear and simple form by Professor Worthington, in a small treatise referred to at the end of this chapter.

SECTION II.—REGULATORS AND METERS.

135. *Preliminary Remarks. Revolving Pendulum.* Centrifugal forces may be employed in machines to do work by energy transmitted from a source, or derived from the kinetic energy of the moving parts. Sometimes the work thus done is the object of the machine, as in certain drying machines where the substance to be dried is caused to rotate with great rapidity so that the fluid is expelled at the outer circumference: or, partially, in centrifugal pumps. More frequently they serve to move a kinematic chain connected with a shifting piece which regulates the speed of the machine. Such mechanisms are called Centrifugal Regulators or, more briefly, Governors.

In Fig. 112 Q is a heavy particle attached by a string to a fixed point O and revolving in a horizontal circle the centre of which is N vertically below O. This will be possible if the centrifugal force due to the motion of the particle is equal to the horizontal component of the tension of the string. Let S be that tension, W the weight of the particle, and let the string make an angle θ with the vertical, then the horizontal and vertical components of S are

Fig. 112.



$$X = S.\sin\theta;\quad W = S.\cos\theta.$$

Let A be the angular velocity of the revolving particle, then it is shown in works on elementary dynamics that the centrifugal force is

$$X = \frac{W}{g}.A^2.QN.$$

Equating these values of X and eliminating S,

$$W.\tan\theta = \frac{W}{g}.A^2.QN.$$

Since $QN = ON.\tan\theta$, this reduces to the simple formula

$$ON = \frac{g}{A^2},$$

which shows that the vertical distance of Q below the point of suspension depends on the speed, not on the length of the string or the magnitude of the weight.

This distance is called the "height" of the revolving pendulum, and will be denoted by h. If t be the period, that is the time of a complete revolution, we find, since $At = 2\pi$,

$$t = 2\pi\sqrt{\frac{h}{g}},$$

showing that the period is the same as that of a double oscillation of a simple pendulum of length h (see Art. 103). The height of a simple revolving pendulum may, as already explained in Art. 101 (p. 201), often be conveniently adopted as a measure of a speed of revolution. It is *inversely* proportional to the *square* of the speed being given in inches at n revolutions per minute by the equation

$$h = \frac{35,232}{n^2}.$$

Instead of supposing the string attached to a point O in the axis of revolution, we may suppose it attached to a point K, rigidly connected by a cross-piece KE, with a revolving spindle ON. The same reasoning applies, O being now an ideal point, found by prolonging the string to meet the axis. The height of the pendulum is still ON, and is found by the same formula.

136. *Speed of a Governor to overcome given Frictional Resistances. Loaded Governors.*—In the simplest centrifugal governors two heavy balls are attached to arms, which are jointed either directly to a revolving spindle, or to the ends of a cross-piece attached to a spindle. Motion is communicated by links from the arms to a piece sliding on the spindle, the movement of which is communicated by a train of linkwork, either to a throttle valve directly controlling the supply of steam, or to an expansion valve which regulates the cut-off. In either case an upward movement of the arms has the effect of diminishing the mean effective pressure, and a downward movement of increasing it. Two forms of this mechanism are shown in the figures of Plate VI.: in one of these (Fig. 1) the weight of the sliding piece is increased by a large additional weight, the governor is then said to be loaded; while in the other (Fig. 2), the arms cross each other, the spindle being slotted, or the arms bent to permit this. The object of these arrangements we shall see presently.

If now the speed of revolution be increased or diminished, the arms move outwards or inwards, and so adapt the mean effective pressure to the work which is being done. If there were no frictional resistances the smallest variation of speed would produce a corresponding motion in the arms; but, as the linkwork mechanism necessarily offers a certain resistance, motion cannot take place until the change of speed has

Plate. VI.

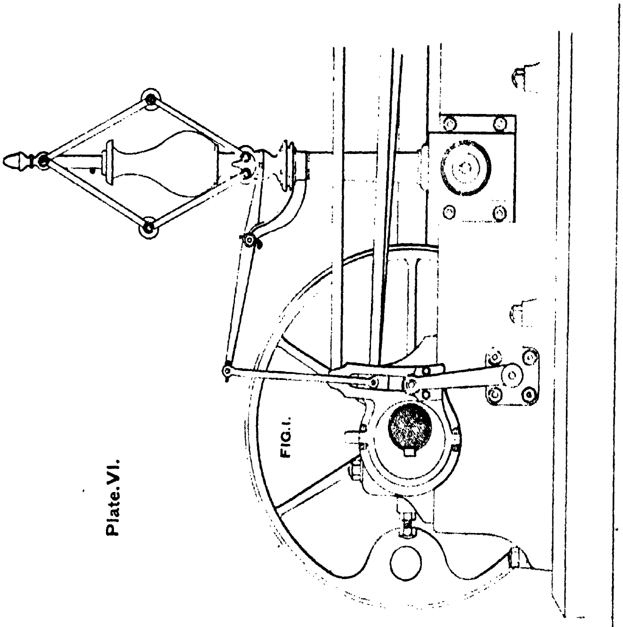


FIG. I.

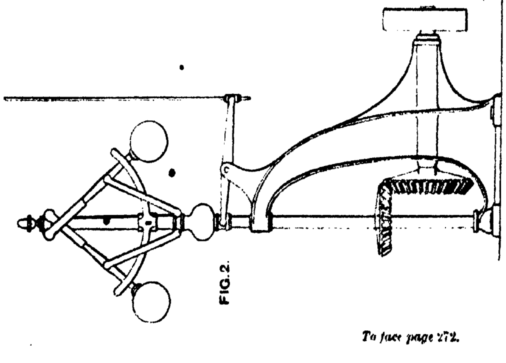


FIG. 2.

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reached a certain magnitude, which is smaller the more sensitive the governor is. These frictional resistances are measured by a certain addition to, or subtraction from, the weight of the sliding-piece, which might be determined experimentally, and therefore will be supposed a known quantity *F*. We first investigate what change of speed will be necessary to overcome them.

In Fig. 113 AQB is a triangle revolving about AB which is vertical, a *heavy particle is placed at Q*, and the weights of the bars *AQ*, *BQ* are small enough to be neglected. If the triangle revolve at a speed corresponding to the height *AN* of a simple revolving pendulum *AQ*, there will be no stress on *BQ*, but if it be greater or less there will be a pull or thrust, the magnitude of which is determined thus:—

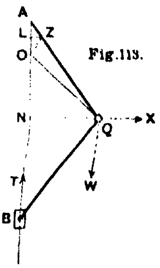


Fig. 113.

Set up *NO* equal to the height due to the revolutions, and join *QO*. Then it appears from what was said in the last article that if *NO* be taken to represent the weight *W* of the particle, *NQ* will represent *X* the centrifugal force, and therefore the resultant force on *Q* must be represented by *QO*. Through *O* draw *OZ* parallel to *BQ*, then *QOZ* is a triangle of forces for the joint *Q* of the triangular frame *AQB*, so that *QZ*, *OZ* must represent the stresses on *AQ*, *BQ* respectively. For our purposes we require the vertical component of the stress on the link *BQ*, which is obtained by drawing *ZL* horizontal: *OL* must be the force in question which we call *T*. In the Figure *T* is an upward force, *O* being below *A*, and the speed of revolution therefore great. In this construction the links need not be actually jointed to the spindle *AB*; they may, as in the simple pendulum, be attached to the extremities of cross-pieces fixed to *AB*. *A* and *B* are then ideal points of intersection of the links with the axis of rotation.

In general *AQ* and *BQ* are equal; we may then obtain a simple formula for *T*. Let $NO = h$, a quantity given by the same formula as before for a given speed, and let *AN*, the actual height of the governor, be denoted by *H*, then $OA = H - h$; but in the case supposed, $OA = 2OL$, therefore

$$2T = W.\frac{H-h}{h}; \quad h = H.\frac{W}{W+2T}:$$

formulæ which give the pull for any speed, and conversely the speed for which the pull will have a given value. In practical applications there are always two balls, so that if *W* be the weight of one, $2T$ will be the pull due to both.

We can now find within what limits of speed the mechanism can be in equilibrium. Let w be the weight of the sliding-piece B, inclusive of any load which may be added to it, h the height due to the speed at which there is no tendency to move the arms, h_1, h_2, the heights due to the speeds at which they are on the point of moving upwards or downwards respectively, then

$$h_1 = H\frac{W}{W+w+F};\quad h = H\frac{W}{W+w};\quad h_2 = H\frac{W}{W+w-F}.$$

In general F will be small compared with $W+w$, and then we have very approximately,

$$h_2 - h = h - h_1 = h\frac{F}{W+w}.$$

These results show that loading a governor gives it a power of overcoming frictional resistances which would otherwise require a weight of ball equal to the sum of the load and the actual weight. Light balls may therefore be used as in the figure (Plate VI.) without sacrificing power, as the load may be made great without inconvenience. The speed of a loaded governor is greater than that of a simple governor of the same actual height, as appears from the formula for h. It may be altered at pleasure by altering the load. This arrangement is known as Porter's governor, from the name of the inventor.

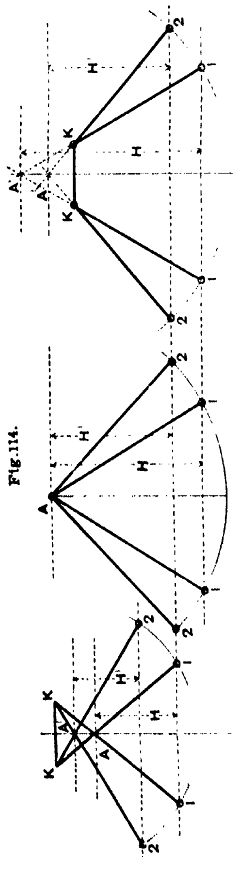


Fig. 114.

137. *Variation of Height of a Pendulum Governor by a Change of Position of the Arms.*—Next suppose the speed to alter so much that the arms actually change their position, then if H remained the same, the tendency to move would also be the same, and the movement must therefore continue until the speed is brought back within the limits for which rest is possible. In the ordinary pendulum governor, however, H alters in a way which depends on the mode of attachment and arrangement of the arms, as will appear from the annexed diagram (Fig. 114) which shows three cases.

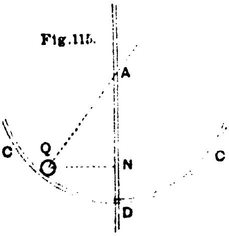
In the centre figure the arms are jointed to the spindle so that their centres of rotation are in the axis, in the two others they are jointed to a cross-piece KK, but differently arranged in the two cases. In all three, as explained in the preceding article, the height H is measured to A, the real or ideal intersection of the arms with the axis of rotation.

Suppose the arms to move from position 1 to position 2 in the figure; H diminishes to H', but the amount of diminution is different in the three cases: in the right hand figure it is greatest, and in the left-hand least. Indeed in the latter case where the arms are crossed it is possible by making KK long enough, to change the diminution into an increase. (Ex. 4, p. 289.) If then, by an increase in the speed, the arms move into a new position, the speed required for equilibrium does not remain the same but increases, so that, when the adjustment has been effected between the energy and the work, the speed is increased, instead of being the same as before. Conversely, after adjustment to suit a diminished speed, the speed actually attained is diminished. Thus the effect of the variation in H is to widen the limits within which the speed can vary.

138. *Parabolic Governors.*—A governor may be constructed in which H does not vary at all.

In Fig. 115 Q is a ball resting on a curve CC attached to a vertical spindle. The curve lies in a vertical plane, and D is the lowest point. When at rest the ball can only be in equilibrium at D, but, if the spindle revolve, it may rest at another point, the position of which depends on the speed of revolution. If the curve be a circle we have only the pendulum governor in a different form, for, drawing the normal QA and the perpendicular QN, A will be a point to which Q might be attached by a string and the curve removed. Hence, AN must be equal to h, the height due to the speed of revolution. But if the curve be not a circle the same thing must be true, only A is now not a fixed point; hence in every case the sub-normal AN of the curve at the point of equilibrium must be equal to h. In general this geometrical condition determines one, and only one, position for a given speed; but if the curve be a

Fig. 115.



parabola with vertex at D, AN will be constant, and therefore Q will rest in any position for one particular speed, but for lower speeds will roll down to D, and for higher speeds will move upwards indefinitely. We have here a governor for which, neglecting frictional resistances, only one speed is possible. Such a governor is said to be "isochronous."

The curve arrangement is inconvenient for constructive reasons, but if it be replaced by a linkwork mechanism the ball still moves in a parabola. An isochronous governor is therefore often said to be "parabolic." The term is preferable, for no governor is actually isochronous on account of frictional resistances. A pendulum governor is much more nearly parabolic when the arms are crossed, and by properly taking the length of the cross-piece (Ex. 4, p. 289) it may be made exactly parabolic for small displacements. This arrangement is called Farcot's governor from the name of the inventor.

139. *Stability of Governors.*—If the curve CC be not a parabola H, which in this case is the sub-normal, will diminish or increase as the ball Q moves outwards. Take the first case and suppose Q in equilibrium at a certain point when the speed of revolution has a given value. Let Q now be moved up or down, then, if released, it will not remain at rest, but will return towards its original position and oscillate about it, or in other words the equilibrium of Q is stable. A governor possessing this property is described as "stable," and its stability is greater the quicker H diminishes. Similarly when H increases for an outward movement of the balls the governor is "unstable," and a parabolic governor may properly be described as "neutral."

A certain degree of stability is necessary for the proper working of a governor, and the amount required is greater the greater the frictional resistances. For assuming the revolutions at which the machine is intended to work to be n, the balls commence to work outward at the speed $n+x$, where x is a small quantity depending on the frictional resistance. After starting, the frictional resistances are not increased, but on the contrary will somewhat diminish; and, in a neutral governor, the balls therefore move outwards with increasing speed until by alteration of the regulating valve the supply of energy is diminished and the speed of the machine lessened. This change however requires time, and besides the balls are in motion and have to be stopped. The consequence is that they move outwards too far, and the supply of energy being too small the revolutions diminish to $n-x$, the speed necessary to move the balls inwards, notwithstanding the frictional resistance. Thus the motion is unsteady, the balls oscillating, and the speed fluctuating, between limits wider than $n \pm x$ without ever settling down to a permanent regime, an action known as "hunting."

The oscillation of the balls may be checked by a suitable brake, but it is preferable to employ a governor possessing a moderate degree of stability; the tendency to move the balls then diminishes as soon as the balls move, and they stop before moving far. The greater the frictional resistances the greater is the change required to enable the balls to return at once if they have moved too far for equilibrium. An important characteristic therefore of a good centrifugal governor is that the stability be capable of adjustment to suit the frictional resistances. Certain forms of compound governors, as for example that known as the "cosine," fulfil this condition and can, probably, be made more perfect than the simple pendulum governor. It should also be remarked that a governor should not be so sensitive as to be called into action by the changes of speed in the course of a revolution consequent on the fluctuation of energy of the moving parts. These changes are regulated by the fly-wheel as already fully described in Ch. IX.

All such mechanisms are however imperfect in principle, for they cannot come into operation till a certain change of speed has actually existed for a perceptible length of time. Where the changes of resistance are sudden and violent the best governor will scarcely prevent violent fluctuations in speed. In screw vessels, where this difficulty is much felt, it has been proposed to employ an auxiliary engine rotating against a uniform resistance; any difference of speed of which and the screw shaft immediately shifts the regulating valve.

140. *Brakes.*—In order that a machine may be under complete control when the changes of resistance are sudden and violent, and especially when it is required to stop it, it is not sufficient to cut off the supply of energy, but it is necessary in addition to have some means of absorbing the energy stored in the moving parts. An apparatus for this purpose is called a Brake. The surplus energy may in some cases be stored by springs or an elastic fluid, and subsequently applied to useful purposes; the brake is then combined with an accumulator. In general, however, this cannot conveniently be accomplished and frictional resistances are then employed to convert the energy into heat, which is dissipated by radiation and conduction. When the amount to be disposed of is not too great the friction of two solids pressing against one another may be used for the purpose, but care must be taken to provide sufficient surface to prevent temperature from rising too high during the process. A brake of this class is generally applied to a rotatory wheel or drum, and consists either of a solid block of wood or metal pressed against the wheel by some suitable mechanism, or else of a strap of metal often lined with small blocks of wood embracing the drum and tightened by a lever or otherwise. Three common forms are shown in Plate VII., two of these are used as dynamometers, and will be referred to again presently.

The most powerful brakes however are those in which hydraulic resistances are employed, some examples of which will be found in a later chapter.

In the "cup governor," invented by Dr. Siemens,* a regulator and an hydraulic brake are combined. A cup containing water rotates within a cylindrical casing; at low speeds the water remains within the cup, but as soon as the speed exceeds a certain limit centrifugal action causes it to pour over the edge of the cup into the space between the cup and the casing. A set of vanes attached to the cup rotate between fixed vanes attached to the casing, and break up the descending water, which re-enters the cup by an orifice in the bottom. There is then a great resistance to the motion of the cup which absorbs surplus energy. Some other forms of governor will be considered hereafter.

141. *Dynamometers.*—Mechanisms employed for the purpose of measuring physical quantities, such as time, speed, etc., are called generally Meters. The subject is very extensive, and would require a complete chapter to deal with even in outline. We can only notice here very briefly the apparatus used for the measurement of Power, a class of instruments known as Dynamometers. They are of very various construction, the most common being those in which the instrument measures the driving effort while the speed is independently determined and the power thence obtained as in Art. 97, page 193.

(1) In Fig. 4, Plate III., page 141, a common form of "transmission" dynamometer is represented. A shaft transmitting power is divided into parts and bevel wheels *BD* attached to each. A lever *A* turning about an axis concentric with the shaft in a plane perpendicular to it carries bevel wheels *C*, gearing with *BD*, through which the power is transmitted. If *A* be held fast a couple will be required to prevent it turning, which is just twice the driving couple being transmitted, and hence if a weight sliding on *A*, as shown in the figure, be so placed by

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* *Phil. Trans.*, 1866.

PLATE VII.

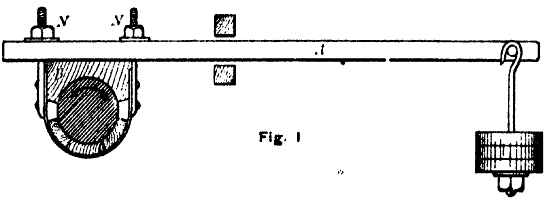


Fig. 1

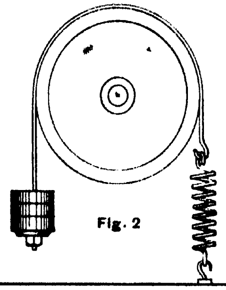


Fig. 2

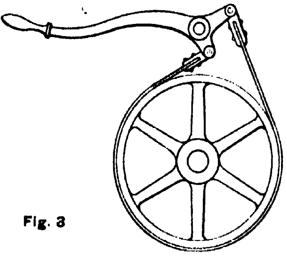
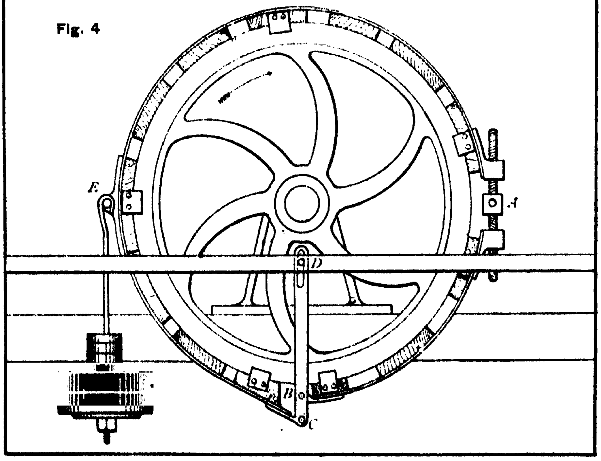


Fig. 3

Fig. 4



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trial that *A* just remains horizontal the driving couple in question will be determined. Hence the revolutions of the shaft being known the power can be found.

(2) Two shafts being connected by a belt some arrangement may be adopted by which the difference of tensions of the two sides of the belt is measured, and thus the driving effort being transmitted may be determined. For example, in the apparatus employed by Froude and Thorneycroft to measure the power required to drive a model screw propeller the two sides of the belt pass round pulleys mounted at opposite ends of a lever turning about a fulcrum at the centre. The force required to prevent the turning furnishes a measure of the difference of tension.

(3) In Fig. 1, Plate VII., a "friction dynamometer" is represented in one of the various forms in which it is applied. *A* is a lever from which a weight is suspended, *B* is a block fixed to *A*, which rest on a revolving drum. A strap passes below the drum and is tightened by the nuts *NN* till the friction just balances the weight, which in its turn is adjusted by trial till it just balances the driving couple tending to turn the shaft. Stops are provided to prevent the lever from moving except within narrow limits, and when the adjustment is perfect the lever remains horizontal without resting against either stop. Here the driving couple and consequently the power are determined as in the preceding example, from which it only differs in the way in which the power is employed. Instead of being transmitted to a machine which is being driven it is all absorbed by a friction brake which replaces the machine for the time being. A modification is shown in Fig. 2, in which the strap passes over a wheel and is tightened by a suspended weight, the difference between which and the tension of a spring balance, to which the other end of the strap is attached, measures the driving effort.

In both these forms of friction dynamometer any variation in the driving effort requires a corresponding adjustment. The more complex form shown in Fig. 4 is provided with a compensating lever *DBC*, which tightens the friction strap embracing the wheel when the driving effort is great and loosens it when the effort diminishes. A self-acting adjustment is thus obtained, but the pressure of the fixed pin *D* fitting into a slot in the end of the lever renders the indications inaccurate, and the error may be serious unless special care is taken.

(4) In both the preceding cases the driving effort and the speed of the driving pair are constant, but in the indicator universally employed to measure the power of steam and other heat engines we find an example in which both vary. The driving effort is now measured for each

position of the piston and a curve drawn which represents it; the area of this curve will be the work done per stroke, and divided by the length of the stroke will give the mean driving effort. This will be further explained in Part V.

SECTION III.—STRAINING ACTIONS ON THE PARTS OF A MACHINE.

142. *Transmission of Stress in Machines.*—We have seen (Art. 94, p. 189) that a mechanism becomes a machine if certain links are added capable of changing their form or size, and so producing forces which tend to move the mechanism combined with other forces which resist the motion. Each link so added exerts equal and opposite forces on the elements it connects, and for a pair of forces the general word "Stress" may be used, which has already been employed in Article 1 in the case of the bars of a framework structure.

When the machine is at rest the forces, being all in pairs, balance each other, and have no tendency to move the machine as a whole. For example, in the direct-acting vertical engines represented in Fig. 1, Plate I., page 109, the driving link is the steam, pressing with equal force, one way on the cylinder cover, and the other way on the piston; the working link is the resistance to turning of the crank shaft, which exerts equal and opposite forces, one way on the crank, the other way on the frame which carries the crank-shaft bearings. The steam pressure and the working resistance may each be described as a "Stress." The forces which make up the stress are transmitted from the piston through the connecting rod to the crank, and, in the opposite direction, from the cylinder cover through the frame to the crank shaft. The horizontal pressure of the cross-head on the guide-bars is in like manner balanced by the equal horizontal thrust of the connecting rod on the crank pin, combined with the moment of the working resistance.

So in every machine, when at rest, or moving slowly and steadily, the stress is transmitted from the driving pair to the working pair, not only through the movable parts of the machine, but in the opposite direction, through the framing; and this is a circumstance which must be always borne in mind in designing the framing. Thus, in our example, the steam cylinder and crank-shaft bearing must be rigidly connected by a frame strong enough to withstand the total steam pressure, and, in addition, the bending due to the lateral pressure on the guide bars.

We have here one of the simplest examples of the transmission of stress; whether in a machine or in a structure it always takes place in a closed circuit.

If the driving pair and the working pair are the same, and acted on by the same stress, the whole state of stress is the same for all the mechanisms which are derived by inversion from the same kinematic chain. All such mechanisms are therefore statically as well as kinematically identical; it is only when we consider machines in motion, or the straining actions due to gravity, that it is necessary to consider which link (if any) is fixed to the earth. For example, the only difference between the direct-acting engine of Fig. 1, and the oscillating engine of Fig. 4, Plate I., is that the working pair is *BA* in the first and *BC* in the second. So again, in Plate III., the only difference between the water wheel of Fig. 2 and the horse gear of Fig. 3 is in the nature of the driving link, which in the first case is gravity acting on the falling water, and in the second a living agent.

A striking example of the balance of forces in a machine occurs in the hydraulic riveting machines. Here the working pair is a small hydraulic cylinder and its ram, between which the rivet is compressed. This cylinder is suspended from a crane by chains, and can be moved into any position, as it communicates with the accumulator (Part V.) by a flexible pipe. Any portable machine, however, is an example of the same kind: machines which require foundations have no complete frame apart from the solid ground which connects their parts together.

143. *Reversal of Stress.*—In many machines the direction in which stress is transmitted through one or more of the moving parts is reversed in the course of the period. For example, in a double-acting engine of the ordinary type the piston rod and the connecting rod are alternately in compression and tension as the crank turns through a revolution. Such a reversal of stress is a cause of shocks which, though they may individually be small, yet from the rapidity with which they recur at high speeds are ultimately destructive, and require in any case to be carefully considered in the design.

Suppose a crank which is rotating uniformly to be connected by a rod with a reciprocating piece such as a piston, but in the first instance let there be no steam admitted to the cylinder. When the piston is at the end of its stroke it is at rest, and has to be set in motion; it consequently drags on the crank with a force which we have already investigated in Art. 109, p. 224. As the piston moves onwards the drag diminishes and becomes zero near the middle of the stroke at the point where the velocity of the piston is greatest. In the second half of the stroke the piston is being gradually reduced to rest, and consequently presses against the crank pin and drives the crank, thus reversing the stress on the rods. A small amount of play is necessary for the purposes of lubrication between the crank pin and the brasses into which it fits, and consequently at the instant of reversal a "knock" occurs, thus damaging the bearing surfaces and wasting energy. The intensity of a knock of this kind depends on the acceleration of the moving piece, and would be small in the case here supposed where there is no steam admitted to the cylinder, so that reversal occurs in the middle of the stroke. Next imagine steam admitted to the cylinder in the usual way, then, as already described fully in the article cited, the pressure on the crank pin is due to the difference between the steam pressure and the force called into play by inertia, and the effect is that reversal occurs at or near the ends of the stroke. If the speed be moderate and the moving parts light, the knock will occur at the ends of the stroke, and if the steam be suddenly admitted and there be no compression, will be of considerable intensity. It may, however, be much diminished by "cushioning," that is, by closing the exhaust port before the end of the return stroke and thus enclosing in the cylinder a mass of steam, the compression of which behind the returning piston furnishes a force which, by its increase, gradually diminishes the stress and renders the reversal at admission less violent. At very high speeds or with heavy moving parts reversal occurs after the stroke has begun; as shown by the point K on the dotted line $L'CL'$ shown in Fig. 100, p. 225, the effect of reversal in the absence of cushioning is then not so great as if it occurred in the absence of cushioning at the ends of the stroke. Heavy reciprocating parts may therefore, under certain circumstances, be advantageous.

When the speed is excessive the forces called into play by inertia are so great that reversal must be avoided altogether. For driving a fan or some similar purpose a small engine of three inches stroke is sometimes run at 1000 or even 2000 revolutions per minute; on making the calculation by the formula of page 224 we find that the force P necessary to start the piston is now 150 times its weight, and the shock at reversal necessarily great. If the engine is made single acting, reversal can be prevented entirely by cushioning. In the Willans high speed engine the piston rod prolonged moves as a plunger in an independent cylinder containing air, which serves as the cushion, an arrangement which admits of any compression being used in the steam cylinder, which may for other reasons be convenient or economical.

The speed in the foregoing case is limited by the amount of cushioning employed, and this is also the case in cam mechanisms with force closure, such as have already been discussed in Ch. VI.

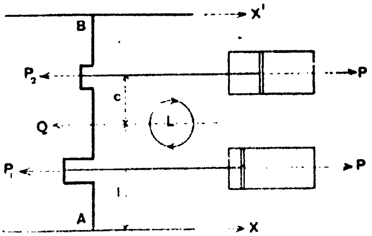
144. *Stability of Machines. Balancing.*—In a machine with reciprocating parts the balance of forces (Art. 142) is destroyed by their inertia when the machine is in motion, and, in consequence, the machine must be attached to the earth or some massive structure by fastenings of sufficient strength. The straining actions on these fastenings will now be briefly considered.

Taking the case of a direct-acting horizontal steam engine, let P be the total pressure of the steam on the cylinder cover, then the pressure (P') transmitted to the crank pin is not equal to P, but there is a difference (S), given by the formula (Art. 109, p. 224): neglecting obliquity,

$$S = P - P' = W \cdot \frac{x}{h}.$$

This difference will be a force acting on the engine as a whole, and straining the fastenings. The direction of this force is reversed twice every revolution, and its maximum value is obtained by putting $x = a$ in the above formula. In slow-moving engines the value of S is small, but at high piston speeds it becomes very great, and must be carefully provided against, especially when, as in locomotives, the engine cannot be attached to the ground.

Fig. 116.



In most cases there are two cranks at right angles, and therefore two forces S, S' given by the equations

$$S = W \cdot \frac{a}{h} \cdot \cos\theta\,;\ S' = \frac{Wa}{h} \cdot \sin\theta,$$

where θ is the angle the first crank makes with the line of centres. These two forces are equivalent to a single force (Fig. 116),

$$Q = S + S' = W \cdot \frac{a}{h}(\sin\theta + \cos\theta),$$

acting midway between them, and a couple

$$L = (S - S')c = W \cdot \frac{a}{h} \cdot c(\cos\theta - \sin\theta),$$

where $2c$ is the distance apart of the centre lines of the cylinders. The total effect therefore is the same as that of a single alternating force combined with an alternating couple, which tends to turn the engine as a whole about a vertical axis. The maximum values are

$$Q_0 = W\cdot\frac{a}{h}\sqrt{2}\,;\quad L_0 = \frac{Wac}{h}\cdot\sqrt{2},$$

and they are each reversed twice in every revolution.

In locomotives this action produces dangerous oscillations at high speeds, and must therefore be counteracted by the introduction of suitably placed balance weights, so as to neutralize both the force and the couple.

Fig. 117.

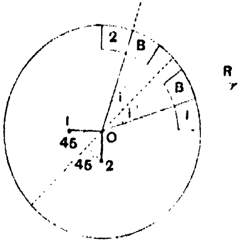


Fig. 117 shows a projection on a vertical plane of the two driving wheels and their cranks. On each wheel a balance weight is placed, occupying a segment between two or more spokes. The centre of gravity of each weight is in a radius nearly, but not exactly, opposite the nearer crank, the angle of inclination to the bisector being an angle i somewhat less than 45°. If B be the weight, r the radius of the circle in which its centre of gravity lies,

$$R = B\cdot\frac{r}{h}$$

is its centrifugal force; and by rightly taking the values of B and i the horizontal components of these forces derived from the two balance weights may be made to counteract both the force and the couple (Ex. 10, p. 290). In practice the weights are fixed approximately by a formula derived in this way, and the final adjustment is performed by trial. The engine is suspended by chains, and its oscillations, when perfectly adjusted, are very small even at very high speeds.

In high speed marine engines similar forces arise, of great magnitude, which must add considerably to the strain upon the fastenings, and which are now known to be the principal cause of vibration of the vessel. The question of balancing these forces has therefore become of great importance, and we shall recur to it hereafter.

When the speed of a machine is excessive, we have already seen that reversal of stress must be avoided, and besides this the greatest care is necessary that the axis of rotation of each rotating piece passes through its centre of gravity, and coincides with one of the axes of inertia of the piece (Art. 132). The magnitude of the forces which arise, in case of any error, may be judged of from the results of Exs. 13, 16, pages 290, 291. The vibrations due to these forces will, however, in some cases be greatest at some particular speed—depending on the natural period of vibration of the frame of the machine—which could only be determined by trial. (Ch. XVI.)

In similar machines the forces due to inertia will be in a fixed proportion to the weight of the pieces, when the revolutions vary inversely as the square root of the linear dimensions of the machine.

144A. *Stability of Machines (continued).* The pressure on a piston necessary to overcome the inertia of a connecting rod has already been investigated, but to complete the subject it is now desirable to study the relation which exists between the pressure thus found and the horizontal momentum of the reciprocating parts upon the change of which the disturbing forces considered in the last article depend.

Referring to page 229 the horizontal velocity of a point on the rod distant μl from the crank pin is $V_0\{\sin\theta + \mu \,.\, \tan\phi \,.\, \cos\theta\}$, which may be written

$$\text{Horizontal velocity} = (1-\mu)V_0 \,.\, \sin\theta + \mu V.$$

Taking therefore as before f as the acceleration of the piston, we have at the point in question

$$\text{Horizontal acceleration} = (1-\mu)\frac{V_0^2}{a} \,.\, \cos\theta + \mu f,$$

as may be easily proved independently. Hence, by summation, the rate of change of the total horizontal momentum of the rod is

$$X = (1-\sigma)\frac{WV_0^2}{ga} \,.\, \cos\theta + \sigma W \,.\, \frac{f}{g},$$

where, as before, σl is the distance of the centre of gravity of the whole rod from the crank pin.

Writing for f its approximate value given on page 228 we find

$$X = \frac{WV_0^2}{ga}\left\{\cos\theta + \frac{\sigma}{n}\cos 2\theta\right\}.$$

If $\sigma = 1$ we have the case of weights actually or virtually attached to the piston, when the inertia-pressure P already found is equal to X. But in the case of the rod P is a smaller quantity, the reason of which is that the angular swing of the rod requires a force at the crank pin in order to produce it. As the rod swings outwards it is gradually stopped, as it swings inwards it is gradually accelerated, by a pull of the crank arm towards the centre. Now if T be this pull, $T\cos\theta$ will be a horizontal force at the crank pin which assists in producing the horizontal acceleration, so that

$$T\cos\theta + P = X.$$

Taking for X the value just given and for P the value already found, we find after dividing by $\cos\theta$,

$$T = \frac{WV_0^2}{ga}\left\{\beta + \frac{\sigma - \gamma}{n} \cdot \cos\theta\right\}.$$

The force T thus found is a tension of the crank arm which furnishes an unbalanced force on the crank centre. The total horizontal disturbing force is therefore not P but the larger quantity X given by the formula above.

In balancing the reciprocating parts of an engine the formula shows that we have two separate sets of horizontal periodic disturbing forces to consider. The first are given by the formula

$$X_1 = \frac{\Sigma W V_0^2}{ga} \cdot \cos\theta,$$

where ΣW is the total weight of all the parts, connecting rod included. These are independent of the length of the rod, and they go through their variations in one revolution of the crank shaft. The second are given by the formula

$$X_2 = \frac{W' + \sigma W''}{n} \cdot \frac{V_0^2}{ga} \cdot \cos 2\theta,$$

where W' is the weight of the piston and all parts attached to it, while W' is the weight of the connecting rod, and they go through their variations in half the time. In the production of vibration the effect of the two sets is entirely different, as will be fully explained hereafter. It is the first set alone which are considered in the preceding article; the second set are a consequence of the obliquity of the connecting rod, and though of much less intensity, for a complete balance require consideration as well as the first.

Besides the horizontal forces there are also periodic forces perpendicular to the line of centres which may here be briefly noticed.

The force T found above has a vertical component $T \sin\theta$, and in addition the pressure of the cross-head on the guide is altered by the inertia of the connecting rod. The effect of these two vertical forces taken together is best seen by dividing the weight W of the rod into two parts σW at the cross-head and $(1 - \sigma)W$ at the crank pin. As the two parts have the same centre of gravity as the original rod their motion gives rise to horizontal and vertical reactions of the same total amount. The difference consists in the couple called into play by the rotation about the centre of gravity. The (radius)^2 of gyration of the two weights about the centre of gravity is easily seen to be $\sigma(1 - \sigma)l^2$, whereas that of the rod is the much smaller quantity $(\gamma - \sigma^2)l^2$. The effect of the rod may be derived from that of the weights by subtraction, and the disturbing forces to be considered are therefore (1) a force

$$S = (1 - \sigma)\frac{W'V_0^2}{ga} . \sin\theta$$

acting at the crank centre perpendicular to the line of centres, and (2) a couple

$$L = \frac{(\sigma - \gamma)W'l^2}{g} \times \text{Angular Acceleration},$$

the angular acceleration being given by Ex. 9, p. 103.

The transverse forces here considered are due to the inertia of the rod alone, and are generally of less importance than the horizontal forces previously found. The method here used to find them may be employed to find the value of T, and thence a value of P may be derived which will be found to agree with that already obtained. It need hardly be added that the terms "horizontal" and "vertical" are simply abbreviations for "parallel" and "perpendicular" to the line of centres. The actual position of that line has no influence on the questions considered.

As described in the last article in the case of a locomotive, the original method of balancing the forces due to the inertia of reciprocating parts was by the introduction of rotating masses suitably placed. The forces in question, however, are not annulled by the action of rotating weights, but simply altered in direction through 90°; if originally horizontal as in the locomotive they become vertical and conversely. Also the secondary forces which are the effect of the obliquity of connecting rod are not balanced at all. With short rods these forces may be of serious importance. If we take the case of a pair of cranks at right angles already considered (p. 283), then since $\cos(180° + 2\theta) = -\cos 2\theta$, the forces arising from each crank are equal and opposite so that the resultant alternating force is zero. The alternating couple, however, is doubled, becoming

$$L' = 2 . \frac{W + \sigma W'}{n} . \frac{ac}{h} . \cos 2\theta.$$

The maximum value of this is when $\cos 2\theta = 1$, and comparing it with the maximum value of the primary couple (p. 284), we obtain

$$\text{Ratio} = \frac{L'_0}{L_0} = \frac{W + \sigma W'}{W + W'} . \frac{\sqrt{2}}{n}.$$

Taking, for example, $(1 - \sigma)W'$ as 20 per cent. of the whole weight $W + W'$, the ratio is $1{\cdot}13/n$, which is not inconsiderable even when as in a locomotive the value of n exceeds 6. If a shaft be available which rotates at exactly double the speed of the crank shaft, the secondary forces may be balanced by counterweights attached to it, but this will rarely be the case. For a complete balance, therefore, other reciprocating weights must be introduced for the special purpose; or else, and preferably, in an engine with four or more cylinders, one-half the reciprocating parts may be made to balance the remainder, the engine thus becoming self-balanced. The conditions to be satisfied in such arrangements are essentially the same as when rotating weights are employed.

145. *Straining Actions on the Parts of a Machine due to their Inertia.*—Another important effect of the inertia of a piece is to produce straining actions upon it. An important example is that of a ring rotating about its centre: the centrifugal force produces a tension on the ring which may be thus determined.

Suppose Fig. 121, p. 302, to represent the ring. Let the velocity of periphery be V, the weight W, and the radius r, then the centrifugal force on the small portion BB' of length z is

$$S = W \cdot \frac{z}{2\pi r} \cdot \frac{V^2}{gr}.$$

Resolve this in a given direction and sum the resolved parts, as in the article to which this figure refers, then the total is

$$P = W \cdot \frac{2r}{2\pi r} \cdot \frac{V^2}{gr} = \frac{W}{\pi r} \cdot \frac{V^2}{g}.$$

The stress to which this gives rise is evidently

$$q = \frac{W}{2\pi r A} \cdot \frac{V^2}{g} = w \cdot \frac{V^2}{g},$$

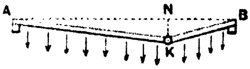
where A is the sectional area of the ring and w is the weight of unit volume. The result here obtained is of great importance; it shows that the "centrifugal tension" of a revolving ring is independent of the radius for a given speed of periphery. Hence the result also applies to every point of a flexible element, such as a belt, whatever be the form of the surfaces over which it is stretched. In high-speed belts the tension is considerably increased by this cause, and additional strength has to be provided (Ex. 12, p. 290).

Another example of the straining actions due to inertia occurs in the motion of a rod, the ends of which describe given curves. Shearing and bending are produced, and at high-speeds the magnitude of the stress thus arising is very great. Two common examples are given on p. 291, but the limits of this work do not permit us to pursue the subject.

In similar machines the intensity of the stress occasioned by the straining actions we are here considering will be the same if the revolutions vary inversely as the linear dimensions of the machine.

146. *Virtual Machines.*—It has already been pointed out (Art. 94) that a machine may be regarded as a mechanism with two additional links applied as straining links, or, what is the same thing, a frame with one straining link (Art. 43). Further, as also remarked in the article cited, the external forces on any structure may be regarded as a set of straining links. It follows then that if in any framework or other structure one of its parts suffer a change of form or size of any kind, the rest remaining rigid, we shall have a machine in which the driving links exert a known stress and the working link is the bar in question. The principle of work then enables us to determine the stress on the bar, for the stress ratio must be the reciprocal of the velocity ratio. A machine thus formed may be called a "virtual machine," its movements being only supposed for the purpose of the calculation, not actually existing. It is especially in applying this method that we find in treatises on statics the principle of work employed under the title "principle of virtual velocities."

Fig. 118.



We must content ourselves with a single example of this method. AB (Fig. 118) is a beam supported at the ends and loaded uniformly. Imagine the beam broken at K, and the pieces united by a stiff hinge, the friction of which is exactly equal to the bending moment M, then if the hinge be supposed gradually to yield under the weight, so that the joint K descends through the small space $KN(=y)$,

$$\text{Energy exerted} = \tfrac{1}{2}yw(AK+BK),$$

$$\text{Work done} \quad = M(i_1+i_2) = M\left(\frac{y}{AK}+\frac{y}{BK}\right),$$

where i_1, i_2 are the angles AK, BK make with the horizontal.

Equating the two,

$$M\left(\frac{1}{AK}+\frac{1}{BK}\right) = \tfrac{1}{2}w(AK+BK),$$

which gives the known value (p. 39),

$$M = \tfrac{1}{2}w \, . \, AK \, . \, BK.$$

The advantage of this method is that it leads directly to the required result, without the introduction of unknown quantities which require to be afterwards eliminated.

EXAMPLES.

1. In Ex. 1, p. 207, suppose the gun to weigh 35 tons, what additional powder will be required to provide for recoil? *Ans.* 1 lb. nearly.

2. Two vessels of displacements 8,000 and 5,000 tons are moving at 6 knots and 4 knots respectively. One is going north and the other south-west; find the energy of collision. *Ans.* 11,700 foot-tons.

3. Find the height of a governor revolving at 75 revolutions per 1′. *Ans.* 6·24″.

4. Find the dimensions of a Farcot governor to revolve at 40 revolutions per 1′, with the arms inclined at 30° to the vertical, and to be parabolic for small displacements. *Ans.* Height of governor = 22″. Length of arms = 34″. Length of cross-piece to which arms are attached = 8½″. More generally, if θ be the inclination, l the length of the arms, the length of cross piece is $2l \cdot \sin^3\theta$.

5. In a simple governor revolving at 40 revolutions per 1′ find the rise of the balls in consequence of an increase of speed to 41 revolutions. Also find the weight of ball necessary to overcome a frictional resistance of ½ lb., the linkwork being arranged so that the slider rises at the same rate as the balls. *Ans.* Rise of balls = 1·1″. Weight of each ball = 5 lbs.

6. The balls of a governor weigh 5 lbs. each and it is loaded with 50 lbs. The linkwork is such that the slider rises and falls twice as fast as the balls. Find the height for a speed of 200 revolutions per 1′, and if the speed be altered 2 per cent., find the tendency to move the regulating apparatus. How much is this tendency increased by the loading? If the engine is required to work at three-fourths its original speed, by how much should the load on the governor be diminished? *Ans.* Height = 9·7″. Tendency = 2·2 lbs. (increased 11 times).

7. A uniform rod is hinged to a vertical spindle and revolves at a given number o revolutions; find its position. Deduce the effect of the weight of the arms of a governor on its height. *Ans.* Height of rod $= \frac{3}{2} \cdot g/A^2$. Height of governor is increased in the ratio $1 + \frac{1}{2}n : 1 + \frac{1}{3}n$ where n is the ratio of the weight of the arm to the weight of the ball.

8. In Ex. 6, p. 124, find the ratio in which the bending moment at each point is affected by the inertia of the rod.

Every point of the rod describes relatively to the engine a circle and the centrifugal force of any portion of the rod = 18·6 times the weight. In the lowest position the centrifugal force acts with gravity, and so in this position the bending action is the same as if the weight of the material of the rod were 19·6 times its true weight.

9. In a horizontal marine engine with two cranks at right angles distant 8 feet from one another, weight of reciprocating parts attached to each crank 10 tons, revolutions 75 per minute, stroke 4 feet. Find the alternating force and couple due to inertia. *Ans.* Alternating force = 54·2 tons. Alternating couple = 216·8 foot-tons.

10. An inside cylinder locomotive is running at 50 miles per hour, find the alternating force and couple. Also find the magnitude and position of suitable balance weights, the diameter of driving wheels being 6 feet, the distance between centre lines of cylinders 2′ 6″, stroke 2′, weight of one piston and rod 300 lbs. Horizontal distance apart of balance weights 4′ 9″. Diameter of weight circle 4′ 6″. *Ans.* Alternating force = 7,871 lbs. Alternating couple = 9,893 foot-lbs. $B = 106{\cdot}5$ lbs. $i = 27\frac{3}{4}°$.

11. A fly-wheel 20 feet diameter revolves at 30 revolutions per 1′. Assuming weight of iron 450 lbs. per cubic foot, find the intensity of the stress on the transverse section of the rim, assuming it unaffected by the arms. *Ans.* 96 lbs. per sq. inch.

12. A leather belt runs at 2,400 feet per 1′; find how much its tension is increased by centrifugal action, the weight of leather being taken as 60 lbs. per cubic foot. *Ans.* 20·5 lbs. per square inch.

13. If r be the radius of the circle described by the centre of gravity of a rotating body, h the height due to the revolutions, show that the centrifugal force is

$$R = W \cdot \frac{r}{h}.$$

Obtain the numerical result (1) for a wheel weighing 100 lbs. with centre of gravity one-sixteenth of an inch out of centre, revolving at 1000 revolutions per minute, (2) for a piece weighing 10 lbs. revolving at 300 revolutions per minute in a circle 1 foot diameter. *Ans.* (1) 178 lbs. (2) 154 lbs.

14. In question 8 suppose the connecting rod of uniform transverse section, find how much the bending moment upon it due to its weight is increased by the effect of inertia.

Here the bending moment is greatest (very approximately) when the crank is at right angles to the connecting rod, and the forces due to inertia then consist (also very approximately) of a set of forces perpendicular to the rod, and varying as the distance from the cross-head pin. At the crank pin we have simply the centrifugal force due to the revolutions and length of crank. Thus the curve of loads is a straight line (p. 62) whence, proceeding by the methods of Chap. III., we find for the maximum moment

$$M = \frac{11\,Wl}{9\sqrt{3}} \cdot \frac{a}{h}$$

where l is the length of rod, a the length of crank, h the height due to the revolutions In the numerical example the effect of inertia is about 9½ times that of the weight W.

15. A body rotates about an axis OE, lying in a principal plane through its centre of gravity G, and inclined to a principal axis OG at an angle θ. Show that the moment of the centrifugal forces about O is

$$L = W\frac{k^2 - k'^2}{h} \cdot \sin\theta \cdot \cos\theta,$$

where h is the height due to the revolutions, and k', k are the radii of gyration about OG and a line through O, perpendicular to OG in the plane GOE, respectively. Deduce the height of a compound revolving pendulum.

16. A disc rotates about an axis through its centre at 1000 revolutions per minute. The disc is intended to be perpendicular to the axis, but is out of truth by $\frac{1}{100}$th of the radius: find the centrifugal couple. *Ans.* If r be the radius in inches the couple in inch-lbs. is

$$L = \frac{Wr^2}{13{\cdot}1}.$$

17. In question 10 find the alternate increase and diminution of the pressure of the driving wheel on the rail due to the inertia of the balance weight. *Ans.* 4,400 lbs.

NOTE. This force of about 2 tons produces great straining actions on both the wheel and the rails.

18. The power of a portable engine is tested by passing a strap over the fly-wheel, which is 4 feet 6 inches diameter, fixing one end and suspending a weight from the other. The weight is 300 lbs., and the tension of the fixed end is found by a spring balance to be 195 lbs.: what is the power when running at 160 revolutions per minute? *Ans.* 7·2 H.P.

19. In question 10, page 235, find the least number of revolutions for which there can be a "knock" after the stroke has commenced. If the steam be cut off at ¼th or earlier, show that a knock will also occur at other points of the stroke. *Ans.* 124.

20. In the cam movement shown in Fig. 1, Plate IV., page 159, suppose the cam a circular disc of radius equal to the stroke of the sliding piece. Supposing the force of the spring twice the weight of the sliding piece; find the greatest number of revolutions per 1′ the cam can make when rotating uniformly. *Ans.* If S be the stroke in inches, n the revolutions,

$$n = \frac{216}{\sqrt{S}}.$$

21. In the original form of the 3-cylinder Brotherhood engine the cylinders communicated with a central chamber containing steam at full pressure. At the further end the

steam was alternately admitted and exhausted. Show that to avoid reversal of stress the weight of piston and rod must not exceed

$$W=70{,}500\frac{P}{n^2S},$$

where P is the total pressure on one piston.

22. Find the alternating force and couple both primary and secondary for a pair of cranks inclined at an angle β.

The simplest way of expressing the results for a pair of cranks is by measuring the angle θ which gives the position of the cranks not to one of the cranks but to the line bisecting the angle β between them. The results then are

$$X_1=A_1 . \cos\theta . \cos\tfrac{1}{2}\beta\,;\quad X_2=A_2 . \cos 2\theta . \cos\beta\,;$$
$$L_1=B_1 . \sin\theta . \sin\tfrac{1}{2}\beta\,;\quad L_2=B_2 . \sin 2\theta . \sin\beta\,;$$

where A_1, B_1 : A_2, B_2 are co-efficients found as in the text pages 283, 287 for the primary and secondary forces respectively. The results show that, when the crank angle β can be chosen at pleasure, the disturbing forces may, to a considerable extent, be regulated. In some cases the alternating couples may be more injurious than the alternating forces, and this is probably true in a locomotive. In others the secondary forces may be less injurious than the primary,

REFERENCES.

For further information on subjects considered in this chapter the reader is referred to

WORTHINGTON. *Dynamics of Rotation.* Macmillan.
KENNEDY. *Mechanics of Machinery.* Macmillan.

PART IV.

STIFFNESS AND STRENGTH OF MATERIALS

PART IV.—STIFFNESS AND STRENGTH OF MATERIALS.

147. *Introductory Remarks.*—The straining actions which tend to cause a body or a structure to separate into parts A and B in the manner explained in Part I. are counteracted by the mutual action between the parts at each point of the real or ideal surface which divides them. In other words (see Art. 1), a STRESS exists at each point of the surface, the elements of which are A's action on B and B's action on A. If we consider the total amount of the stress, these elements each form one element of the straining actions on A and B respectively; but for our present purpose it is needful to consider, not the total amount, but the intensity of the stress. This in general varies from point to point, and at each point is measured by the stress per unit of area on any small area enclosing the point.

Either element (say A) may be regarded either as A's action *on* B, or as the resistance which A offers to the action *of* B; in other words, stress may be regarded in two aspects, either as the cause tending to produce separation into parts, or as the resistance to such separation. It is under the first aspect that we shall chiefly regard stress, generally employing the word resistance when we wish to express the second idea. Stress then may be described as the straining action on the ultimate particles of a body. Conversely a straining action as defined in Ch. II. may also be described as the "resultant stress" on the section we are considering.

If the stress exceeds a certain limit, separation into parts occurs, and this limiting intensity of stress varies for different material and measures the Strength of the material.

Accompanying the tendency to separation into parts we invariably find changes of dimension in the body and each of its parts, for no body in nature is absolutely rigid. Such changes are called STRAINS, and are of two kinds, changes of volume and changes of figure, or, in other words, changes of size and changes of shape. Changes of size in any dimension are measured by the ratio of the change to the original dimension considered; changes of shape consist in the alteration of relative angular position or distortion of the parts considered, and are measured by the absolute magnitude of the alterations in question. In most cases which concern us, both kinds of change take place together, and are of exceeding smallness.

The strains produced in solid bodies by the action of forces depend on the nature of the material and on the kind of stress.

Bodies are either solid or fluid. A fluid may be defined as material which offers no resistance to change of shape, but only to change of volume, especially diminution of volume, so that any distorting stress, however small, will cause indefinite change of shape if sufficient time be allowed. On the other hand a solid body will resist a distorting stress for an indefinite time, provided that stress be not too great. In a fluid body at rest only one kind of stress can exist, namely, a pressure equal in all directions: hence often called "fluid" stress.

There are two extreme conditions in which a solid body may exist, the Elastic state and the Plastic state. Elasticity is the power a body possesses of returning to its original shape and dimensions after the forces which have been applied to it are removed. All bodies possess this property to a greater or less extent, and most (perhaps all) possess it to a great degree of perfection if the strains to which it has been exposed are not too great. Even so unlikely a material as soft clay is elastic if the force applied to it is very small. This may be shown by suspending a long filament, formed by forcing clay through a small orifice, by one end and twisting the other, to which an index is attached; on release the index returns to its original position.* In perfectly elastic material the recovery of size and shape on removal of the forces is complete, unless the temperature has meanwhile varied; and the materials of construction may be regarded as approximately satisfying this condition, provided a certain limit stress be not overpassed. This is called the Elastic Strength of the material. It is also described as the "limit of elasticity."

When, on the other hand, the forces applied to the body are comparatively great, the material in many cases approaches the other extreme condition, the plastic state. In this state any forces causing a distorting stress beyond a certain limit, and so applied that disruption does not occur, will produce indefinite distortion, so that the material behaves like a fluid. Thus soft clay, lead, copper, or even malleable iron may

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*See Robison's *Mechanical Philosophy*, vol. I., page 375. The original observation is said to have been made by Coulomb. Though frequently quoted it does not appear to have been verified.

be moulded into different shapes or drawn out into wire. In intermediate cases a body may exhibit the properties of the elastic and the plastic states combined.

We commence by studying matter in the perfectly elastic state. There are two different kinds of elasticity,—Elasticity of Volume and Elasticity of Figure. A fluid possesses the first kind only, since by definition it has no power of resisting change of shape: the second is characteristic of solids. In general a change of dimensions involves both a change of size and a change of shape, so that both kinds of elasticity are called into play together. In perfectly elastic material the strain produced by a given stress is always proportional to the stress, being found by dividing the stress by a co-efficient or "modulus" of elasticity, depending on the kind of stress and the nature of the material. This property having been discovered by Robert Hooke, is known as Hooke's Law. Further, if the stress be relaxed in the slightest degree the strain diminishes, that is, in perfectly elastic material, the elastic forces are completely "reversible" (p. 186).

The magnitude of the stress produced by the action of given forces upon a body depends very much on whether they are applied all at once or are supposed to be at first very small and gradually to increase to their actual amounts. The next four chapters will be limited to the action of a gradually applied load on perfectly elastic material, after which the effect of sudden application and of impact is considered. The experimental part of the subject is placed in the last chapter (Ch. XVIII.), but should be referred to constantly as required.

CHAPTER XII.

SIMPLE TENSION, COMPRESSION, AND BENDING OF PERFECTLY ELASTIC MATERIAL.

SECTION I.—TENSION AND COMPRESSION.

148. *Simple Tension.*—The effect of forces acting on a bar has already been explained in Chapter II. to consist in the production of certain straining actions which we called Tension, Compression, Bending, Shearing, and Twisting, and we now go on to consider the changes of form and size which the bar undergoes and the stress produced at each point on the supposition that the material of the bar is perfectly elastic.

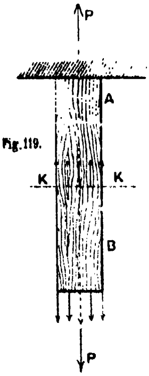


Fig. 119.

Let AB (Fig. 119) be a bar subjected to the action of equal and opposite forces applied at the ends in the same straight line. At any transverse section KK there will be a tendency to separate into two parts A, B, which is counteracted by a mutual action between the parts at each point of the section which, in accordance with our previous definitions, is called the Tensile Stress at the point. The total amount of the stress will be P; but the intensity will depend on the area of the section (A), so that P/A is the mean intensity of stress, or the stress per unit of area. The stress may be the same at all points of the section. We then say it is uniformly distributed, and the intensity at all points $= P/A$.

Stress is commonly expressed either in pounds or in tons per sq. inch. The second method is on the whole the most convenient, and will be chiefly employed in this treatise. In metric measures the unit commonly employed is the kilogramme per square centimetre, which is connected with the British system by the relations:

One kilogramme per sq. cent. = 14·233 pounds per sq. inch.

One ton per sq. inch = 157·5 kilogrammes per sq. cent.

In order that the intensity of the stress may be the same at every point of every transverse section of the bar, it is *theoretically* necessary that the load P should be applied in a uniformly distributed manner all over the end B. Then if the material is perfectly homogeneous each elementary portion of KB will be strained alike, and the uniformly distributed load at B will be balanced by a uniformly distributed stress over any section KK. In such a case the line of action of the resultant of the applied load P passes through the centre of gravity or centre of position of the transverse section KK. Unless it does so the equilibrium of the portion KB is not possible by means of a uniformly distributed stress over the section. But from experience it appears that for uniformity of stress it is not absolutely necessary for the load to be applied in this distributed manner. It may be applied for instance by pressure on a projecting collar; and yet *if the line of application of the load traverses the centre of gravity of the sectional area*, the material, if homogeneous, will so yield as *practically* to produce at a section a little distant from the place of application of the load a stress of uniform intensity. This is a particular case of a principle which will be further referred to hereafter.

If the applied load is increased, the stress on the section is proportionately increased, until at last the material yields under it and the bar breaks. If $W =$ breaking load, the corresponding stress measured by W/A is a quantity which depends on the nature of the material. If we call it f, then the breaking or ultimate load $= Af$.

Accompanying the application of the load producing a tensile stress, an increase of length and diminution of transverse dimension is observed. In metallic bodies the alterations are exceedingly small if the limit of elasticity is not exceeded (see Table II., Ch. XVIII.), and therefore in estimating the stress on the section it is not worth while to take account of the slight alteration in the area of the transverse section. Under the same load the change of length is proportional to the length. If x be the total change of length, and l the original length, then the extension per unit of length is

On account of the smallness of e it is immaterial, so long as the limit of elasticity is not exceeded, whether l is taken as the original or altered length of a metallic bar.

As already stated (Art. 147), it is usual to restrict the word *strain* to mean the alteration of the dimension and form which bodies undergo, and to use the word *stress* when referring to the elastic forces which accompany the strain. Thus e is a measure of the tensile strain produced in the bar, whilst p is a measure of the accompanying tensile stress. Since by Hooke's law the extension of the bar is proportional to the force producing it, it follows that the strain is proportional to the accompanying stress. Thus p and e may be connected by some constant the value of which depends on the nature of the material. We may write

$$p = Ee,$$

in which E is called the modulus of elasticity of the material, a quantity which is of the same kind and expressed in the same units as the stress p. When the stress p is expressed in pounds per square inch, the value of E for wrought iron may be taken as about 29,000,000. This is about 13,000 tons per square inch, but in many kinds of iron the value of E is considerably less than this.

Putting for e its value x/l, we have the general relation,

$$\frac{p}{E} = \frac{x}{l}.$$

The transverse strain, that is, the contraction per unit of transverse dimension, is from one-third to one-fourth the longitudinal strain.

149. *Work done in Stretching a Rod.*—Having found the relation between the tensile stress and strain, we will now consider how much work must be done in order to stretch it.

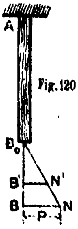


Fig. 120

Let the load of gradually increasing amount be applied to the bar, the bar will stretch equal amounts for equal increments of load: or the elongation of the bar will for all loads be proportional to the load. This may be represented graphically. Suppose the load P' produces the extension shown, greatly exaggerated, by B_0B' (Fig. 120), and we set off an ordinate $B'N'$ to represent P' on some scale, and do that for any number of loads, taking, for example, BN to represent P, which produces the extension $B_0B = x$; then all the points N will lie on the sloping line passing through B_0. Having done this, the area of the triangle B_0BN will represent the quantity of work done in stretching the bar by the amount $B_0B = x$. Thus

$$\text{Work done} = \tfrac{1}{2}Px.$$

The energy thus exerted is stored up in the stretched bar, and may be recovered if the bar is allowed under a gradually diminished load to contract. In the perfectly elastic bar the contraction will be exactly the same as the extension, and there will be no loss of energy in stretching it. In other words the elastic forces are "reversible." But if the elasticity is imperfect, some of the energy expended in stretching the bar is employed in producing molecular changes, as, for example, change of temperature. On contraction this amount of energy will not be restored.

The energy stored may be described as the Elastic Energy of the bar, and we may express it in a different form. For P put its value $= pA$, and for x its value $= pl/E$. The substitution of these values of P and x will give

$$\text{Elastic Energy} = \tfrac{1}{2}pA\,.\,\frac{pl}{E} = \frac{p^2}{E}\,\frac{Al}{2} = \frac{p^2}{E} \times \frac{\text{Volume}}{2}.$$

Otherwise, replacing p by Ee, we find

$$\text{Elastic Energy} = \tfrac{1}{2}Ee^2 \times \text{Volume}.$$

Thus the work required to produce a given stress p or strain e is proportional to the volume, or, what is the same thing, to the weight, of the bar.

If the stress produced is increased up to the elastic limit, or, as it is often called, the *proof stress*, so that $p = f$, then $\frac{f^2}{E}\,.\,\frac{\text{Volume}}{2}$ expresses the greatest amount of work which can be done on, and stored in the bar without injuring it or impairing its elasticity. This is called the *resilience* of the bar. The quantity f^2/E, the value of which depends on the nature of the material, is called the *modulus of resilience*; it is *double* the resilience per unit of volume (see Appendix), and, as we shall see hereafter, furnishes a measure of the resistance of the material to impact when the limits of elasticity are not exceeded (Chap. XVI.). A table of co-efficients of strength and elasticity for materials commonly used in construction will be found at the end of Chapter XVIII.

150. *Thin Pipes and Spheres under Internal Fluid Pressure.*—We now pass on to consider an important case of simple tension: that of a thin cylindrical shell subjected to internal fluid pressure. A cylinder with rigid ends and a sphere are cases of a vessel under internal fluid pressure which tends to preserve its form. The equilibrium in these two cases is stable, for if the vessel suffers deformation the internal pressure tends to make it recover its original true form. Vessels, the sides of which are flat, tend, by bulging, to assume these forms, and the tendency must be resisted by staying the surfaces in some way. If, as generally happens, there is acting also an external fluid pressure less than the internal, then, in what follows, the intensity of the internal pressure must be taken to be the excess of the internal over the external pressure.

Let p be the intensity of the fluid pressure in pounds per square inch, d the diameter, t the thickness of the shell, and l the length of the cylinder. Suppose in some way that the ends are maintained perfectly rigid, and for convenience let them be flat. There are two principal ways in which the strength of the shell can be estimated.

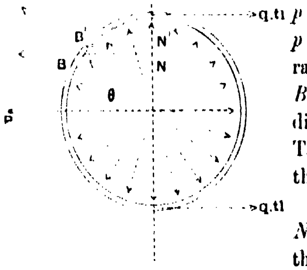
First, consider the tendency to tear asunder longitudinally, parallel to the axis of the cylinder. Imagine the cylinder divided into two parts by a plane passing through the axis of the cylinder. On each half cylinder there is a pressure P due to the resultant fluid pressure on that half which tends to produce a separation at the section imagined. The separation is prevented by the resistance to tearing which the metal of the shell offers, calling into action a uniform tensile stress at the two sections made by the imaginary plane through the axis of the cylinder.

Let q = intensity of tensile stress produced; then the area over which the stress acts being $2tl$, the total resistance to tearing is $q \times 2tl$, which must be equal to P the tendency to tear. In a transverse section take two points B, B' (Fig. 121) near together. The surface of the shell, $BB' \times l$, is acted upon by a normal pressure p per unit of area. The pressure $p \,.\, BB' \,.\, l$ may be taken to act in a radius drawn to the middle point of BB', making an angle θ with the direction of the resultant force P. The resolved part of this pressure in the direction of P

$$= pl \,.\, BB' \,.\, \cos\theta = pl \,.\, NN',$$

NN' being the projection of BB' on the plane of section. Summing up the pressures on all the small arcs BB', composing the semicircle, we obtain the total separating force.

Fig. 121.



$$P = pl \,.\, \Sigma NN' = p \,.\, l \,.\, d,$$

$$\therefore\ 2qtl = pld,$$

$$\text{or}\quad q = \frac{pd}{2t};$$

thus the tensile stress is directly proportional to the diameter, and inversely proportional to the thickness of the cylindrical shell. For greatest accuracy d should be taken as the mean of the internal and external diameters. The formula just obtained is true only when the thickness is small compared with the diameter. If t is large, the stress is not uniform over the section; the formula will then give the mean stress if d be understood to mean the internal diameter.

We next consider the tendency for the cylinder to tear across a transverse section when there are no longitudinal stays to take the pressure on the ends. The total pressure on each end of the cylindrical shell is the separating force, and in the absence of stays the resistance to separation is due to the tensile stress, q' suppose, called into action over the annular area $\pi d \,.\, t$ of the transverse section.

$$\therefore\ \pi dt \,.\, q' = \frac{\pi}{4} d^2 p\ ; \quad \text{or } q' = \frac{pd}{4t}.$$

This is just half the stress on the longitudinal section. If the vessel is spherical in form, the stress produced on all sections of the sphere through the centre is the same as at the transverse section of the cylinder.

The formula just obtained is used to estimate the strength of a boiler which is more or less cylindrical; but since the boiler is made up of plates overlapping each other, connected together at the edges by rivets, and since also a line of rivets in a longitudinal section is generally found only for a portion of the length of the boiler, the question of strength is complicated. But a longitudinal section through the greatest number of rivet holes is the weakest section, and if for q we write f, where f is a co-efficient of strength to be determined from experience, the value of it depending, among other things, on the form of joint, then the formula

$$p = \frac{2ft}{d}, \quad \text{or } t = \frac{pd}{2f},$$

may be used as a semi-empirical formula to determine the greatest pressure which can be employed in a given boiler, or the thickness of metal required to sustain a given pressure. The value of the co-efficient for iron boilers with single rivetted joints is about 4,000 lbs. per square inch, or, when double rivetted, as is usual in large boilers, 5,500. With steel the value is about one-third greater. In large boilers at high pressure these values, however, have of late been very greatly exceeded, for reasons which will be considered in a subsequent chapter.

151. *Remarks on Tension.*—The results obtained in the present section are, strictly speaking, only applicable when the piece of material considered is of uniform transverse section, but they nevertheless may be used when the transverse section is variable, provided the rate of variation be not too great and the other conditions mentioned are strictly fulfilled. The intensity of the stress is then different at different parts of the bar, varying inversely as the transverse section and in determining the elongation this must be taken into account.

In many cases of tension the effect of the weight of the tie and other circumstances introduces an additional stress, the amount of which is often imperfectly known. This is allowed for, either by making a certain addition to the theoretical diameter, or by the use of a factor of safety adapted to the particular case. On the other hand it also often happens, as in the case of ropes for example, that the strength of the material is greater in small sizes than large ones for reasons connected with the mode of manufacture.

152. *Simple Compression.*—When the forces applied to the ends of a bar act in a direction towards one another the bar is in a state of *compression.* If the bar is long compared with its transverse dimensions, then any slight disturbance from uniformity will cause it to bend sideways under the compressive force, and we have then, *not* simple compression, but compression compounded with bending, an important case to be considered hereafter. To obtain simple compression the ratio of length to smallest breadth should not exceed certain limits which depend on the nature of the material, viz., cast iron 5 to 1, wrought iron 10 to 1, steel 7 to 1. These values, however, depend to some extent on the type of section. Further, it is necessary that the material be perfectly homogeneous, and that the line of action of the load should be in the axis of the bar. Then the results we have obtained for simple tension apply to this case of simple compression

$$p=\frac{P}{A},$$

and the strength of the column is given by $P=Af$, where f is the co-efficient of strength. The compression x which the column undergoes is connected with the stress by the equation

$$p=E\frac{x}{l}.$$

The modulus of elasticity E would, in a perfectly elastic body, be the same as for tension. In actual materials it sometimes appears to be less; but within the elastic limit only slightly less.

EXAMPLES.

1. A rod of iron 1 inch in diameter and 6 feet long is found to stretch one-sixteenth inch under a load of 7½ tons. Find the intensity of stress on the transverse section and the modulus of elasticity in lbs. and tons per square inch.

Stress = 21,382 lbs. = 9·55 tons.

Modulus of elasticity = 24,638,000 lbs. = 11,000 tons.

2. What should be the diameter of the stays of a boiler in which the pressure is 30 lbs. per square inch, allowing one stay to each 1½ square feet of flat surface and a stress of 3,500 lbs. per square inch of section of the iron? *Ans.* 1½ inch.

3. In Example 1 find the work stored up in the rod in foot-pounds. *Ans.* 43¾.

4. If in the last question the rod were originally 2″ diameter and half its length were turned down to a diameter of 1″, compare the work stored in the rod with the result of the previous question.

Ans. Ratio = ⅝.

5. In Example 1 assume the given load of 7½ tons to be the proof load; find the modulus of resilience. *Ans.* 18·56 in inch-lb. units.

6. Find the thickness of plates of a cylindrical boiler 4′ 2″ diameter to sustain a pressure of 50 lbs. per square inch, taking the co-efficient of strength of plate at 4,000 lbs. *Ans.* $\frac{5}{16}$″.

7. A spherical shell 4′ diameter ¼″ thick is under internal fluid pressure of 1,000 lbs. per square inch. Find the intensity of stress on a section of the sphere taken through the centre. *Ans.* 48,000 lbs. per square inch.

8. Find the necessary thickness of a copper steam pipe 4″ diameter for a steam pressure of 100 lbs. above the atmosphere, the safe stress for copper being taken as 1,000 lbs. per square inch. *Ans.* ·2″.

9. A circular iron tank, diameter 16 feet, with vertical sides ½″ thick, is filled with water to a depth of 12 feet: find the stress on the sides at the bottom. How should the thickness vary for uniform strength throughout? *Ans.* 1,024 lbs. per square inch.

10. What length of iron suspension rod will just carry its own weight, the stress being limited to 4 tons per square inch, and what will be the extension under this load?

Ans. 2,700 feet. Extension = 5″.

11. The end of a beam 10″ broad rests on a wall of masonry; if it be loaded with 10 tons what length of bearing surface is necessary, the safe crushing stress for stone being 150 lbs. per square inch? *Ans.* 15″.

12. Find the diameter of bearing surface at the base for a column carrying 20 tons, the stress allowed being as in the last question. *Ans.* 20″ nearly.

13. Compare the weight of the shell of a cylindrical boiler with the weight of water it contains when full. *Ans.* Ratio = 15·5 p/f.

SECTION II.—SIMPLE BENDING.

153. *Proof that the Stress at each Point varies as its Distance from the Neutral Axis.*—The nature of the straining action producing bending has been sufficiently explained in the third section of Chapter II., and we shall now consider the kind of stress which results on the ultimate particles of a solid bar of uniform transverse section and of perfectly elastic material. The bar is supposed symmetrical about a plane through its geometrical axis, and the bending is supposed to take place in this plane, which may be called the Plane of Bending. In the first instance the bending is supposed to be "simple," that is, it is not combined with shearing as is most often the case in practice, but is due to a uniform bending moment (see Art. 21). The curvature of the beam is then uniform, that is to say, it is bent into a circular arc. The investigation consists of three parts.

Fig. 122 shows a longitudinal section AB and a transverse section LL through the centre of the beam; by symmetry it follows that if the bending moment be applied to both ends in exactly the same way, that transverse section, if plane before bending, will be still plane *after bending, for there is no reason for deviation in one direction rather than another. It will be seen presently that if the bending moment be applied to the ends of the beam in a particular way all transverse sections will be in the same condition, and we may therefore assume that not only the central section, but any other sections KK we please to take, will remain plane notwithstanding the bending of the beam. All such sections, if produced, will meet in a line the intersection of which by the plane of bending will be a point O, which is the common centre of the circular arcs KL, PP, NN, etc., formed by* the intersection of the same plane with originally plane longitudinal layers. These layers after bending have a double curvature, one in the plane of bending, the other in the transverse plane; the transverse bending however need not be considered at present, and the transverse section of the layers may be treated as straight lines. Before bending, the layers were all of the same length, being cut off by parallel planes, but now they will vary in length since they lie between planes radiating from an axis O. We shall find presently that some layers must be lengthened and some shortened, an intermediate layer, NN in the figure, being unaltered in length. This layer is called the Neutral Surface, and the transverse section of that layer SS is called

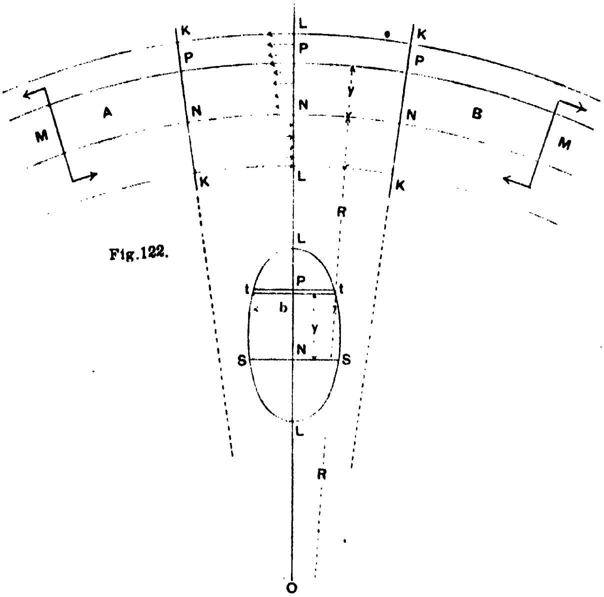


Fig. 122.

the Neutral Axis, the last expression being always used in reference to a *transverse* section, *not* a longitudinal section. Let the radius of the neutral surface be R. The more the beam is bent, that is the less R is, the greater will be the stress produced by the bending action; and the first step in the investigation is to obtain the relation between the stress produced at any point of a transverse section and the radius of curvature R. If we bisect SS in N and draw LNL at right angles to SNS, it is necessary that the section of the beam should be symmetrical on each side of LNL; with this restriction the section may be any shape we please.

Now consider any layer PP of the beam between the planes LL and KK which is at the distance y from the neutral surface NN or neutral axis SNS. This layer will be curved to a circle whose radius is $R+y$, and it must undergo an alteration of length from NN which it had before bending, to PP which it now has. Thus the alteration of length per unit of length, that is, the strain $e=\frac{PP-NN}{NN}$, but since arcs are proportional to radii $\frac{PP}{NN}=\frac{R+y}{R}$,

$$\therefore \text{ the strain } e=\frac{PP-NN}{NN}=\frac{y}{R}.$$

If the layer we are considering is taken below the neutral surface, the strain, which will then be compression, will be given by the same expression $e=y/R$, e and y both being negative.

Accompanying the longitudinal strain just estimated there must be a longitudinal stress proportional to the strain. Let p be the intensity of that stress, then

$$p=Ee,$$

where E is a modulus of elasticity. If we imagine the beam divided into elementary longitudinal bars, and if we imagine each of those bars independent of the others, it will follow that E is the same modulus of elasticity as we have previously employed in Section I. of this chapter. This, however, implies that the bar can freely contract and expand laterally when stretched and compressed, and we therefore could not be sure *a priori* that the union of the bars into a solid mass would not cause the value of E to be different from that for simple stretching, and to vary for different layers of the beam. It will be seen hereafter, however, that there are good reasons for the assumption.

Accordingly we write

$$p=E\cdot\frac{y}{R},$$

where E is the ordinary (also called Young's) modulus of elasticity.

If y is taken below the neutral axis then p is negative, signifying that the stress is now compressive. In perfectly elastic material the value of E is the same for compression as for tension, and so, within the limits of elasticity, the same equation will apply for all parts of the transverse section.

Thus the stress at any point of the transverse section of the bar is proportional to its distance from the neutral axis.

154. *Determination of Position of Neutral Axis.*—The second step in the investigation is to find the position of the neutral axis, which may be done by dividing the beam into two portions, A and B, by a section LL, and considering the horizontal equilibrium of either portion, say B. The external forces, being vertical, have no horizontal component, and we have therefore only to take account of the internal molecular forces which act at the section LL. Above the neutral axis the action of LA is a tendency to pull B to the left; but below the neutral axis, the tendency is to thrust B to the right. In order that it may remain in equilibrium, and not move horizontally, it is necessary that the total pull should equal the total thrust; or the total horizontal force at the section must be zero. To estimate the horizontal force, consider the force acting on a thin strip of the transverse section, of breadth b, and thickness t, distant y from the neutral axis. The thrust or pull on this elementary strip $= p \,.\, b \,.\, t$.

Summing the forces on all the strips composing the sectional area, we must have

$$\Sigma p \,.\, bt = 0\,;$$

but $p = Ey/R$ where E and R are the same for all strips of the section.

$$\therefore\ \frac{E}{R} \,.\, \Sigma bt \,.\, y = 0.$$

That is to say, the sum of the products of each elementary area into its distance from the neutral axis must be zero.

This can be true only if the axis passes through the centre of gravity of the section; for it is the same thing as saying that the moment of the area about the neutral axis is to be zero.

155. *Determination of the Moment of Resistance.*—The third and last step in the investigation is to obtain the connection between the bending moment applied, and the stress which is produced by it. Again, considering either portion, AL or BL, of the beam, say AL, the external forces on A produce a bending moment or couple, M, which has to be resisted by the internal stresses called into action at the section K; so that the total moment of these stresses must be equal to M. The moment of the resisting stresses, being a couple, may be estimated about any axis with the same result. For convenience we will estimate it about the neutral axis of the section.

Let us again consider the elementary strip of area bt, distant y from neutral axis, on which the intensity of stress is p, the force, pull, or thrust, on this strip being pbt. The moment of the force $= p\,.\,bt\,.\,y$. Seeing that forces on all elementary strips, whether pull or thrust, all tend to turn the piece AL the same way, the total moment of the stresses will be found by summing all terms, $p\,.\,bty$, for the whole area of the section.

$$\therefore\ M = \Sigma p\,.\,bty.$$

Since $p = Ey/R$, substitute, and remember that E/R is the same for all strips, then

$$M = \frac{E}{R}\Sigma b\,.\,t\,.\,y^2.$$

In this formula the area of each strip has to be multiplied by the square of its distance from the neutral axis and the sum of the products taken. This, or an analogous sum, is of constant occurrence in mechanics, and has a name assigned to it. Σbty is the simple moment of an area about an axis. Σbty^2 may be called the moment of the second degree, but the common name is the *Moment of Inertia*: because a similar sum (differing only from this in involving the mass) occurs in dynamics under that name. To distinguish the two cases area-moment and mass-moment, the former is sometimes called the geometrical moment of inertia.

Let I denote the moment of inertia, so that $I = \Sigma bty^2$, the value of which for any form of section can be obtained by geometry, then

$$M = \frac{E}{R}I, \text{ or } \frac{M}{I} = \frac{E}{R},$$

thus connecting the curvature of the beam with the moment producing it. Having previously found $p/y = E/R$, we can now connect the moment with the stress by writing

$$\frac{p}{y} = \frac{M}{I}.$$

This equation may be employed to determine the strength of a beam to resist bending. The limit of strength is reached when either the greatest safe tensile stress on one side of the neutral axis, or the greatest safe compressive stress on the other side of the neutral axis is called into action. Thus in the equation $p/y = M/I$ we must put $p = f_1$, the co-efficient of strength under tension, or $p = f_2$, the co-efficient of strength under compression; and for y, either y_1, the distance of the most remote point on the stretched side, or y_2, the distance of the most remote point on the compressed side, so that

$$M=\frac{f_1}{y_1}I, \text{ or } \frac{f_2}{y_2}I.$$

The strength of the beam, or maximum moment of resistance to bending, is measured by the least of these quantities.

y_1 or y_2 is readily determined from geometry, the form of the section of the beam being given. It may be most conveniently expressed as a fraction of the depth of the beam. Thus y_1 or y_2 may be put $=qh$, where the co-efficient q has different values. In a rectangular section $q=\frac{1}{2}$, in a triangular section $q=\frac{1}{3}$ or $\frac{2}{3}$, and so on.

Next to express the value of I. It will be found that whatever be the form of the section, I may always be written $=nAh^2$, A being the area of the section of the beam, h the depth in the direction of bending, and n a numerical co-efficient, the value of which depends on the form of the section.

For a rectangular section,

$$n=\tfrac{1}{12} \text{ so that } I=\tfrac{1}{12}Ah^2.$$

For an elliptical or circular section,

$$n=\tfrac{1}{16}, \text{ so that } I=\tfrac{1}{16}Ah^2.$$

For a triangular section,

$$n=\tfrac{1}{18}, \text{ so that } I=\tfrac{1}{18}Ah^2,$$

and so on.

Therefore assuming q and n known, we can write

$$M=\frac{f}{qh}nAh^2=f\frac{n}{q}\,.\,Ah,$$

a formula which shows that for sections in which n/q is the same, the moment of resistance to bending is proportional to the product of the area and depth of the beam. Sections with the same n and q are said to be of the *same type*. They are often, but not correctly, said to be *similar*.

In estimating the numerical value of M, care must be taken with the units. It is generally advisable to use the inch unit throughout.

156. *Remarks on Theory of Bending.*—In the foregoing theory of simple bending it is supposed

(1) That the bar is homogeneous and of uniform transverse section and perfectly elastic;

(2) That sections plane before bending are plane after bending, for which it is theoretically necessary that the bending moment should be uniform, and applied at the ends of the bar in a particular way;

(3) That longitudinal layers of the beam expand and contract laterally in the same way, as if they were disconnected from each other.

These assumptions are not obvious *a priori*, and require justification, which at the present stage of the subject we are not in a position to give; for the present it may be stated that if the material be homogeneous and perfectly elastic, the equations hold good with certain qualifications to be considered hereafter (Chap. XVII.), even though the transverse sections and the curvature vary and however the bending moment is applied. The *strength* of the material, however, is not generally the same, as if the layers were disconnected, and co-efficients of strength require therefore to be determined by special experiment on transverse strength (Chap. XVIII.).

157. *Calculation of Moments of Inertia.*—We have frequently to deal with beams of complex section, in which case to determine I it is convenient to divide the section up into simple areas, the I of each of which is known, and the total moment of inertia of the section will be the sum of these I's. In employing this process we require to know the relation between the moments of inertia of an area about two axes parallel to one another, one being the neutral axis. We make use of a general theorem which may be thus proved.

Let A be an area of which we know the moment of inertia about the neutral axis, SS (Fig. 123), and we require to know the moment of inertia about any parallel axis, XX, distant y_0 from SS. Dividing the area into strips of breadth b, and thickness t,

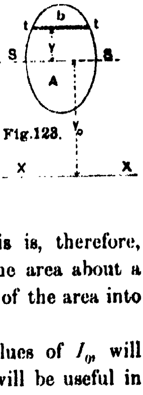


Fig. 123.

Moment of Inertia required $I = \Sigma b \,.\, t \,.\, (y + y_0)^2$

$$= \Sigma bty^2 + 2y_0 \Sigma bt \,.\, y + y_0^2 \Sigma b \,.\, t.$$

Now Σbty^2 = moment of inertia about neutral axis, $\Sigma bt \,.\, y = 0$, because the neutral axis passes through the centre of gravity of the section, and Σbt = Area A;

$$\therefore \; I = I_0 + Ay_0^2.$$

The moment of inertia of an area about any axis is, therefore, determined by adding to the moment of inertia of the area about a parallel axis through the centre of gravity the product of the area into the square of the distance between the two axes.

This theorem, together with previously quoted values of I_0, will enable us to determine the following results, which will be useful in application to beams—

Rectangle of height y about its base, ... $I = \frac{1}{3}Ay^2$.

Triangle „ „ ... ... $I = \frac{1}{6}Ay^2$.

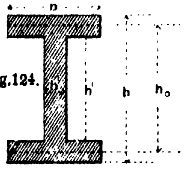
Triangle about a parallel to its base through vertex, $I = \frac{1}{2}Ay^2$.

Many other forms will divide up into rectangles or triangles, or both; for example, the moment of inertia of a trapezoid about the neutral axis may be readily determined by taking, for the area above the neutral axis, the I for a rectangle about one end, and triangles about the base. For the area below, a rectangle about one end and triangles about the vertex, and add the results.

158. *Beams of I Section with equal Flanges.*—The case of a beam of I section is very important.

First, suppose the flanges of equal breadth and thickness, and the web of uniform thickness b', the depth being h', b being the breadth of the flange, and h the whole depth of the beam. The moment of inertia of the section may be taken as the difference of the moments of inertia of two rectangles (see Fig. 124).

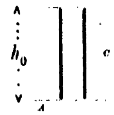
Fig. 124.



$$I = b\tfrac{1}{12}h^3 - \tfrac{1}{12}(b - b')h'^3.$$

This is the accurate value of I, and when the flanges are thick this expression for I must be used; but if the flanges are thin compared with the depth, an approximation can be obtained by supposing each flange to be concentrated in its centre line, and taking for the depth of the beam the distance h_0 to the centre of flanges.

If A = area of each flange and C = area of web,



$$\text{then } I = A\frac{h_0^2}{4} + A\frac{h_0^2}{4} + \tfrac{1}{12}Ch_0^2 = \frac{h_0^2}{2}\left(A + \frac{C}{6}\right).$$

Putting $p = f$ and $y = \tfrac{1}{2}h_0$, in the formula $\dfrac{p}{y} = \dfrac{M}{I}$,

$$M = \frac{f}{\tfrac{1}{2}h_0}\,\frac{h_0^2}{2}\left(A + \frac{C}{6}\right) = fh_0\left(A + \frac{C}{6}\right).$$

Since the total area of the flanges is $2A$ it appears that, area for area, the web has only one-third the resistance to bending of the flanges. The result given by this formula is too large, the excess being greater the thicker the flanges, partly because a part of the web is reckoned twice over, and partly for the reason mentioned below.

We previously deduced an approximate expression for the strength of an I beam, viz.,

$$M = Hh = fhA \text{ (see Art. 27)},$$

in which the effect of the web in resisting bending was neglected, the whole of the bending action being supposed to be taken by the flanges. The present formula shows the amount of the error involved in that assumption. In using this approximation when h the effective depth is reckoned from centre to centre of the flanges, two errors are made, one in supposing the resistance to bending of the web neglected, and the other, in supposing the mean stress on the flange equal to the maximum. When the web is very thin the first of these errors is the least, and the effective depth is more nearly h_0^2/h', where h' is the outside depth and h_0 the depth from centre to centre of flanges. This is little greater than the *inside* depth. On the other hand, in beams rolled in one piece the web is thick. The first error is then the greater, and Prof. Philbrick has pointed out that the approximation gives fairly accurate results if h be taken as the *outside* depth.* Such approximate rules are useful in rough preliminary calculations of dimensions, but always require verification.

159. *Ratio of Depth to Span in I Beams.*—The formula just obtained for the moment of resistance of a beam of I section shows that the greater the depth of the beam and the thinner the web the stronger will the beam be for the same weight of material, or in other words that the best distribution of material is as far away from the neutral axis as possible. The practical limitation to this is that a certain thickness of web is necessary to hold the flanges together and give sufficient power of resistance to lateral forces and to the direct action of any part of the load which may rest on the upper flange. Hence the weight of web rapidly increases as the depth increases, and a certain ratio of depth to span is best as regards economy of material (see Ex. 17, page 319). This is especially important in large girders in which economy of material is the primary consideration. In smaller beams the proper ratio of depth to span is generally in great measure a question of stiffness, a part of the subject to be considered in Chapter XIII. The moment of resistance of I sections of practical proportions is generally nearly double that of a rectangular section of equal area and mean depth. The straining actions on the web will be considered in Chapter XV.

160. *Proportions of I Beams for Equal Strength.*—Materials in general are not equally strong under tension and compression, so that a beam whose section is symmetrical above and below the neutral axis will yield on one side before the material on the other side of the neutral axis has reached its limiting stress. Accordingly we might obtain a more economical distribution of material if we were to take some from the stronger side and put it on the weaker, so that the limiting tensile on one side and the limiting compressive stress on the other may be produced simultaneously. The section of the beam will be different above and below the neutral axis, which will not now be at the centre of depth of the beam, but in such a position that the distance to the top and bottom of the beam are in the proportion of the greatest allowed stresses to one another. The neutral axis in all cases must pass through the centre of gravity of the section.

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* *Van Nostrand's Magazine.* Nov., 1886.

Let f_A, f_B be the co-efficients of strength under compression and tension respectively, y_A, y_B distances of the most strained layer from the neutral axis, then the beam will be strongest when

$$\frac{y_A}{f_A}=\frac{y_B}{f_B}=\frac{y_A+y_B}{f_A+f_B}=\frac{h}{f_A+f_B}.$$

For simplicity of calculation we will consider a beam (Fig. 125) in which the web is of uniform thickness throughout the depth, and so of rectangular section, and each flange also of rectangular section, and determine the relation which should hold between the areas of flanges and web for maximum strength of beam, and the moment of resistance to bending where this condition is satisfied. We will further suppose each flange to be concentrated in its centre line.

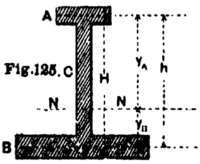
Fig. 125.

Let A = area of compressed flange, B = area of stretched flange, C = area of web. Since the neutral axis is at the centre of gravity of the section, we obtain, by taking moments about the axis,

$$A\,.\,y_A+C\frac{y_A-y_B}{2}=By_B;$$

or, substituting the previously given values of y_A and y_B,

$$Af_A+C\frac{f_A-f_B}{2}=Bf_B.$$

Supposing f_A and f_B known, A, B, and C must be such as to satisfy this relation. We have some liberty of choice between these quantities and frequently find one of the flanges omitted, so producing a beam of T or ⊥ section.

In a cast-iron beam, where the resistance to compression is greater than for tension, the compressed flange A may be omitted. Putting $A=0$ we get $C=\frac{f_A-f_B}{2f_B}B$, and supposing $\frac{f_A}{f_B}=4$; $C=\frac{2}{3}B$, or $B=1\frac{1}{2}C$. In a wrought-iron beam on the other hand, if we take f_A/f_B to be $\frac{3}{5}$, the stretched flange B is to be omitted. Putting $B=0$, we find

$$A=\frac{f_B-f_A}{2f_A}C=\frac{1}{3}C.$$

Otherwise we may assume the depth and thickness of the web to be given (Art. 159), then the equation

$$Af_A + C\cdot\frac{f_A - f_B}{2} = Bf_B$$

furnishes a relation between the areas of the flanges. For example, in cast iron, if we assume $f_A = 4f_B$, we find

$$B = 4A + \tfrac{3}{2}C.$$

Having decided on the proportions between the parts of the section we can now calculate the moments of inertia and resistance. Still considering the flanges concentrated in their centre lines,

$$I = Ay_A^2 + By_B^2 + \tfrac{1}{3}C\cdot\frac{y_A}{h}\cdot y_A^2 + \tfrac{1}{3}C\cdot\frac{y_B}{h}\cdot y_B^2$$

$$= Ay_A^2 + By_B^2 + \tfrac{1}{3}C\cdot\frac{y_A^3 + y_B^3}{h},$$

a result which admits of ready calculation. Further

$$\frac{M}{I} = \frac{f_A}{y_A} = \frac{f_B}{y_B} = \frac{f_A + f_B}{h},$$

whence we obtain $M = (f_A + f_B)\frac{I}{h}$.

The calculation just now made is one which has been frequently given in dealing with beams of I section,* but in applying it to actual examples it should be remembered that the results are obtained on the supposition that the flanges are concentrated in their centre lines, and are consequently only approximate when the co-efficients f_A, f_B mean the intensities of the stress at those centre lines, *not* at the surface of the beam where the stress is greatest. If, for example, F_A be the maximum stress on the flange A,

$$F_A = f_A\cdot\frac{y_A + \frac{1}{2}t_A}{y_A},$$

where t_A is the thickness of the flange. The difference is especially great in the case of the larger flange of cast-iron beams, and the true ratio of maximum compressive and tensile stress is much less than it appears in the preceding article. On the other hand, in extreme cases, such as we are now considering, the stress may not be uniformly distributed along a line parallel to the neutral axis.

Extensive experiments were made on cast-iron beams by Hodgkinson with the object of determining the best proportions between the flanges, with the result that rupture always took place by tearing asunder of the lower flange, unless it was at least six times the size of the compressed flange. This proportion is rarely adopted in practice, from the difficulties of obtaining a sound casting, and the necessity of having sufficient lateral strength. Nor is it certain that the proportions which are best for resisting the ultimate load are also best in the case of the working load; it is, in fact, probable that a smaller proportion is better even on the score of strength. If we take $f_A = 2\frac{1}{2}f_B$, instead of $4f_B$ we find

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* See Rankine's *Civil Engineering*, page 257.

$$B = 2\tfrac{1}{2}A + \tfrac{3}{4}C,$$

which agrees more closely with practice. The ratio of maximum compressive and tensile strength is in this case about 2, which, according to some authorities, is the ratio of *elastic* strengths in the two cases.

In wrought-iron beams the areas of the flanges are usually equal, and this is correct if the elastic strength, and not the ultimate strength, is regarded as fixing the proper proportions, and if there be sufficient provision against the yielding of the top flange by lateral flexure. Small-sized beams of this kind are rolled in one piece, while large girders are constructed of iron or steel plates and angle irons, rivetted together. Some of the forms they assume are shown in Plate VIII., Chapter XVIII.

In making calculations respecting girders, approximate methods may be used for preliminary tentative calculations, but should be checked by a subsequent accurate determination of the neutral axis and moment of inertia. A previous reduction of the section to an equivalent solid section is required when, as is often the case, all parts of the section do not offer the same elastic resistance to the stress applied to them, either because they are not sufficiently rigidly connected or from the material being different. This is especially the case in determining the resistance to the longitudinal bending of a vessel occasioned by the unequal distribution of weight and buoyancy already considered in Chapter III. On this important question the reader is referred to a treatise on Naval Architecture by Mr. (now Sir) W. H. White. In many cases of built-up girders the shearing action which generally exists has considerable influence, a matter for subsequent consideration (Ch. XV.). The effect of the weight of the girder itself has been considered in Chapter IV. (See also Ex. 13, p. 319, and Art. 192.)

161. *Beams of Uniform Strength.*—A beam of uniform strength is one in which the maximum stress is the same on all sections. For beams of the same transverse section throughout, this can only be the case when the bending moment is uniform, but, by properly varying the section, it is possible to satisfy the condition however the bending moment vary. For this purpose we have only to consider the equation

$$M = f \cdot \frac{n}{q} \cdot Ah,$$

which must now be satisfied at all sections. Suppose

$$A = kbh,$$

where k is a numerical factor depending on the type of section, then

$$M = f . \frac{nk}{y} . bh^2.$$

All sections of the beam being supposed of the same type we have only to make Ah or bh^2 vary as M, that is, as the ordinates of the curve of bending moments. The principal cases are—

(1) Depth uniform. Here the breadth must vary as the bending moment, whence it is clear that the curve of moments may be taken as representing the half plan of the beam.

(2) Sectional area uniform. Here the depth must vary as the bending moment, that is, the curve of moments may be taken to represent the elevation or half elevation of the beam.

(3) Breadth uniform. Here the elevation or half elevation of the beam must be a curve, the co-ordinates of which are the square roots of the co-ordinates of the curve of moments.

(4) Ratio of breadth to depth constant. Here the half plan and half elevation are each a curve, the ordinates of which are the cube roots of the ordinates of the curve of moments.

The first, third, and fourth of these cases are common in practice with some modifications occasioned by the necessity of providing additional material at sections of the beam where the bending moment vanishes, as it usually does at one or both ends.

162. *Unsymmetrical Bending.*—It occasionally happens that the plane of the bending moment is not a principal plane of the beam, as for

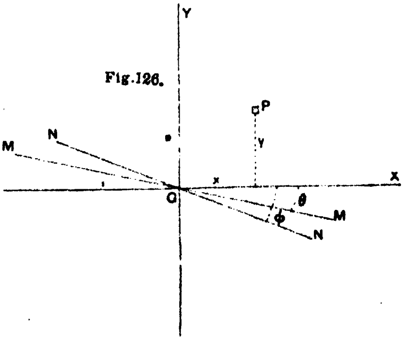
Fig. 126.

example when a vessel heels over, the plane of longitudinal bending will not coincide with the plane of symmetry of the vessel which is obviously the plane of the masts. The neutral axis does not now coincide with the axis of the bending couple, though in other respects the theory of binding still holds good.

In Fig. 126 let MM be the axis of the bending moment M, inclined at an angle θ to the principal axes of inertia GX, GY of the plane section. Then the couple M may be resolved into two components $M \cos\theta$ and $M \sin\theta$, each of which will produce stress at any point P as if the other did not exist. Let p be the stress, x, y the co-ordinates of P referred to the axes GX, GY, the moments of inertia about which are I_1, I_2, then

$$p = \frac{M . \cos\theta . y}{I_1} + \frac{M . \sin\theta . x}{I_2}.$$

The position of the neutral axis NN is found by putting $p = 0$, then the angle ϕ which it makes with GX is given by

$$\tan\phi = -\frac{y}{x} = \frac{I_1}{I_2} . \tan\theta.$$

This equation shows that the neutral axis is parallel to a line joining the centres of the circles into which the beam would be bent by the component couples supposed each to act alone.

The neutral axis being thus determined and laid down on the diagram the points can be found which lie at the greatest distance from that axis. At these points the stress will be greatest, and if X, Y be their co-ordinates, still referred to the axes GX, GY, the moment of resistance will be determined by the equation

$$f = M\left\{\frac{Y . \cos\theta}{I_1} + \frac{X . \sin\theta}{I_2}\right\}.$$

For a different method of expressing the moment of resistance see Rankine's *Applied Mechanics*, p. 314.

EXAMPLES.

1. A bar of iron 2″ diameter is bent into the arc of a circle 372′ diameter. Find in tons per square inch, 1st, the greatest stress at any point of the transverse section; 2nd, the stress on a line parallel to the neutral axis half an inch from the centre, E being taken = 29,000,000. *Ans.* Maximum stress = 5·8. Stress at ½″ from centre = 2·9.

2. Find the diameter of the smallest circle into which the bar of the last question can be bent; the stress being limited to 4 tons per square inch. *Ans.* Diameter = 540 feet.

3. Find the position of the neutral axis of a trapezoidal section; the top side being 3″, bottom 6″, and depth 8″. Also find the ratio of maximum tensile and compressive stresses. *Ans.* Neutral axis 3·56 inches from bottom. Ratio of stresses 5 to 4.

4. A cast-iron beam is of I section with top flange 3″ broad and 1″ thick and bottom flange 8″ broad and 2″ thick; the web is trapezoidal in section ½″ thick at top and 1″ at bottom; total outside depth of beam 16″. Find the position of the neutral axis and the ratio of maximum tensile and compressive stresses. *Ans.* Neutral axis 4·81 inches from bottom. Ratio of stresses 3 to 7.

5. A wrought-iron beam of rectangular section is 9″ deep, 3″ broad, and 10 feet long.

Find how much it will carry loaded in the centre, allowing a co-efficient of 3 tons per square inch. Also deduce the load the same beam will bear when set flatways. *Ans.* When upright load = 4·05 tons. When set flatways load = 1·35 tons.

6. A piece of oak of uniform circular section is 16″ diameter and 12 feet long. It is supported at the two ends and loaded at a point 5 feet from one end. How great may the load be, allowing a stress of ½ ton per square inch? *Ans.* Load may be 5·74 tons.

7. In Example 5 suppose half the weight of metal formed into a beam of I section, of the same depth, each flange being equal to the web: what load will the beam carry? *Ans.* Load may then be 4⅔ tons.

8. Find the moment of resistance to bending of the section given in Example 4, the co-efficient for tension being 1 ton per square inch. *Ans.* I = 798 inch units. Moment of resistance to bending = 166·4 inch-tons.

9. Suppose the skin and plate deck of an iron vessel to have the following dimensions at the midship section, measured at the middle of the thickness of the plates. Find the position of the neutral axis and moment of resistance to bending. Breadth 48′ and total depth 24′, the bilges being quadrants of 12′ radius. Thickness of plate ⅝″ all round and co-efficient of strength 4 tons in compression.

Ans. Neutral axis 13″ above centre of depth.

Moment of resistance to hogging = 32,500 ft.-tons, and to sagging 39,000.

10. What should be the sectional area of a T beam of wrought iron to carry 4 tons uniformly distributed? Span 20′, depth of beam 10″. Co-efficient for compression 3 tons, and for tension 5 tons. *Ans.* Area = 13·7 square inches.

11. If, in the last question, the flange is made equal to the web instead of being proportioned for equal strength, show that to carry the same load the beam must be about one quarter heavier.

12. In Example 8 find the moments of inertia and resistance on the supposition that the flanges are concentrated at the centre lines, and thus by comparison with previous results show the amount of the error involved in the assumption. *Ans.* Moment of inertia = 861·5 inch units. Moment of resistance = 227 inch-tons.

13. Show that the limiting span (Art. 41) of a beam of uniform transverse section is

$$L = \lambda \cdot \frac{8n}{Nq},$$

where N is the ratio of span to depth, and the rest of the notation is the same as on pages 81 and 310. Obtain the numerical result for a wrought iron beam of rectangular section, taking λ from Table I., Ch. XVIII., and supposing $N = 12$.

Ans. $L = 336$ ft.; in an ordinary I section the result would be doubled. For the case of large girders see Art. 192.

14. If l be the length of an iron rod in feet, d its diameter in inches, just to carry its own weight when supported at the ends, show that when the stress allowed is 4 tons per square inch $l = \sqrt{224d}$.

15. If I_1, I_2 be the moments of inertia of two plane areas, A_1, A_2, about their neutral axes which are supported parallel at distance apart z, show that the moment of inertia of their sum or difference about their common neutral axis is $I = I_1 \pm I_2 + z^2 \cdot \frac{A_1 A_2}{A_1 \pm A_2}$. Apply this formula to the trapezoidal section of Question 3. *Ans.* $I = 185$ inch units nearly.

16. Find the moment of resistance to bending of a beam of I section, each flange consisting of a pair of angle irons 3½″ × ½″ rivetted to a web ·37″ thick and 16″ deep between them. Assuming it 24 feet span, find the load it would carry in the middle, using a co-efficient of 3 tons per square inch. *Ans.* $M = 288$ inch-tons. $W = 4$ tons.

17. If it be assumed that for constructive reasons the thickness of web of an I beam with equal flanges must be a given fraction of the depth, show that for greatest economy of material the sectional area of the web should be equal to the joint sectional area of the flanges. Prove that in this case $M = \frac{1}{3}f \cdot Sh$.

18. In a cast-iron beam of I section of equal strength for which $f_A=2\frac{1}{2}f_B$; if it be assumed that for constructive reasons the thickness of the web should be a given fraction of the depth, show that for greatest economy of material the large flange, the web, and the small flange should be in the proportion 25, 20, 4. Prove also that the moment of resistance is given by the same formula as in Question 17, supposing $2/f=1/f_A+1/f_B$.

19. A beam of rectangular section of breadth one-half the depth is bent by a couple the plane of which is inclined at 45° to the axes of the section. Find the neutral axis, and compare the moment of resistance to bending with that about either axis. *Ans.* Ratio $=2\sqrt{2}/3$ and $\sqrt{2}/3$.

20. If a beam be originally curved in the form of a circular arc of radius R_0 instead of being straight, show that the neutral axis does not pass through the centre of gravity of the section. In a rectangular section of depth h show that the deviation is, approximately,

$$z=\frac{h^2}{12R_0}.$$

21. In the preceding question, if R_0 is large show that the equations of bending are

$$\frac{p}{y}=E\left(\frac{1}{R_0}-\frac{1}{R}\right)=\frac{M}{I}.$$

CHAPTER XIII.

DEFLECTION AND SLOPE OF BEAMS.

163. *Deflection due to the Maximum Bending Moment.*—It is not only necessary that a beam should be strong enough to support the load to which it is subjected, it is also necessary that its changes of form should not be too great, or in other words, that it should be sufficiently stiff, and we next proceed to determine under what conditions this will be the case.

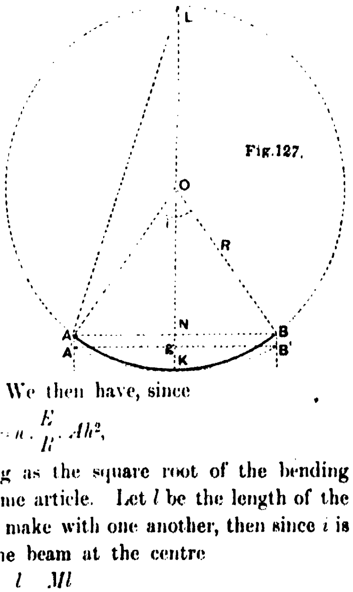

Fig. 127.

The question is simplest when the beam is bent into an arc of a circle; we have then

$$\frac{p}{y} = \frac{M}{I} = \frac{E}{R} = \text{constant}.$$

Two cases may be especially mentioned—

(1) Depth uniform. We then have p constant, that the beam is of uniform strength. (See Case 1 of Art. 161.)

(2) Sectional area uniform. We then have, since

$$M = \frac{E}{R} I = n \cdot \frac{E}{R} \cdot Ah^2,$$

the depth of the beam varying as the square root of the bending moment, as in case 3 of the same article. Let l be the length of the beam, i the angle its two ends make with one another, then since i is also the angle subtended by the beam at the centre

$$i = \frac{l}{R} = \frac{Ml}{EI}.$$

If the beam be supported at the ends i is twice the angle which the ends make with the horizontal, an angle called the Slope at the ends. Let AB be the beam (Fig. 127), O the centre of the circle into which it is bent KL, the diameter of the circle through K the middle point of the beam. Then KN is the deflection which is given by a known proposition of Euclid,

$$KN.NL = AN^2.$$

Hence remembering that the diameter of the circle is very large* we have, if δ be the deflection,

$$\delta = \frac{l^2}{8R} = \frac{Ml^2}{8EI}.$$

This formula gives the deflection in any case where the curvature is uniform.

When the transverse section is uniform the curvature varies. Unless the bending moment be likewise uniform, the deflection curve is not then a circle AKB, but for the same maximum bending moment a flatter curve $A'KB'$. Thus the deflection is less than that calculated by the above formula, which may be described as the "deflection due to the maximum moment." The actual deflection may conveniently be expressed as a fraction of that due to the maximum moment. It is possible to construct the deflection curve graphically by observing that the curvature at every point is proportional to the bending moment. We have then only to strike a succession of arcs with radii inversely proportional to the ordinates of the curve of bending moment. It is however more convenient to proceed by an analytical method.† The fraction is least when the beam is least curved, which is evidently the case when it is loaded in the middle, and we shall show presently that it is then two-thirds, while, when uniformly loaded, it is five-sixths.

164. *General Equation of Deflection Curve.*—It was shown above that

$$i = \frac{M}{EI}.l.$$

If the bending moment vary, then we must replace l by an element of the length ds and i by the corresponding element of the angle; we shall then have an equation

$$\frac{di}{ds} = \frac{M}{EI},$$

which by integration will furnish i. It will generally be convenient to reckon i from a horizontal tangent and it then means the slope of the beam at the point considered. To perform the integration it is in most cases necessary to suppose the slope of the beam small, as it actually is in most important cases in practice, and we may then replace ds the element of arc by dx, the corresponding element of a horizontal tangent AN (Fig. 128) taken as axis of x, whence

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* For clearness it is made small in the figure.

† Readers who have no knowledge of the Calculus may pass over the next four articles.

$$\frac{di}{dx}=\frac{M}{EI}\text{ approximately,}$$

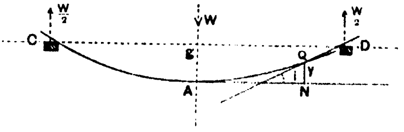
an equation which can generally be integrated because M is usually a function of x.

The deviation y of any point Q of the beam from the straight line AN can now be found since $dy/dx=i$, from which we further obtain the fundamental equation

$$\frac{d^2y}{dx^2}=\frac{M}{EI},$$

which applies to all cases where the bending of the beam is occasioned by a transverse load. We shall first give some elementary examples of the determination of the deflection and slope of a beam and then consider the question more generally.

Fig. 128.



165. *Elementary Cases of Deflection and Slope.—Case I.* Suppose a beam supported at the ends and loaded in the middle.

In Fig. 128 CD is the beam resting on supports at C, D, and loaded in the middle with a weight W. Take the centre A as origin and the horizontal tangent at A as axis of x, then if l be the whole length

$$\frac{d^2y}{dx^2}=\frac{M}{EI}=\frac{\frac{W}{2}\left(\frac{l}{2}-x\right)}{EI}.$$

$$\therefore\ i=\frac{dy}{dx}=\frac{\frac{W}{2}\left(\frac{l}{2}x-\frac{1}{2}x^2\right)}{EI}$$

is the slope of the beam at Q, no constant being required since i is zero when $x=0$.

If $x=l/2$ we get the slope at the ends of the beam

$$i_1=\frac{Wl^2}{16EI}.$$

Integrating a second time

$$y=\frac{\frac{W}{2}\left(\frac{1}{4}lx^2-\frac{1}{6}x^3\right)}{EI}.$$

As before no constant is required because $y=0$ when $x=0$.

If now we put $x=l/2$ we get the elevation of D above AN or, what is the same thing, the depression Ag of A below the level of the supports. This is called the Deflection of the beam; if we denote it by δ,

$$\delta=\frac{\frac{W}{2}\left(\frac{1}{16}l^3-\frac{1}{48}l^3\right)}{EI}=\frac{Wl^3}{48EI},$$

a result which we may also write

$$\delta=\frac{2}{3}\cdot\frac{M_0l^2}{8EI}=\frac{2}{3}\cdot\delta_0,$$

where M_0 is the maximum moment and δ_0 the deflection due to it.

Case II. Let the beam be supported at the ends and loaded uniformly with w pounds per foot-run. It will be sufficient to give the results, which are obtained in precisely the same way, remembering that the bending moment is now $\frac{1}{2}w(a^2-x^2)$ where a is the half span. We have

$$i_1=\frac{wa^3}{3EI}=\frac{Wl^2}{24EI};\quad \delta=\frac{5}{24}\cdot\frac{wa^4}{EI}=\frac{5}{384}\cdot\frac{Wl^3}{EI}.$$

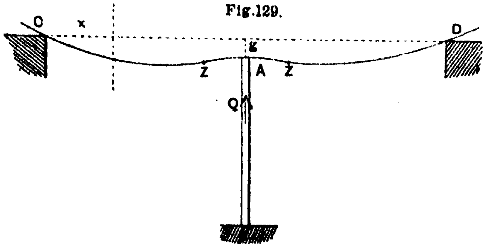
The value of δ may be expressed as in the previous case in terms of the deflection due to the maximum moment. We have $\delta=\frac{5}{6}\cdot\delta_0$.

166. *Beam propped in the middle.*—When a beam is acted on by several loads the deflection and slope due to the whole is the sum of those due to each load taken separately. An important example is

Case III. Beam supported at the ends and propped in the middle, uniformly loaded. (Fig. 129.)

Here the deflection of the beam is the difference between the downward deflection due to the uniform load and the upward deflection due to the thrust Q of the prop. Hence we write down at once for the deflection at the centre,

Fig.129.



$$\delta=\frac{5}{384}\cdot\frac{Wl^3}{EI}-\frac{Ql^3}{48EI},$$

an equation which may be used to determine the load carried by the prop when its length is given, and conversely.

First suppose the centre of the beam propped at the same level as the supports, then $\delta = 0$, and

$$Q = \frac{5 \times 48 W}{384} = \tfrac{5}{8} W,$$

so that the prop in this case carries five-eighths of the weight of the beam, the supports C, D only carrying three-eighths. Each supporting force is $\frac{3}{16}wl$, l being as before the whole length of the beam; hence the bending moment at a point distant x from C is given by the formula

$$M = \tfrac{3}{16} wlx - \tfrac{1}{2} wx^2 = \tfrac{1}{2} wx(\tfrac{3}{8} l - x),$$

from which it appears that the beam is bent downwards until a point Z is reached, such that

$$CZ = \tfrac{3}{8} l = \tfrac{3}{4} AC.$$

Here the bending moment is zero, that is Z is a "point of contrary flexure" or "virtual joint." (Compare Art. 38.)

Beyond Z the beam is bent upwards, and at the centre A we get, by putting $x = \frac{1}{2}l$,

$$-M_0 = \tfrac{1}{32} wl^2.$$

The case here discussed is also that of a beam, one end of which is fixed horizontally and the other supported at exactly the same level.

Let us next inquire what will be the effect of supposing the centre of the beam propped somewhat out of the horizontal line through the supports at the ends. Let us suppose δ to be $1/n^{\text{th}}$ the deflection of the beam when the prop is removed, then

$$\frac{1}{n} \cdot \frac{5}{384} \cdot \frac{Wl^3}{EI} = \frac{5}{384} \cdot \frac{Wl^3}{EI} - \frac{Ql^3}{48EI},$$

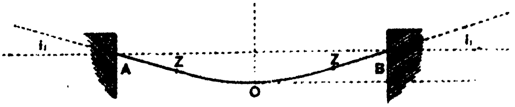
that is

$$Q = \tfrac{5}{8} W \left(1 - \frac{1}{n}\right),$$

a formula which gives the load on the prop. If, for example, $n = 5$, $Q = \frac{1}{2} W$, or if $n = -5$, $Q = \frac{3}{4} W$; thus if the centre of the beam be out of level, by as much as one-fifth the deflection when the prop is wholly removed, the load on the prop will vary between $\frac{1}{2} W$ and $\frac{3}{4} W$, a result which shows the care necessary in adjustment to obtain a definite result.

167. *Beams fixed at the Ends—Case IV.* Uniformly loaded beam, with ends fixed at a given slope.

Fig. 130.



In Fig. 130 AB is a uniformly loaded beam, with the ends A, B fixed not horizontally but for greater generality at a slope i. Here the central part of the beam will be bent downwards and the end parts upwards; at Z, Z there will be virtual joints; let $OZ = r$, then taking O as origin the bending moment at any point between O and Z is

$$M = \tfrac{1}{2}w(r^2 - x^2),$$

a formula which will also hold for points beyond Z, as can be seen from Art. 38, or proved independently. We have then

$$\frac{d^2y}{dx^2} = \frac{\frac{1}{2}w(r^2 - x^2)}{EI};$$

$$i = \frac{\frac{1}{2}w(r^2x - \frac{1}{3}x^3)}{EI}.$$

No constant is required, because i is zero at O. Let a be the half span OA, or OB, then putting $x = a$, we get for the slope at the ends

$$i_1 = \frac{\frac{1}{2}w(r^2a - \frac{1}{3}a^3)}{EI},$$

a formula from which r can be determined if i_1 be given. If $r = a$, we get the case where the ends are free; let the slope then be i_0, we have

$$i_0 = \frac{wa^3}{3EI} \text{ as before (p. 324).}$$

Now, assume the actual slope to be $1/n^{\text{th}}$ of this, we get

$$\frac{1}{n} \cdot \frac{wa^3}{3EI} = \frac{\frac{1}{2}w(r^2a - \frac{1}{3}a^3)}{EI};$$

that is, $$r^2 = \tfrac{1}{3}a^2\left(1 + \frac{2}{n}\right).$$

If the ends are fixed exactly horizontal, then

$$r^2 = \tfrac{1}{3}a^2,$$

and by substitution we find for the bending moment at the centre and the ends

$$M_0 = \tfrac{1}{6}wa^2; \quad M_A = M_B = \tfrac{1}{3}wa^2.$$

If the ends were free, the bending moment at the centre would have been $\frac{1}{2}wa^2$, so that the beam will be strengthened in the proportion 3 : 2. The formula obtained above, however, shows that a small error in adjustment of the ends will make a great difference in the results.

It is theoretically possible so to adjust the ends that the bending moments at the centre and the ends shall be equal, in which case the beam will be strongest. For this we have only to put

$$\tfrac{1}{2}wr^2 = \tfrac{1}{2}w(a^2 - r^2),$$

that is, $$r^2 = \tfrac{1}{2}a^2,$$

whence by substitution we get

$$n = 4;$$

that is, the ends should be fixed at one-fourth the slope which they have when free, and the strength of the beam will then be doubled.

By proceeding to a second integration the deflection of the beam can be found. In particular when the ends of the beam are horizontal it can be shown that the deflection is only one-fifth of its value when the ends are free. On the effect of shearing see page 332.

The graphical representation of the bending moments in Cases III., IV., is easily affected, as in Fig. 42, page 77.

168. *Stiffness of a Beam.*—The stiffness of a beam is measured by the ratio of the deflection to the span. In practice, the deflection is limited to 1 or 2 inches per 100 feet of span when under the working load; that is, the ratio in question is $\frac{1}{600}$th to $\frac{1}{1200}$th. It appears from what has been said that if M_0 be the maximum moment the deflection is given by

$$\delta = k \frac{M_0 l^2}{8EI},$$

where k is a fraction, which in beams of uniform section, varies from two-thirds to unity, depending on the way in which the beam is loaded.* Hence the greatest moment which the beam will bear consistently with its being sufficiently stiff is

$$M_0 = \frac{8E\delta}{kl} \cdot \frac{I}{l}.$$

If we express I as usual in terms of the sectional area and depth, we get

$$M_0 = s \frac{n}{k} A \frac{h^2}{l},$$

where s is a co-efficient depending on the material and on the admissible deflection which may be called the "Co-efficient of Stiffness."

We thus obtain a value for the moment of resistance of a beam which depends on its stiffness, not on its strength, and if that value be less than that previously obtained for strength (p. 310), we must evidently employ the new formula in calculating dimensions. On comparing the two, we find that they will give the same result if

$$\frac{sh}{kl} = \frac{f}{q}; \text{ or } \frac{h}{l} = \frac{fk}{qs};$$

that is to say, for a certain definite ratio of depth to span, and if there is no other reason for fixing on this ratio, it will be best to choose the value thus determined. The two formulæ then give the same result. In large girders a greater depth is generally desirable, then the strength formula must be used; while in small beams it may often be convenient or necessary to have a smaller depth, and then the stiffness formula must be employed.

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* When the transverse section is not uniform the co-efficient k may be greater than unity.

169. *General Graphical Method.*—The foregoing simple examples of the determination of the deflection and slope of a beam are perhaps those of most practical use, but, by the aid of graphical processes, there is no difficulty in generalizing the results which are of considerable theoretical interest. We can, however, afford space only for a hasty sketch.

The general equations given in Art. 164 show that the angle (i) between two tangents to the deflection curve of a beam is proportional to the area of the curve of bending moments intercepted between two ordinates at the points considered. Starting from the lowest point of the deflection curve, let us now imagine a curve drawn, the ordinate of which represents that area reckoned from the starting point, then that curve will represent the slope of the beam at every point, and may therefore properly be called the "Curve of Slope." But referring again to the general equations we see that the ordinate of the deflection curve reckoned upwards from the horizontal tangent at the lowest point, is connected with the slope in the same way as the slope with the bending moment, and is consequently proportional to the area of the curve of slope. Thus it appears, on reference to Chapter III., that the curves of Deflection, Slope, and Bending Moment are related to each other in the same way as the curves of Bending Moment, Shearing Force, and Load. The five curves, in fact, form a continuous series each derived from the next succeeding by a process of graphical integration.

We now see that any property connecting together the second three quantities must also be true for the first three. For example, we know, from the properties of the funicular polygon, that two tangents in the curve of moments intersect in a point vertically below the centre of gravity of the area of the corresponding curve of loads. (See Arts. 31, 35.) It must therefore be true that two tangents to the deflection curve intersect vertically below the centre of gravity of the corresponding area of the curve of moments, a useful property, which can be proved directly without much difficulty.

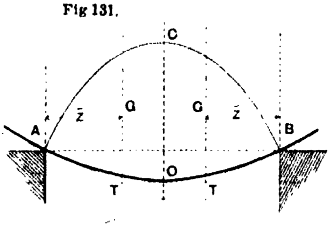
The deflection curve of a beam may therefore be constructed in the same way that the funicular polygon is constructed in Art. 35, the perpendicular distance (H) of the pole from the load line in the diagram of forces being made equal to EI. To do this we have only to divide the moment curve into convenient vertical strips and regard each as representing a weight. Set down these ideal weights as a vertical line and choose a pole at a distance from the line equal to EI, measured (on account of the largeness of E) on a scale less in a given ratio. Now, construct the polygon and draw its closing line, the intercept multiplied by the scale ratio is the deflection of the beam. A parallel to the closing line in the diagram of forces gives the slopes at the extremities of the beam which correspond to the supporting forces of the loaded beam in the original case.

We have hitherto supposed the beam to be of uniform stiffness throughout; if not, let the quantity EI, which is now variable, be E_0I_0, at some datum section. Reduce the ordinates of the curve of moments in the proportion E_0I_0 to EI, then the reduced curve is to be employed in the way just described for the original curve.

170. *Examples of Graphical Method. Theorem of Three Moments.*—Let us now take some examples.

Case I. Symmetrically loaded beam, of flexibility also symmetrical about the centre. Let ACB (Fig. 131) be the curve of moments, reduced if necessary, AOB the deflection curve; both curves, of course, will be symmetrical about the centre vertical, then from what has been said, tangents at A, B to the deflection curve intersect the tangent at O in points T vertically below the centres of gravity of the two equal areas ACO, BCO. Hence if S be the area of the whole curve of moments, $\bar{z}$ the horizontal distance of either point T from the nearer end,

Fig 131.



$$i_0 = \frac{S}{2EI}; \quad \delta = \bar{z} \cdot i_0 = \frac{S \cdot \bar{z}}{2EI}$$

must be the slope of the ends of the beam and its deflection.

Case II. Beam continuous over several spans loaded in any way. (Fig. 132.) Let ACO', BDO' be the moment curves due to the load on two spans AO', BO' of a beam AOB, continuous over three supports A, O, B, of which the centre O is somewhat below the level of A, B. Being continuous, there will be bending moments at A, O, B, which are represented in the diagram by AE, $O'L$, BF. Joining EL, FL, the actual bending moment at each point of the beam will be represented by the intercept between the line ELF and the curves of moments due to the load and corresponding supporting forces. (See Art. 38.) The curve AOB is the deflection curve, AT, BT are the tangents at A, B and TOT is the tangent at O, intersecting AT, BT in the points T.

Now, let i_A be the angle between the tangents at O and A, then, as before,

$$i_A = \frac{S}{EI},$$

where S is the area of a curve representing the actual bending moment at each point. In the present case S is the difference of two areas,

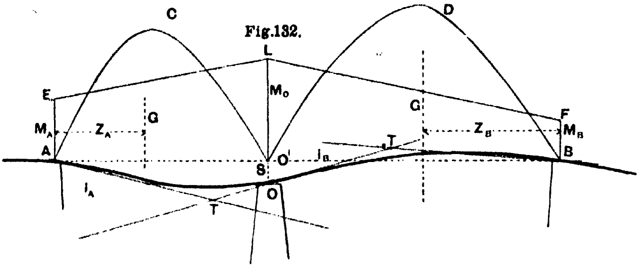


Fig.132.

one the moment curve for the load, the other the trapezoid EO' for the moments M_A, M_0.

$$\therefore\ S = A - \frac{M_A + M_0}{2} \cdot l_A,$$

where A is the area of the moment curve ACO' and l_A is the span AO'. Let the horizontal distance from A of the common centre of gravity of the two curves be x; then, as before, x is also the horizontal distance of T from A and

$$y_A = \frac{Sx}{EI}, \text{ as before.}$$

To find x let z_A be the horizontal distance of the centre of gravity of ACS from A, then

$$Sx = Az_A - M_A l_A \cdot \frac{l_A}{2} - \frac{M_0 - M_A}{2} \cdot l_A \cdot \tfrac{2}{3} l_A;$$

$$= Az_A - \tfrac{1}{6} M_A \cdot l_A^2 - \tfrac{1}{3} M_0 \cdot l_A^2.$$

We have thus found y_A the distance of A from the tangent through O; and y_B the corresponding distance of B, is written down by change of letters.

Assuming now the depression of O, the centre of the beam, below the level of the two other supports to be δ, it appears from the geometry of the diagram that

$$\frac{y_A - \delta}{l_A} = \frac{-y_B + \delta}{l_B};$$

or

$$\frac{y_A}{l_A} + \frac{y_B}{l_B} = +\delta\left(\frac{1}{l_A} + \frac{1}{l_B}\right);$$

hence dividing the values of y_A, y_B, by l_A, l_B respectively, and adding

$$A\frac{z_A}{l_A} + B \cdot \frac{z_B}{l_B} - \tfrac{1}{3} M_0 (l_A + l_B) - \tfrac{1}{6} M_A l_A - \tfrac{1}{6} M_B l_B = \delta\left(\frac{1}{l_A} + \frac{1}{l_B}\right) EI.$$

This equation connects the bending moments at three points of support of a continuous beam, the centre support being below the end supports by the small quantity δ. It can readily be extended to the case where the flexibility of the beam is variable by reducing the moment curves as previously explained, then the moments M, which are the results of the calculation, will, in the first instance, be reduced, and can afterwards be increased to their true values.

The above equation is the most general form of the famous Theorem of Three Moments, originally discovered by Clapeyron, which is much employed in questions relating to continuous beams—a somewhat large subject, on which we have not space to enter. The general method of Art. 169 can, however, be applied directly without using the Theorem of Three Moments. Further information on this point will be found in Mr. R. H. Graham's work on the *Geometry of Position*. (Macmillan, 1891.)

171. *Elastic Energy of a Bent Beam.*—The work done in bending a beam by a uniform bending moment M is evidently $\frac{1}{2}Mi$, where i is the angle which the two ends of the beam make with each other, as in Art. 163; hence by substitution for i we find for the elastic energy U,

$$U=\frac{M^2}{2EI}\cdot l;$$

and if the bending moment vary,

$$U=\int\frac{M^2}{2EI}\cdot dx.$$

An important case is when the beam is of uniform strength, then we have

$$p=\frac{My}{I}=\text{constant}=\frac{M_0y_0}{I_0},$$

where the suffix 0 refers to a datum section. Then

$$U=\frac{M_0^2}{2EI_0}\int\frac{I}{I_0}\cdot\frac{y_0^2}{y^2}\cdot dx.$$

Assuming now the section (A), though varying, to remain of the same type

$$\frac{I}{I_0}=\frac{Ay^2}{A_0y_0^2}.$$

If, therefore, we call V the volume of the beam,

$$U=\frac{M_0^2}{2EI_0}\cdot\frac{V}{A_0}=\frac{p^2}{2E}\cdot\frac{I_0}{A_0y_0^2}V.$$

With the notation of Art. 155 this gives

$$U=\frac{p^2}{E}\cdot\frac{n}{2q^2}\cdot V.$$

For the resilience we have only to change p into f, the proof strength. It thus appears that in beams of uniform strength with *transverse sections of the same type the resilience is proportional to the volume, and less than that of a stretched or compressed bar, as might have been foreseen from general considerations.* The ratio of reduction is $q^2 : n$, being $3 : 1$ in rectangular sections, $4 : 1$ in elliptic sections. When the beam is not of uniform strength the ratio of reduction must be greater for the same type of section. The reduction is of course least in I sections of uniform strength.

The elastic energy U is a function of great importance in the theory of continuous beams and other similar structures, the relative yielding of the several parts of the structure being always such that this function is less than it would be for any other distribution of stress and strain. It may also be called the Elastic Potential, and when known all the equations necessary to determine the distribution of stress may be found by simple differentiation. (See Appendix.)

In the case of a beam supported at the ends and loaded at a given point, the elastic energy may also be expressed in the form

$$U = \tfrac{1}{2} W\delta,$$

where W is the load and δ the deflection of the loaded point. Taking the load in the middle and substituting by the formula on page 324, we find

$$U = \frac{W^2 l^3}{96EI} = \frac{24EI \, . \, \delta^2}{l^3} ;$$

results which we shall have occasion to use hereafter.

172. *Concluding Remarks.*—Throughout this chapter it has been supposed that the deflection and slope of a beam are exclusively due to the bending action of the load, and this supposition is sufficiently accurate when the object is solely to estimate the stiffness of a beam in practical cases. The effect of the shear, which nearly always accompanies bending, will be briefly noticed in a later chapter (Art. 190, Ch. XV.), and it need only here be added that in some of the examples discussed in this chapter, where the results depend on a nice adjustment of the slope of the ends of a beam or the level of the supports on which it rests, the effect of shearing may be very considerable. Structures, the straining actions on which depend on a delicate adjustment should, like frames with redundant parts (Art. 26), be avoided when possible, but when employed the effect of shearing should be carefully examined.

EXAMPLES.

1. If l be the length of an iron rod in feet, d its diameter in inches, just to carry its own weight with a deflection of 1 inch per 100 feet of span, show that

$$l = \sqrt[3]{233d^2}.$$

Compare this result with that of Ex. 14, p. 319, and state what formula is to be used when both stiffness and strength are required.

2. Find the ratio of depth to span in a beam of rectangular section loaded in the middle, assuming stress $= 8000$, $E = 28{,}000{,}000$, deflection $= \frac{\text{span}}{1200}$. *Ans.* $\frac{1}{17 \cdot 5}$.

3. A beam is supported at the ends and loaded at a point distant a, b from the supports with a weight W. Show that the depression of the weight below the points of support is $\frac{Wa^2b^2}{3EI(a+b)}$.

4. In the last question deduce the work done in bending the beam, and verify the result by direct calculation. (See Art. 20.)

5. A dam is supported by a row of uprights which take the whole horizontal pressure of the water. The uprights may be regarded as fixed at their base at the bottom of the water while their upper ends at the water level are retained in the vertical by suitable struts sloping at 45°, the intermediate part remaining unsupported. Find the bending moment at any point of the upright, and show that the thrust on the struts is about two-sevenths the horizontal pressure of the water.

6. A timber balk 20 feet long of square section supports 160 square feet of a floor, find the dimensions that the deflection of the floor, when loaded with 60 lbs. per square foot, may not exceed ½ inch. *Ans.* 12¾".

7. A shaft carries a load equal to m times its weight (1) distributed uniformly, (2) concentrated in the middle. Considering it as a beam fixed at the ends, find the distance apart of bearings for a stiffness of $\frac{1}{1200}$th. *Ans.* If l be the distance apart in feet, d diameter in inches, then for a wrought-iron or steel shaft

$$(1)\ l = 10 \cdot 5\sqrt[3]{\frac{d^2}{m+1}}; \qquad (2)\ l = 8 \cdot 3\sqrt[3]{\frac{d^2}{m+\frac{1}{2}}}.$$

8. A beam originally curved, as in Ex. 21, p. 320, is fixed at one end and loaded in any way. If i be the change of slope at any point and X, Y the displacements parallel to axes of x, y of the point consequent on any load, prove that

$$\frac{di}{ds} = \frac{M}{EI}; \quad \frac{dX}{dy} = -i; \quad \frac{dY}{dx} = i.$$

Apply these formulæ to find the straining actions at any point of one of the rings of a chain of circular links.

9. A weight W is fixed to the centre of a vertical rotating shaft, and, by its centrifugal force when the shaft is slightly bent, tends to increase its lateral deflection. Show that the number of revolutions of the shaft per minute must not approach that given by the equation

$$n = 1300\sqrt{\frac{EI}{Wl^3}}$$

all dimensions being in inches.

Note.—This is the simplest case of what is known as "centrifugal whirling," a question considered in Art. 203A, Ch. XVI.

CHAPTER XIV.

TENSION OR COMPRESSION COMPOUNDED WITH BENDING. CRUSHING BY BENDING.

173. *General Formula for the Stress due to a Thrust or Pull in combination with a Bending Moment.*—The bars of a frame and the parts of other structures are often exposed, not only to a pull or thrust alone, or to a bending action alone, but to the two together; and the total stress at any point of a transverse section is then the sum of that due to each taken separately. That is to say, if H be the thrust, reckoned negative if a pull, M the bending moment, the stress at any point distant y from the neutral axis of the bending (see Art. 155), reckoned positive on the compressed side, must be given by

$$p=\frac{H}{A}+\frac{My}{I}=\frac{H}{A}\left\{1+\frac{q}{n}\cdot\frac{M}{Hh}\right\},$$

the notation being as in the article cited.

This formula shows how the effect of a thrust or pull is increased by a bending action: it has many important applications, some of which we shall now briefly indicate.

174. *Strut or Tie under the Action of a Force parallel to its Axis in cases where Lateral Flexure may be neglected.*—*Case I.* Bar under the action of a force in a principal plane parallel to its axis.

Let z be the distance from the axis of the line of action of the force, then

$$M=Hz:\ p=\frac{H}{A}\left(1+\frac{q}{n}\cdot\frac{z}{h}\right).$$

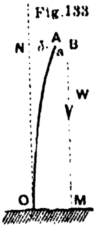
For example, let the section be circular, then $n=\frac{1}{16}$, $q=\frac{1}{2}$, and we find

$$p=\frac{H}{A}\left(1+\frac{8z}{h}\right),$$

from whence it appears that a deviation from the axis of $\frac{1}{16}$th the diameter of a rod increases the effect of a thrust or pull 50 per cent. Similarly it can be shown that if the line of action of the force lie outside the middle fourth of the diameter of a circular section, or the middle third of a rectangular section, the maximum stress will be more than double the mean, and at certain points the stress will be reversed. In designing a structure, then, the greatest care must be exercised that the line of action of a thrust or pull lies in the axis of the piece which is subjected to it; to effect which, the joints, through which such straining actions are exerted, must be so designed that the resultant stress at the joint is applied at the centre of gravity of the section of the piece. This is a condition which cannot always be satisfied, and allowance in any case must be made for errors in workmanship. In practical construction it is the joints which require most attention, being most often the cause of failure. In frames which are incompletely braced the friction of pin joints causes the line of action of the stress to deviate from the axis.

The effect is increased in the case of a thrust and diminished in the case of a pull by the curvature of the piece, which increases or diminishes x. Fig. 133 shows the axis of a column, under the action of a weight W, suspended from a short cross piece of length a. The column bends laterally, as shown in an exaggerated way in the figure. The inclination of AB to the horizontal is so small that the difference between the actual and the projected length of AB may be disregarded; the bending moment at O is therefore $W(a+\delta)$, where δ is the lateral deviation AN of the top of the pillar. This deviation we will in the first instance suppose small compared with a, and then determine the condition that this may actually be the case. Neglecting it, the axis of the pillar is bent by the uniform bending moment Wa into a circular arc of radius R, and as in Art. 163,

Fig. 133



$$\delta . 2R = l^2 ;$$

substituting for R its value (Art. 155) we get

$$\delta = \frac{Ml^2}{2EI} = \frac{Wal^2}{2EI} ;$$

whence we find

$$\frac{\delta}{a} = \frac{Wl^2}{2EI}$$

The condition, then, that the lateral deviation should be small is that W should be much less than $2EI/l^2$, and if this condition be satisfied the stress will not be much increased beyond that indicated by the formula given above. The very important cases in which W is large will be treated presently.

In the case of a pull this restriction on the use of the formula need not be attended to, the deviation diminishing the stress.

Case II. Uniformly loaded beam supported at the ends and subject to compression.

Let the load be W and the thrust H, then

$$p = \frac{H}{A}\left\{1 + \frac{q}{n} \cdot \frac{\frac{1}{8}Wl}{Hh}\right\}.$$

For example, let the section be rectangular, then $q = \frac{1}{2}$, $n = \frac{1}{12}$, and we find

$$p = \frac{H}{A}\left\{1 + \frac{3l}{4h} \cdot \frac{W}{H}\right\}.$$

Let us further suppose the ratio of depth to span one-sixteenth, then

$$p = \frac{H}{A} \cdot \left(1 + 12\frac{W}{H}\right) = \frac{W}{A}\left(12 + \frac{H}{W}\right),$$

which shows how greatly the effect of a thrust is increased by a moderate bending moment.

If the deflection be supposed 1 inch in 100 feet then H will in consequence produce an additional bending action at the centre equal to $Hl/1200$, which will be equivalent to an addition to W of $H/150$. For safety H ought not to exceed $3W$, and the stress due to the bending action of the uniform load on the beam will then be increased about 25 per cent. This calculation shows why it is often necessary to support a beam at points not too far apart by suitable trussing even when support is not required to give sufficient stiffness. Theoretically a proper "camber" given to the beam will counteract the bending action, and, conversely, a small accidental deflection will increase it.

175. *Remarks on the Application of the General Formula.*—The formula given in Art 173 is much used in questions relating to the stability of chimneys, piers, and other structures in masonry and brickwork. The stress on horizontal sections of such structures varies uniformly or nearly so, and the formula then shows where the stress is greatest and also where it becomes zero, tension usually not being permissible. It must be borne in mind however that the bending is frequently unsymmetrical, so that the axis of the bending moment will not coincide with the neutral axis of the bending stress on the section (Art. 162). The stability of blockwork and earthwork structures is a large subject which will not be considered in this treatise. The use of the term "neutral axis" to denote the line of zero stress, a line which varies in position according to the proportion between the thrust and the bending, though common, is better avoided.

176. *Straining Actions due to Forces Normal to the Section.*—The reasoning of this section shows that when a structure is acted on by forces some or all of which have components normal to a given section, the straining actions due to the normal components will in general depend on the relative yielding of the several parts of the section (Art. 42). These normal components however can always be reduced to a single force, acting through any proposed point in the section, and a couple, and if the point be properly chosen according to the nature of the structure at the section that single force will be a simple thrust or pull; thus in the cases we have mentioned the point is the centre of gravity of the section. Having done this the couple will be so much addition to the bending action. An important example of this is the case of a vessel floating in the water in which the horizontal longitudinal component of the fluid pressure generally produces bending, the arm of the bending couple being the distance of the intersection of the line of action of the resultant with the section considered, from the neutral axis of the "equivalent girder."

177. *Maximum Crushing Load of a Pillar.*—When the compressing force is sufficiently great it produces a strong tendency to bend the pillar even though there be no lateral force. We have already seen that the condition that this shall not be the case is that W shall be small compared with the quantity $2EI/l^2$, and we now proceed to inquire the effect produced when W has a larger value. All these cases come under the head of what is called Crushing by Bending, and are very common and important in practice.

As in the case of the deflection of a beam the question is much more simple when the pillar bends into an arc of a circle, which it will do in various cases explained in Art. 163. The case which we select is that in which the sectional area remains constant and the thickness varies. Such a pillar is of uniform strength when very slightly bent, and when more bent the weakest point is at the base. When the load is applied exactly at the centre the elevation of such a pillar is a semi-ellipse with vertex at the summit; when not exactly at the centre the ellipse is truncated. As in other cases of uniform strength the section is ideal, requiring modification at the summit when applied in practice.

Assuming then the form of the bent pillar to be a circular arc we have as before

$$\delta = \frac{Ml^2}{2EI'}$$

but we have now, since we cannot neglect δ,

$$M = W(a + \delta).$$

Hence by substitution we find

$$\delta = \frac{W(a+\delta)l^2}{2EI},$$

where I is the moment of inertia at the base, from which we find

$$\delta = \frac{a}{\frac{2EI}{Wl^2} - 1}.$$

This result shows that the pillar bends laterally more and more as W increases, and breaks with some value of W which we will find presently by substitution in the formula of Art. 172.

First, however, observe that if $a = 0$, that is, if the line of action of the load pass through the centre of the pillar at its summit, then $\delta = 0$ unless the denominator of the fraction be also zero, that is, unless

$$W = 2 \cdot \frac{EI}{l^2}.$$

The interpretation of this is, that if W be less than the value just given the pillar will not bend at all, but if disturbed laterally will return to the upright position when the disturbing force is removed. If W have exactly that value then, when put over into any inclined position the pillar will remain there in a state of neutral equilibrium, while the smallest increase of W above this limit will cause the pillar to bend over indefinitely and so break. Thus the foregoing equation may be regarded as giving the crushing load of the pillar under certain conditions to be defined more exactly presently.

If the form of the bent pillar be not a circular arc but some other given curve, the corresponding type of section can be found by use of the general equation given on page 323. A formula of the same form is then obtained, but the co-efficient 2 is replaced by some not very different number depending on the form assumed.

In Fig. 134 let y be the deviation from the vertical BB of any point in the pillar BAB at a distance x from the summit, then Wy is the bending moment M at that point, and the equation may be written

$$Wy = EI \cdot \frac{d^2y}{dx^2}.$$

The case of most importance is that in which the curve BAB is a curve of sines given by the equation

$$y = \delta \sin \frac{\pi}{2} \cdot \frac{x}{l},$$

$2l$ being the height of the pillar and δ the deviation from the vertical at the centre. Differentiating, substituting and dividing by y,

$$W = \frac{\pi^2}{4} \cdot \frac{EI}{l^2},$$

an equation which shows that a pillar of uniform transverse section when bent into a curve of sines will be in equilibrium for this value of W and no other; a result the interpretation of which is the same as in the preceding case, from which it only differs in the number 2 being replaced by $\pi^2/4$ or 2·47. In Fig. 134 both ends of the pillar are rounded so as to be free to change their direction while remaining in the same vertical, the whole height L of the pillar is then $2l$ and

$$W = \pi^2 . \frac{EI}{L^2}.$$

In Fig. 134b the pillar is fixed in direction at both ends and consequently there are two points of contrary flexure or "virtual joints" BB. The position of these joints is easily foreseen, for the four pieces CB, BA, AB, BC are all acted on by the same compressing force applied virtually in the same way and are therefore all of equal length. The whole height L of the pillar must consequently now be taken as $4l$ instead of $2l$ and π^2 replaced by $4\pi^2$.

Fig. 134.

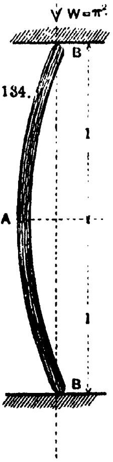


Fig. 134a.

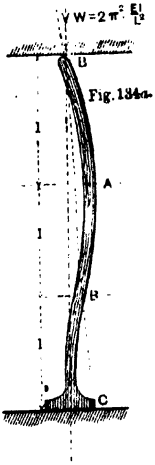
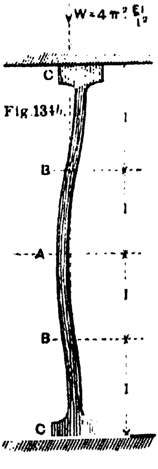


Fig. 134b.



In Fig. 134a we have an intermediate case, the summit being rounded and the base fixed in direction. The two ends are still supposed in the same vertical, so that the pillar now bends into the form $BABC$, having only one point of contrary flexure near the base while the upper portion BAB is in the condition of a pillar with rounded ends. To find the length $2l$ of this upper portion in terms of the whole height L of the pillar, we must observe that the point of contrary flexure is not in the same vertical as the ends but deviates from it by a small quantity which we will call y_0. So that the deviation of any point distant x from the summit is now

$$y = \delta \,.\, \sin \frac{\pi}{2} \,.\, \frac{x}{l} + y_0 \,.\, \frac{x}{2l},$$

a formula which applies to points below B as well as above.

To determine the position of B we have only to observe that when $x = L$ both y and dy/dx must be zero; hence differentiating and eliminating y_0,

$$\tan \frac{\pi L}{2l} = \pi \,.\, \frac{L}{2l},$$

a transcendental equation which when solved by trial gives

$$\pi \,.\, \frac{L}{2l} = 4{\cdot}493,$$

from which we find that the upper portion BAB of the pillar is about 70 per cent. of the whole height and that, in the formula for W, π^2 should be replaced by $2{\cdot}047\pi^2$. For the purposes to which this formula is applied $2\pi^2$ is sufficiently accurate; Rankine employed in Gordon's Formula (p. 344) a coefficient obtained by supposing BB instead of BC vertical, which corresponds to the value $2\frac{1}{4}\pi^2$.

We thus obtain the three formulæ known as Euler's Formulæ,

$$W = \pi^2 \,.\, \frac{EI}{L^2}\,; \quad W = 2\pi^2 \,.\, \frac{EI}{L^2}\,; \quad W = 4\pi^2 \,.\, \frac{EI}{L^2},$$

for the three cases in question with a uniform section. If the pillar be bent into a circle as above, then π^2 is to be replaced by 8.

178. *Manner in which a Pillar crushes. Formula for Lateral Deviation.*—The value of W here found is the maximum load, consistent with stability, which a pillar, free to deflect laterally, can sustain under any circumstances; but, in order that it may actually be sustained, the pillar must be perfectly straight, the material must be perfectly homogeneous, and the line of action of the load must be exactly in the axis. These conditions cannot be accurately satisfied, and consequently a lateral deflection is produced, which increases indefinitely as the load approaches the theoretical maximum. This may be expressed by supposing that a is not zero, but some known quantity depending on the degree of accuracy with which the conditions are satisfied, and which may be called the "effective" deviation; since, when the pillar is straight and homogeneous, it will be the actual deviation of the line of action of the load from the axis. Let W_0 be the theoretical maximum load as calculated from the preceding formulæ, and W the actual load, then

$$\delta = \frac{a}{\frac{W_0}{W} - 1} = a \cdot \frac{W}{W_0 - W} \text{ (p. 338)};$$

thus we see that a load of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$ the theoretical maximum produces a lateral deflection of $1a$, $2a$, $3a$, increasing the deviation of the load from the axis of the column to $2a$, $3a$, $4a$. These numbers are only exact when the pillar is so formed as to bend into the arc of a circle; when this is not the case they follow a more complicated law of the same general character, depending on the type of pillar and the nature of the deviation. For our purpose the simple case is sufficient. It is convenient to express the load in pounds per square inch of the area (A) of the pillar at its base, then we may write with the notation of Art. 155

$$p_0 = \frac{W_0}{A} = \pi^2 \cdot nE \cdot \frac{h^2}{L^2}$$

for the case where the pillar is rounded at both ends, the number π^2 being replaced by $2\pi^2$ or $4\pi^2$ in the two other cases of the last article. Similarly writing $p = W/A$ for the actual load on the pillar, we get by substitution

$$\delta = a \cdot \frac{p}{p_0 - p}, \quad \text{or} \quad a + \delta = a \cdot \frac{p_0}{p_0 - p}.$$

The deviation is accompanied by an increase in the maximum stress (f) on the transverse section, which is given by the formula

$$f = \frac{H}{A}\left(1 + \frac{q}{n} \cdot \frac{M}{Hh}\right) \quad \text{(p. 334)},$$

from which we get, replacing H by W and M by $W(a + \delta)$,

$$f = p\left(1 + \frac{qa}{nh} \cdot \frac{p_0}{p_0 - p}\right),$$

a result which shows that f increases indefinitely as p approaches p_0, so that the pillar must break before the theoretical maximum is reached, however small the original deviation is. The greatest value of f must be the elastic strength, for as soon as this is past an additional lateral deviation at the most compressed part will occur, sooner or later accompanied by rupture.

The formula may be written in the more convenient form

$$\left(\frac{f}{p} - 1\right)\left(1 - \frac{p}{p_0}\right) = \frac{qa}{nh},$$

in which it is worth while to observe that the right-hand side is unity for the deviation necessary to produce double stress when the pillar is so short that no sensible augmentation of the deviation is produced by lateral bending. In materials like cast iron which have a low tenacity, very long pillars give way by tension on the convex side: the formula then becomes

$$\left(\frac{f'}{p}+1\right)\left(1-\frac{p}{p_0}\right)=\frac{qa}{nh},$$

where f' is the tensile stress at the elastic limit. The two formulæ give the same result if

$$p=\frac{f-f'}{2}.$$

For loads greater than this the first formula applies, and for small loads the second. In pillars flat, but not fixed at the ends, without capitals f' may be zero.

179. *Actual Crushing Load.*—We thus see that if a pillar were *absolutely straight and homogeneous it would crush, by direct compression* if p_0 were greater than f, and by lateral bending if p_0 were less than f, the crushing load being the least of these two quantities; but that the smallest deviation will be augmented by lateral bending, so that the actual crushing load will be less than the least of these quantities. Experience confirms this conclusion. When a long pillar is loaded we do not find that it remains straight till a certain definite load p_0 is reached, and then suddenly bends laterally. We find, on the contrary, that a perceptible lateral deflection is produced by a small load, which gradually increases as the load is increased, till rupture takes place, showing, as we might anticipate, that some small deviation existed originally. And as that deviation evidently depends upon accidental circumstances it is impossible, from imperfection of data, to find the actual crushing load of a pillar for those proportions of height to thickness, for which its effect is greatly augmented by a small deviation. The augmentation is on the whole greatest when

$$f=p_0=\pi^2 . n . E . \frac{h^2}{L^2};$$

that is, when

$$\frac{L}{h}=\sqrt{\frac{\pi^2 n E}{f}}.$$

This gives, by taking the values of E and f from Table II., Chapter XVIII.,

Wrought Iron,	$L=36\sqrt{\pi^2 n} . h=28h$	(Circular Section).
Mild Steel,	$L=29\sqrt{\pi^2 n} . h=23h$	,,
Hard Steel,	$L=23\sqrt{\pi^2 n} . h=18h$	,,
Cast Iron,	$L=20\sqrt{\pi^2 n} . h=16h$	,,

In the case of cast iron there is a difficulty in determining the value of f, but if we suppose that the elasticity of the material is not greatly impaired at half the ultimate crushing load, we get the value given. The case of timber is exceptional, and will be referred to further on. For pillars fixed or half-fixed at the ends the number π^2 is to be replaced by $4\pi^2$ or $2\pi^2$ as before.

Let us assume this condition satisfied, and let us imagine the pillar loaded with three-fourths the theoretical maximum crushing load, then by substitution we find, $qa/nh = \frac{1}{3} \cdot \frac{1}{4}$, or since $n/p = \frac{1}{8}$ for a circular section,

$$\frac{a}{h} = \frac{1}{96},$$

from which it will be seen how small a deviation will cause the pillar to crush under three-fourths the theoretical maximum load, when the proportion of height to thickness is that just given. With a pillar of double this height the magnitude of the original deviation (a), always supposing it small, has little influence, and with a pillar of one-third this height lateral flexure has little influence, on the resistance to crushing.

On the whole, then, it would seem that the most rational way of designing pillars would be to calculate the theoretical maximum load, and then adopt a factor of safety depending on the value of the deviation found from the above formula: it is obvious that in some cases a much larger deviation may be considered likely than in others. For example the probable deviation from straightness may easily be imagined to be proportional to the length of the pillar. The Gordon-Rankine formula given in the next article may be regarded as a formula for the average factor of safety necessary on account of the exaggerated influence of errors of workmanship on the strength of pillars in cases where the deviation is not greatly influenced by the length. For the case of thin tubes see Chapter XVIII.

180. *Gordon's Formula.*—A considerable part of our experimental knowledge respecting the strength of pillars is due to Hodgkinson.* His results show that in cast iron pillars with flat ends, the length of which exceeds 100 diameters, the theoretical maximum is closely approached, while with shorter lengths the strength falls off considerably, as might be expected. In other respects the theoretical laws are approximately fulfilled, the principal difference being that columns with one or both ends rounded are somewhat stronger relatively to columns with flat ends than theory would indicate, an effect which may be partly due to imperfect fixing of the ends. Various empirical formulæ have

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* *Phil. Trans.*, 1840, Part II. An abridgment is given in Hodgkinson's work on *Cast Iron.* Weale, 1846.

been given to express the results of experiment on the crushing of pillars. That which has been most used was originally devised by Navier, but is commonly known as Gordon's. It is so constructed as to agree in form with the theoretical formulæ in the extreme cases in which those formulæ give correct results. As employed by Rankine, only replacing r^2, the square of the radius of gyration, by nh^2 in the notation of this work the formula is

$$\frac{W}{A}=\frac{f}{1+\frac{l^2}{cnh^2}},$$

which becomes, when l/h is small,

$$W=Af,$$

and when l/h is large,

$$W=\frac{cnfAh^2}{l^2}=cf\cdot\frac{I}{l^2};$$

while for intermediate values it gives smaller results.

If we compare this last with Euler's formula for a column with flat ends, we get

$$c=4\pi^2\frac{E}{f},$$

and this may be called the "theoretical" value of the constant c. The values actually used for c are somewhat different, being deduced from such experiments as have been made, and the results for different forms of section are not always consistent. Rankine gives in his *Useful Rules and Tables*,

VALUE OF CONSTANTS.

	Value of f.	Value of c.
Wrought Iron, ...	36,000	36,000
Cast Iron, ...	80,000	6,400
Dry Timber, ...	7,200	3,000

These values refer to struts fixed at the ends and to the crushing load. If one end be rounded, the value of c must be divided by 2, and if both ends are rounded, by 4.

Rankine's formula has been very extensively tested for the case of wrought-iron columns of large size of various transverse sections, constructed of rivetted plates, and has been found to give good results.*

In the case of timber Hodgkinson found, from a limited number of experiments on struts of oak and red pine of small dimensions, a formula which agrees with the formula for the theoretical maximum crushing load when the value of E in that formula is taken as about 900,000 lbs. per square inch. It is possible that the low lateral tenacity of this material increases its flexibility under a heavy crushing load. The values just given of the constants for timber in Gordon's formula appear rather low. Recent good authorities give 9400 for f and 6700 for c.

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* *Minutes of Proceedings of the Institution of Civil Engineers* for May, 1878, vol. liv., page 200.

In the case of steel the value of f may be expected to be increased and the value of c diminished in the ratio of the direct resistance to crushing of steel and wrought iron respectively, conclusions on the whole borne out by experience.*

Calculations made by Gordon's formula may be tested by calculating the deviation a by the formula on p. 341; the magnitude of this will be to some extent a measure of the safety of the proposed load. In all cases of struts of large size subject to a heavy load, special care is necessary in considering all the circumstances—if a deflection be occasioned by the unsupported weight of the strut itself, or if, as is often the case, it be constructed of rivetted plates, a large margin of safety is desirable. So also in pieces forming part of a machine in which a bending action may be produced by inertia and friction, or which are subject to shocks, the simple thrust alone is often a very imperfect measure of the stress to which they are subject.

Returning to the case of a long slender column we observe that the resistance to crushing depends solely on the stiffness and not on the strength being proportional to the modulus of elasticity. Hence a long column is stronger when made of wrought iron than when made of cast iron, although with short columns the reverse is true. It appears from Gordon's formula that for a ratio of length to diameter of about 26½ the two materials are equally strong. In very long columns steel is not stronger than iron, for its modulus of elasticity is not very different; in shorter lengths, however, the greater resistance to direct crushing of steel gives it an advantage.

180A. *Partial Fixture of Ends.*—The condition of the ends of a pillar has great influence on its resistance to crushing; thus by Euler's formula the crushing load of a pillar fixed at the ends is four times that of a pillar with both ends rounded. The ends of a pillar in practical cases can hardly ever be regarded as either rounded or fixed in a mathematical sense, and the influence which different methods of fixing may have is a matter of much importance.

(1) The most effectual method of obtaining, for experimental pur-

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* *A Practical Treatise on Bridge Construction*, by T. C. FIDLER, page 180. Griffin, 1887.

poses, a pillar, the ends of which are freely movable in direction while remaining exactly in the same vertical, is to make its ends wedge-shaped, or, *still* better, conical. Experiments on pillars with conical ends were carried out in 1887 by the late Professor Bauschinger, a well-known authority on strength of materials. The test-pieces were of rolled iron of various sections, among which may be especially mentioned some pieces of I section of sectional areas ranging from 10½ to 63½ square centimetres and of lengths from 1 to 4 metres. The results show irregularities arising partly from causes already mentioned and probably partly from the difficulty of obtaining the moments of inertia with sufficient accuracy, but on the whole show a crushing load of more than 85 per cent. of that given by Euler's formula for a pillar with rounded ends. In 1887-88 similar experiments on pillars with conical ends were made by Herr Tetmajer, the test-pieces being iron bars of circular section about 2 inches diameter and also pieces of wood. On comparison with Euler's formula similar results were obtained.

These experiments point to the conclusion that the deviation a instead of increasing slowly with the length, as it would do (Ex. 9, p. 349) if the Rankine formula were satisfied, increases much more rapidly, so that the "constant" c in that formula diminishes rapidly with the length when the ends of the pillar are rounded.

(2) At the same time Bauschinger also made experiments on pillars with flat ends simply butting against the compressing pieces without any attachment. The test-pieces were similar to those in the preceding case, but the results of the experiments now showed a comparatively constant value of c instead of the rapid diminution previously found. In this case also, however, c is not constant, but diminishes with the length.

The same diminution of the constant c as the length increases has been found in many experiments on columns of larger size, as shown by the formulæ proposed by Mr. Cooper.*

(3) The ends of a pillar are very frequently pin-jointed, the load being then transmitted by the pressure of the pin upon its circular-bearing surface. The crushing load now depends on the diameter of the pin, because the *total* diviation cannot exceed the radius of the friction circle (p. 240) of the pins; as soon as this is overpassed, the pillar instantly crushes in consequence of the release of the ends.

Very instructive experiments were carried out at Watertown Arsenal, U.S.A., in 1883, by means of the well-known testing machine there stationed.* The test-pieces were bars of iron about 3 inches square, of lengths ranging from 10 to 60 diameters, some with pin ends, the diameters of the pins ranging from ¾ inch to 2¼ inches, and some with flat ends. The results of these experiments, tabulated on page 118 of the report cited, show that pins 2¼ inches diameter are nearly equivalent to flat ends, but that ¾ inch pins give a much reduced crushing load. The Rankine formula appears to agree fairly well with these experiments, small pin ends being treated as rounded and large ones as fixed.

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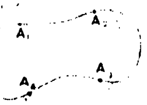
* *Engineering Construction*, by W. H. Warren, p. 196. Longmans, 1894.

(4) The facts described in this article show clearly the empirical character of the Rankine formula; the approximate truth of which, under the complicated conditions in which most experiments have been made, being due to the effects of increasing initial deviation as the length of the column increases, being partially compensated by the increasing influence of the partial fixture of the ends. How far the conditions of the experiment resemble the conditions of practice must always be carefully considered in each individual case.

181. *Collapse of Flues.*—There are other cases of crushing by bending, some of which will be considered in a later chapter; but it will be convenient to mention the important practical problem of the yielding of a thin tube under *external* fluid pressure. The strength of a tube under external fluid pressure is as different from that of a tube under internal pressure as the strength of a bar under compression is different to its strength under tension.

Fig. 135.



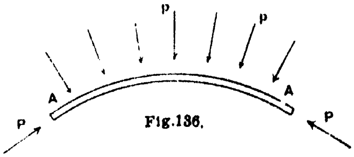


Fig. 136.

A tube perfectly uniform in thickness made of perfectly homogeneous hard material, and subject to perfectly uniform normal pressure externally would theoretically maintain its form until it yielded by the direct crushing of the material. But when the pressure exceeds a certain limit the tube is in a state of unstable equilibrium, and any deviation from perfect accuracy in the above conditions will cause the tube to yield by collapsing, the collapsing being accompanied by bulging. If the tube is very long it will collapse in the manner shown in Fig. 135, the circumference dividing itself up into four arcs, two

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* *Report of Tests on Structural Material made at the Watertown Arsenal.* 1883.

of which are concave outwards and the other two convex. A want of exactness in the construction will in practice generally prevent the collapsing from being symmetrical. Each portion of tube between the points A is under the action of forces applied at the ends towards one another, which crush it by lateral bending just as a long column is crushed. Just before collapsing, each segment AA (Fig. 136), of length s say, will be under the action of a thrust P suppose, applied at the ends tangentially. Equilibrium is maintained by fluid pressure of intensity p on the convex side. When the pressure exceeds a certain limit the equilibrium is unstable, some accidental circumstance determining the position of the point A of contrary flexure, and the consequent length s of any arc.

As shown on page 302 the thrust per inch length of the tube may be taken as approximately proportional to pd. Thus if t = thickness of tube, we may expect that the collapsing pressure would be given by a formula like that which expresses the crushing load of a long slender rod of rectangular section, namely, $pd = k't^3/s^2$ where k' is an elastic co-efficient. All other things being equal, the diameter alone varying, the length s of an arc AA would be proportional to the diameter of the tube d, and, under those circumstances, the collapsing pressure of a thin tube (see Appendix), would probably vary with t^3/d^3. But the length of the tube, as well as the diameter, influences the value of s. In all practical cases, as in all those on which experiments were made, the ends of the tube are rigidly constructed, and very much support the tube in the neighbourhood from collapsing; thus the proximity of the ends has an important effect in determining the length of the arcs into which the circumference divides itself. If the length of the tube is decreased a limit will be reached below which the tube on collapsing divides itself up into six arcs, three concave and three convex, as shown in Fig. 137. Then the length of each arc will bear a smaller proportion to the diameter than in the long tube. A still shorter tube will, when it collapses, divide it into eight arcs, and so on. Thus the length s is in some way dependent on the length of the tube. The correctness of this reasoning is borne out by experiments made by Fairbairn and others. In Fairbairn's experiments the tubes were made of rivetted wrought-iron plates. The ends were made rigid by a strong stay placed within the tube, keeping the ends apart. The tube thus constructed was placed in a larger cylinder of wrought iron and external pressure was applied by forcing water in. The pressure being gradually increased the tube will at last suddenly collapse, making a noise which indicates the instant of the occurrence. The results of

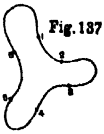


Fig. 137

the experiments showed that the collapsing pressure may be approximately expressed by the formula

$$p = k\frac{t^2}{ld},$$

the dimensions being all in inches, the co-efficient $k = 9{,}672{,}000$. This formula must not be used for extreme cases nor for tubes of thickness less than $\frac{3}{8}$ inch.

Since a short tube is so much stronger than a long one, we have an explanation of the advantage of rivetting a T-iron ring around a boiler furnace tube, which amounts to a virtual shortening of the length of the tube. Other formulæ have been proposed, some of which represent the results of experiment more closely, but the materials at present available do not admit of the construction of a satisfactory formula. Some further remarks on the subject will be found in the Appendix.

EXAMPLES.

1. Find the thickness of metal of a cast-iron column fixed at the ends, 1 foot mean diameter, 20 feet high, to carry 100 tons. Factor of safety, 8. *Ans.* Thickness 1″.

2. Find the crushing load of a wrought-iron pillar 3″ diameter, 10 feet high, rounded at the ends. *Ans.* Crushing load = 66,218 lbs. 30 tons nearly.

3. If in last question the pillar were of rectangular section of breadth double the thickness, what sectional area would be required for equal strength? *Ans.* Sectional area = 9·4 square inches instead of 7 square inches as before.

4. Assuming the crushing resistance of steel to be $1\frac{1}{2}$ times that of wrought iron, and its modules of elasticity 10 per cent. greater, find the probable values of f and c in the Gordon-Rankine formula. *Ans.* $f = 54{,}000$, $c = 26{,}400$.

5. Find the crushing load in tons of a timber pile 12 inches square, 30 feet long, fixed at one end, rounded at the other. *Ans.* $56\frac{1}{2}$ tons.

6. Find the collapsing pressure, according to Fairbairn's formula, of a cylindrical boiler flue $\frac{7}{16}$″ thick, 48″ diameter, and 30 feet long. *Ans.* Collapsing pressure = 107 lbs.

7. In Ex. 1 calculate the deviation of the line of action of the load from the axis to produce a maximum stress of 10,000 lbs. per square inch. *Ans.* 1·8″.

8. In Ex. 2 calculate the deviation to produce a maximum stress of 9,000 lbs. per square inch with a load of 11,000 lbs. or of 22,000 lbs. *Ans.* 1·5 or ·5.

9. Assuming the crushing load of a pillar to be given by the Gordon-Rankine formula with the theoretical values of the constants, show that the deviation is given by the formula

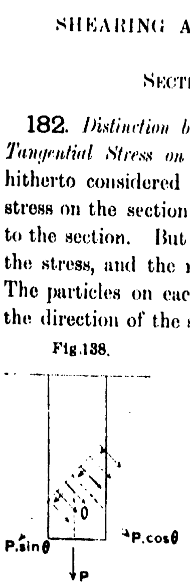
CHAPTER XV.

SHEARING AND TORSION OF ELASTIC MATERIAL.

SECTION I.—ELEMENTARY PRINCIPLES.

182. *Distinction between Tangential and Normal Stress.—Equality of Tangential Stress on Planes at Right Angles.*—In the cases we have hitherto considered of simple tension, compression, and bending, the stress on the section under consideration has been at all points normal to the section. But we may take our section inclined at any angle to the stress, and the mutual action is then not normal to the section. The particles on each side of the section partly act on one another in the direction of the section itself, and so constitute a stress analogous to friction, resisting the sliding of one portion relatively to the other. Such a stress is called *tangential* or *shearing stress*, being the stress called into action by shearing.

Fig. 138.



Let us return to the case of the stretched bar carrying a load P (Fig. 138). On a transverse section of the bar only a normal stress is produced. Now suppose we take an oblique section, whose normal makes an angle θ with the axis of the bar, and let us resolve the force P into two components, one perpendicular and the other parallel to the section. The normal component $P\cos\theta$ tends to produce a direct separation at the section, producing a tensile stress similar in character to that on a transverse section, but of less intensity.

If A = area of transverse section of bar, then $A\sec\theta$ = area of oblique section; the intensity of the normal stress

$$p_n = \frac{P\cos\theta}{A\sec\theta} = \frac{P}{A}\cos^2\theta = p\cos^2\theta, \text{ where } p = \frac{P}{A};$$

the other component $P \sin \theta$ produces a tangential or shearing stress of intensity

$$p_t = \frac{P \sin \theta}{A \sec \theta} = p \sin \theta \cos \theta.$$

Similarly if the bar is subjected to a compressive instead of a tensile load.

Many materials which offer great resistance to direct compression yield by sliding across an oblique plane. Now p_t is a maximum when $\theta = 45$, this is therefore approximately the angle of separation. The same maximum stress, the value of which is $p/2$, occurs on another plane sloping the other way at an angle of 45°. We sometimes find fracture to occur across two oblique planes; sometimes across one only.

If in $p_t = p \sin \theta \cos \theta$ we change θ into $90 + \theta$, p_t has the same value; so that the intensity of the tangential stresses on two planes at right angles to one another is the same. This is true generally in all cases of stress, as will be seen presently.

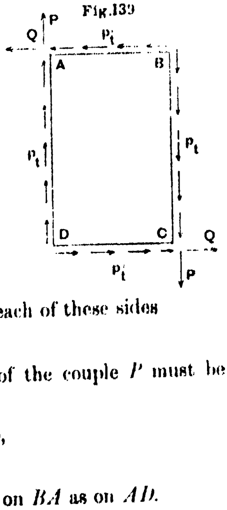
183. *Tangential Stress equivalent to a Pair of Equal and Opposite Normal Stresses. Distorting Stress.* In the example we have just considered we have both shearing and normal stress; but there are cases in which there is only a shearing stress. Let $ABCD$ (Fig. 139) be a rectangular plate of thickness t. Over the surfaces BC and AD suppose a tangential stress to be applied of intensity p_t. Calling b and a the length of the sides of the plate, the total amount of the tangential stress on each side is

$$P = p_t . bt.$$

To prevent the turning of the plate, suppose the forces P balanced by the application of an uniform stress over the surfaces BA and DC of intensity p'_t. The amount of the force on each of these sides

$$Q = p'_t . at.$$

Fig. 139



Since equilibrium is produced, the moment of the couple P must be equal to the moment of the couple Q.

$$\therefore p_t . bt . a = p'_t . at . b,$$

$$\text{or} \qquad p_t = p'_t ;$$

that is, the intensity of the stress is the same on BA as on AD.

Shearing therefore cannot exist along one plane only. It must be accompanied by a shearing stress of equal intensity along a plane at right angles. Such a pair of stresses unaccompanied by normal stress constitute a Simple Distorting Stress, so called because it distorts the elements of the body.

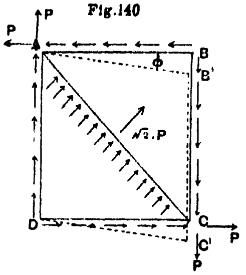


Fig. 140

Let us now assume, for simplicity, the plate to be square (Fig. 140). The effect of the forces is to produce a change of form, which, in perfectly elastic bodies, is exactly proportioned to the shearing force which produces it. The square $ABCD$ becomes a rhombus $AB'C'D$, the angle of distortion ϕ being proportional to the stress p_t. We may write

$$p_t = C\phi,$$

where the co-efficient C is a kind of Modulus of Elasticity, but of a different nature from that previously employed. The volume of the elastic body A is in general practically unaltered. Under the action of the forces it has simply undergone a change of form or figure, and the co-efficient C which connects the change of form with the stress producing it, is a co-efficient of elasticity of figure. It is sometimes called the *modulus of transverse elasticity*, but preferably the *co-efficient of rigidity*.

The ordinary (Young's) modulus of elasticity E connects the stress and strain in a bar when it undergoes changes both of volume and figure. The co-efficient of rigidity C for metallic bodies is generally less than $\frac{2}{5}E$, and for wrought-iron bars may be taken as 10 to 10½ millions, or in torsion somewhat greater.

Let us now take a section of the square plate (Fig. 140) along one of the diagonals and consider the forces which act on the two sides of the triangular upper portion. Resolve these forces parallel and perpendicular to the diagonal. The components of the two P's along the diagonal balance one another, and there will be no tendency for this triangular portion to slide relatively to the other; that is to say, there is no shearing stress on the diagonal section. But the other components, perpendicular to the diagonal, cause the upper triangular portion to press on the lower with a force

$$2\frac{P}{\sqrt{2}} = \sqrt{2}\,.\,P.$$

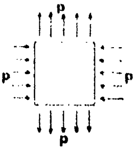
If we divide this force by the area of the diagonal section over which it is distributed, we obtain the intensity of this normal stress,

$$p_n = \frac{\sqrt{2}\,.\,P}{\sqrt{2}\,.\,at} = p_t.$$

On the diagonal section AC which we have been considering, this stress is compressive, but if we take the section along BD, the other diagonal, we find by the same reasoning a stress of the same magnitude, but tensile.

Thus it appears that a shearing stress on any plane necessarily involves tensile and compressive stresses of equal intensity on planes at 45° to this plane, so that a simple distorting stress, which was defined above as a pair of shearing stresses on planes at right angles, may also be defined as a pair of normal stresses of equal intensity and of opposite sign, as shown in Fig. 141.

Fig. 141.

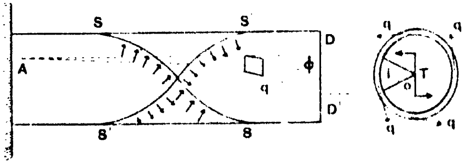


SECTION II.—TORSION OF SHAFTS. SPRINGS.

184. *Torsion of a Tube. Round Shafts.*—We now proceed with various examples of this kind of stress, commencing with the case of torsion. Torsion was mentioned as one of the five simple straining actions to which the bar as a whole may be exposed. It is produced by a pair of equal couples applied at the ends of the bar, the axis of the couples being the axis of the bar.

When we consider the nature of the elastic forces called into action amongst the particles of the bar, Torsion reduces to a case of Shearing. To understand this, we will begin with a simple case. Imagine a thin tube (Fig. 142) with one end fixed, and the other acted on by an uniform tangential stress of intensity q. Let t be the thickness and d the mean diameter of the tube, then

Fig. 142.



Sectional area of tube $= \pi dt$ approximately;

Total shearing force $= q\pi dt$;

and since the force on each unit of area of the section acts approximately at the same distance from the centre of the tube, the total twisting moment $= q\pi dt \times \frac{1}{2}d = \frac{1}{2}q\pi d^2t$. This twisting moment is balanced by the resistance to turning offered at the fixed end. At any transverse section *KK* of the tube there will be produced an uniform stress of intensity q.

Let us now consider a small square traced on the surface of the tube, with two sides on two transverse sections. If we take the square small enough we may treat it as a plane square. To balance the shearing stress q, which acts on the sides of the square lying in the transverse planes, a shearing stress of equal intensity is, as explained above, called into action on the other two sides of the square, in the direction of the length of the tube, so that, if the tube were cut by longitudinal slits, the power of resistance to torsion would be as effectually destroyed as if it were cut by transverse slits. But if we made spiral slits at an angle of 45°, as shown at SS in Fig. 142; supposing the slits indefinitely fine, and no material removed, the strength of the tube to resist torsion in the direction shown would not be impaired. The material of the tube would then be divided into spirally-bent ribands, which would be in tension along their length, and in compression laterally, the ribands being caused to press against one another. Along a second set of spirals such as $S'S'$, longitudinal compression and lateral tension exist; the lateral forces are indicated in both cases by arrows in the figure.

So much for the state of stress induced in the tube by the torsion. Next as to the change of form which accompanies the stress. The square will be distorted into a rhombus. A straight line AD, drawn on the surface parallel to the axis of the tube passing through the centre of the square, will be twisted into a spiral AD', the angle of the spiral being the angle of distortion of the square. Let θ be that angle, then

$q = C\theta$, where C is the co-efficient of rigidity.

The effect of this is that, relatively to the end A, the end D is twisted round through an angle $DOD' = i$ suppose, called the angle of torsion.

In circular measure $i = \frac{\text{arc } DD'}{r}$ (r = radius of tube). Also arc $DD' = l\theta$,

θ being a small angle. Therefore $i = l\theta/r$. Since also $\theta = q/C$, we have the angle of torsion $i = ql/Cr$, in terms of the stress. From this we may express the angle of torsion in terms of the twisting moment producing the torsion.

We now pass on to the consideration of the torsion of a solid or hollow cylindrical shaft. First, let us imagine the shaft to be made up of a number of concentric tubes exactly fitting one another, and let us further imagine that at the end of each tube a suitable twisting moment is applied, so that each tube is twisted round through exactly the same angle. This effect will be produced by applying over the section at the end of each elementary tube a tangential stress, which is proportional to the radius of the tube. If we make $q/r = q_1/r_1$, where q_1 and r_1 refer to the outside tube, then the angle of torsion will be the same for all the tubes, and they will not tend to turn relatively to one another, but altogether. We may then suppose them united together again in a solid mass. If the stress applied be proportional to the distance from the centre, the shaft will twist just as if it were a set of tubes, each being subjected to the same stress and strain as if it were an independent tube.

Now in the actual case of the twisting of a solid shaft, all portions from the outside inwards to the centre must turn through the same angle, and hence the shearing stress at any point of the section of the shaft must be proportional to its distance from the centre. This is true except very near the point of application of the twisting moment. Suppose, for example, the twisting moment is applied by means of a wheel keyed on the shaft, then in the immediate neighbourhood of the key-way, the stress will not be as stated, but at a short distance along the shaft the stress distributes itself in the manner described. This is another instance of the general principle already employed in the case of stretching and bending.

The total resistance to torsion of the solid shaft is the sum of the twisting moments of all the concentric tubes into which it may be imagined to be divided. Thus

$$T = \Sigma 2\pi r^2 t q\ ; \text{ in which } q = r \,.\, \frac{q_1}{r_1}.$$

$$\therefore\ T = \Sigma_0^{r_1} 2\pi r^2 t r \frac{q_1}{r_1} = \frac{q_1}{r_1} \Sigma_0^{r_1} 2\pi r t \,.\, r^2,$$

that is, the product of the sectional area of each tube multiplied by the distance squared of the area from the axis of the shaft must be taken and summed. The result is called the Polar Moment of Inertia and will be denoted by I, so that

$$T = \frac{q_1}{r_1} I.$$

The same formula applies to hollow shafts, the summation now extending from the internal radius r_2 to the external radius r_1 and the value of I is then

$$I = \frac{\pi}{2}(r_1^4 - r_2^4),$$

being double the corresponding value in the case of bending.

Since $i = ql/Cr$ we can eliminate q and thus obtain

$$T = CI \,.\, \frac{i}{l},$$

a formula which gives the twisting moment in terms of the torsion per unit of length.

Dropping the suffixes, taking r to be the outside radius, we can write the moment of resistance to torsion of a solid shaft,

$$T = \tfrac{1}{2}\pi f r^3, \quad \text{or} \quad \tfrac{1}{16}\pi f d^3\ ;$$

where f is the co-efficient of strength of the material to resist shearing.

Thus the strength under torsion is proportional to the cube of the diameter. The formula shows that, assuming f to be the same in each case, the strength of a shaft to resist a twisting moment is double its strength to resist a bending moment.

Having determined the diameter of shaft required to take a given twisting moment we are now able to obtain a solution of the practical question, What diameter of shaft is required to transmit a given horse-power at a given number of revolutions per minute?

Let $T_0 =$ *mean* twisting moment transmitted in inch-tons, then $T_0 \times 2\pi N =$ work transmitted per minute in inch-tons, where $N =$ revolutions per minute of shaft.

Let $H.P.$ denote the horse-power to be transmitted, then

$$T_0 \times 2\pi N = \frac{33000 \times 12}{2240} H.P.$$

$$\therefore\ T_0 = \frac{33000 \times 12}{2240 \times 2\pi} \frac{H.P.}{N}.$$

Now the shaft must be strong enough to take not only the mean but the maximum twisting moment.

We may express the maximum in terms of the mean by writing $T = KT_0$, where K is a co-efficient whose value is different in different cases and $T =$ maximum twisting moment, but

$$T = \frac{\pi}{16} f d^3 \quad \text{or} \quad d^3 = \frac{16T}{\pi f}.$$

$$\therefore\ d^3 = \frac{16 \times 33000 \times 12}{2\pi^2 \times 2240} \frac{K}{f} \frac{H.P.}{N},$$

and

$$d = 5{\cdot}233 \sqrt[3]{\frac{K}{f} \frac{H.P.}{N}}.$$

The value of f depends in some measure on the fluctuation to which the twisting moment is subject, but under ordinary circumstances should not exceed 3¼ tons per square inch (Art. 221) for wrought iron, 4¼ tons for steel, and (see Art. 229) 1¼ tons for cast iron. The value of K, the ratio of maximum to mean twisting moment, depends on the circumstances discussed in Chapter X. We may assume it equal to 1⅓ when the number of cranks is 2, allowing a small addition for the bending due to the weight of the shaft. On substitution we obtain for wrought iron

$$d = 4\sqrt[3]{\frac{H.P.}{N}}.$$

This formula agrees closely with the best practice in screw-propeller shafting.

When the amount of bending to which the shaft is subject is considerable, as in the case of crank shafts, the diameter determined by this formula is too small. It will be seen hereafter that when all the forces acting on the shaft are known, a value of K can be calculated which gives the effect of bending. If we assume $K = 2$, the co-efficient 4 in the above formula will be replaced by 4·5, and this agrees closely with practice in the crank shafts of marine screw engines when made of iron, the number of cranks being 2.

In the formula for the angle of torsion,

$$i = \frac{ql}{Cr};$$

if we replace q by its working value for wrought iron (7200 lbs.), C by 5000 tons, and i by the circular measure of 1°, we find

$$l = 13·6d,$$

showing that under the working stress the shaft twists through 1° for each $13\frac{1}{2}$ diameters in its length. For many purposes this is much too small, and the dimensions of a shaft then depend on stiffness, not on strength, as in the case of beams (Art. 168). The greatest angle of torsion permissible depends in great measure on the irregularity of the resistance, and no general rule can therefore be laid down for it. If the angle of torsion be given and the length, the diameter will depend on the fourth root of the twisting moment as shown by the formula already given which connects the two. In this, as in other cases where dimensions depend on stiffness, not on strength, steel has no advantage over iron, because the co-efficients of elasticity of the two materials are the same or nearly so.

A hollow shaft is both stronger and stiffer than a solid shaft of the same length and weight, the central portion of a solid shaft not being twisted sufficiently to develop its full strength.

The distance apart of the bearings of a shaft depends on the stiffness necessary to resist the bending due to the weight of the shaft itself, and of any pulleys or wheels upon it, together with the tension of belts and other similar forces. If the total load be equivalent to m times the weight of the shaft itself uniformly distributed, the length between bearings for a wrought-iron or steel shaft d inches diameter will be given approximately for a stiffness of $\frac{1}{1200}$th by Ex. 7, p. 333.

When, as in screw propeller shafting, the bearings are liable to get out of line, too great stiffness in a shaft will produce great straining actions upon it.

185. *Elliptic and other Sections.*—In the cases hitherto considered the stress called into play at each point of the transverse section is proportional to the distance of the point from the centre, and its direction is tangential to a circle drawn through the point, but in non-circular sections this is no longer the case. In particular the direction of the stress at points near the circumference is necessarily tangential to the contour of the section, for by the principle of Art. 183 (p. 351), in the absence of shearing stress on the outer surface, there can be none on the transverse section in a direction perpendicular to the contour.

Commencing as before with the case of a tube the thickness (t) of which is small compared with the radius of curvature, but which now may vary according to any law; the shear (q) at any point is along the circumference. Taking Δs a small element of the circumference, $t\,.\,\Delta s$ will be the area of the element, and $qt\,.\,\Delta s$ will be the whole shear upon it, while qt will be the shear (S) per unit of length of the circumference. Next take two cross sections of the tube distant Δx apart and consider the rectangular portion of thickness t the sides of which are Δs and Δx. Reasoning in the same way as in Art. 183 already cited, the shear S on the face Δs of the cross section is necessarily accompanied by an equal shear (S') on the longitudinal face Δx. Now if the cross sections are free from constraint caused by the neighbourhood of rigid ends, the state of stress of all cross sections may be taken as the same, and the normal stress (if any) on the two faces forming part of the two cross sections cannot be different: from which it follows that S' is constant, since any change in S' can only be balanced by a corresponding change in the normal stress on the faces at right angles to it. Hence S, that is qt, is also constant, and we infer that the intensity of the shear at any point varies inversely as the thickness of the tube at that point. Now take a point O inside the tube in the plane of the section and join O to the extremities P, P' of the element Δs at the middle of the thickness, thus forming a triangle OPP' on the base Δs. Then if p be the perpendicular dropped from O on the tangent to the circumference drawn through PP'

$$p\Delta s = 2 \times \text{Area } OPP',$$

and if A be the whole area of the cross section of the tube measured to the middle of the thickness

$$\Sigma p\Delta s = 2A.$$

But $S\,.\,\Sigma p\Delta s$ is the moment of stress about O and is therefore equal to the twisting moment (T) on the tube, from which it follows that

$$T = 2SA \text{ or } q = \frac{T}{2At},$$

a formula which, subject to the limitation stated, is true for a tube of any shape, the thickness of which varies continuously in any way.

We now suppose the internal and external peripheries to be similar curves with regard to the centre O, that is, if the radius of the inner periphery drawn in a given direction be r that of the outer periphery is $(1+e)r$ where e is constant, er being the thickness measured along the radius vector, then

$$\frac{t}{er}=\frac{p}{r} \text{ or } t=ep,$$

and hence it appears that in such a tube the shear q varies inversely as the perpendicular dropped from O on the tangent.

The greatest value of q then will be when p is least, in which case evidently $p=r$ so that $t=er$. Imagine now a nest of such tubes fitting one inside the other, it is easily proved (see Art. 216A) that if each tube be twisted through the same angle the shear at corresponding points must be proportional to the radius, that is, q/r is the same for all. Hence if T be now the twisting moment for all the tubes taken together, q/r will be constant in the equations

$$T=2\Sigma qAt=2\Sigma qAer,$$

where r is the least radius and q the greatest shear for any tube. Suppose q_1, r_1, A_1 corresponding quantities for the outermost tube, and write Δr for er, then

$$T=\frac{2q_1A_1}{r_1}\cdot\Sigma\frac{Ar}{A_1}\cdot\Delta r=\frac{2q_1A_1}{r_1}\int\frac{r^3}{r_1^2}dr.$$

Hence, by integration we obtain for a complete set of tubes

$$T=\tfrac{1}{2}q_1A_1r_1.$$

This formula determines the moment of torsion in any case in which the solid shaft can be regarded as split up into a number of similar tubes. Taking for example a circular section it will be seen that the formula is equivalent to that already found on page 355, and it may be modified in the same way for a hollow shaft. But the point O must be taken as the centre of the circle, for otherwise the connection of the tubes together would alter the distribution of stress. So in any case the centre of similarity O of the ideal tubes must be the geometrical centre of the section, and the section must be supposed symmetrical about a centre. For elliptical sections the formula is exact, and it may be taken as a probable approximation whenever the curvature of the profile is continuous.

Whenever the necessary conditions for its truth are satisfied, the investigation shows that the stress at any point near the surface of a twisted shaft varies inversely as the perpendicular distance of the centre from the tangent at the point, and that the strength is proportional to the product of the sectional area and the least diameter. Thus for a given weight of metal the circle is the strongest form, and for other forms the strength is diminished in proportion to the least diameter. If c be that diameter the moment of resistance will be

$$T = \tfrac{1}{4} fAc.$$

When the curvature of the profile of the section is not continuous but contains an angle, as for example the corner of a square, the stress at the corner is necessarily zero, for by the principle already several times used, the stress resolved perpendicular to each of the two edges meeting in the corner is zero. Hence the stress at various points along a radius within the thickness of a tube, the mean value of which has already been found is not sensibly constant, but varies greatly. If for example a nest of rectangular tubes be considered, the stress at the outer corner of each tube will be zero, while that of the inner corner of the next tube in contact with it will be large: the state of stress in the solid shaft is therefore widely different from that of the tubes. Our knowledge of the distribution of stress in these and other cases is entirely derived from the laborious calculations of the late M. St. Venant, an account of which is given by Professor Pearson in a treatise, some notice of which will be found in the Appendix. It will be sufficient here to quote some of the simpler and more important results.*

(1) In a bar of rectangular section, area A, sides b and c, the point of maximum stress is at the centre of the longer side b, and the moment of resistance is

$$T = f \cdot \frac{bc^2}{3 + 1{\cdot}8\beta} = f \cdot \frac{A^{\frac{3}{2}}\sqrt{\beta}}{3 + 1{\cdot}8\beta} = \frac{fAc}{3 + 1{\cdot}8\beta},$$

where β is the ratio of the sides c/b. In a square section of side b this becomes $\cdot 208\ fb^3$, being little more than 6 per cent. greater than the value previously given for a circular section of the same diameter. On comparison it will be found that the strength of a square section is only 73·8 per cent. of that of a circular section of the same area, the co-efficient being the same in the two cases. When β is very small the moment of resistance is $\cdot 333\ fbc^2$. The formula is empirical, being devised by St. Venant to represent accurately in the two extremes cases just mentioned, and with fair approximation in intermediate cases, results of calculation from infinite series. When β is not less than ·3 a somewhat better approximation is obtained by writing $3{\cdot}2 + 1{\cdot}6\beta$ in the denominator. When β is about ·5 the result is the same as for a nest of tubes.

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* *History of Elasticity*, vol. ii., Part I., pp. 19-39; also p. 195.

(2) Of less importance in practice, but interesting from its simplicity, is the formula for an elliptic section which is that already given for a nest of tubes.

(3) The angle of torsion i for a bar of length l under a twisting moment T is given by the formula

$$T = \frac{1}{4\pi^2} \cdot \frac{A^4}{I} \cdot \frac{Ci}{l},$$

where A is the area and I the polar moment of inertia of the section. For elliptic sections this formula is exact (see Art. 216A), agreeing with that given in the preceding article when the ellipse becomes a circle. For rectangles and sectors of a circle it is shown by St. Venant to be a tolerable approximation, and it may be taken as such for any section, the contour of which does not contain re-entering angles. The divisor $4\pi^2$ should, however, be replaced by a slightly different number according to the type of section, the average being taken by St. Venant as 40. The exact value for a square section on comparison with St. Venant's table of results appears to be 42·66 and for rectangles $(\beta > \cdot 3)$ may be taken as 42.

Since the elastic energy U is necessarily equal to $\frac{1}{2}Ti$ the foregoing formula gives

$$U = \frac{C}{8\pi^2} \cdot \frac{A^4}{Il} \cdot i^2 = \frac{2\pi^2 Il}{A^4} \cdot \frac{T^2}{C},$$

in which as before $4\pi^2$ should be replaced by 42 in rectangular sections.

(4) The maximum stress q_1, due to a twisting couple T, may be found from the value of the moment of resistance already given in (1), (2), whence by substitution for T the elastic energy, or its limiting value the resilience, per unit of volume may be found.

It is, however, also important to know the stress q_2 at a point 2 situated at the middle of the shorter side of a rectangular section, for when the section is exposed to bending as well as torsion the combined effect of the two is frequently greatest at this point. On examination of St. Venant's results, given on page 39 of the work already cited, it is found that q_2 is never less than 74 per cent. of q_1, and that it is given with fair approximation by the formula

$$q_2 = q_1 \cdot \frac{3 + \beta^2}{4}.$$

In an elliptic section the point 2 is at either extremity of the major axis and the stress there is βq_1, diminishing to zero as the ratio of axes is reduced. This shows that the distribution of stress for small values of β is quite different from that in a rectangular section; a point further illustrated by comparing the strength of the two for the same area and ratio of axes; when it will be found that for small values of β the rectangle is the stronger form.

Further remarks on torsion will be found in the Appendix.

186. *Crank Shafts.*—The twisting of a shaft is due to the action of transverse forces which have a moment about its axis. The common crank shaft is a case which may here conveniently be considered as an example of the way in which such forces strain the shaft.

In Fig. 143 *ACB* is a shaft turning in bearings *A* and *B* and acted on by twisting moments T_1, T_2 at its ends. The sides of the crank are generally at right angles to the shaft, but in the figure are shown inclined at an angle θ, a case which sometimes occurs. The crank pin is acted on by the thrust of a connecting rod not shown in the figure, which together with other forces (if any) passing through the axis of the shaft and corresponding reactions of the bearings form a system of forces the straining actions due to which are now to be studied: the graphical methods explained in Part I. being employed as most suitable for the purpose.

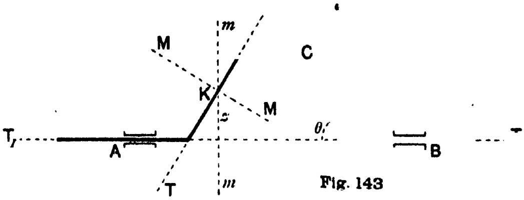


Fig. 143

The first step is to resolve the forces into two sets, one set in the plane of the crank, the other perpendicular to that plane. The first set produce shearing and bending only, which actions may be represented by polygons in the usual way and need not for the present be further considered; the second set alone produce twisting. As regards the straight part of the shaft: if S be the force on the crank pin perpendicular to the plane of the crank and a the crank-radius,

$$Sa = T_1 - T_2,$$

then the difference of the twisting moments T_1, T_2 is determined, but the actual magnitudes depend on the twisting transmitted from the parts of the shaft lying beyond the bearings. If one end B of the shaft be free the corresponding moment T_2 will of course be zero. If the turning moment Sa supplied by the connecting rod furnish energy at both ends of the shaft, as is often the case, T_2 will be negative.

Taking any point K (Fig. 143), on the crank arm at a distance z from the axis, let a polygon of moments be drawn, the force S for this purpose being taken as passing through the axis. The result at K is a bending moment m, the axis of which is shown in the figure by a dotted line perpendicular to the axis. In like manner a polygon of shearing force may be drawn giving the shear at K which we will call F. Taking a transverse section of the crank arm at K the shear on this section will be F while the bending moment M and the twisting moment T will be determined by the equations

$$Fz - M . \sin\theta + T . \cos\theta = T_1,$$
$$M . \cos\theta + T . \sin\theta = m,$$

from which we obtain the values of T and M, namely,

$$T = (T_1 - Fz)\cos\theta + m . \sin\theta,$$
$$M = (Fz - T_1)\sin\theta + m . \cos\theta.$$

For the crank pin we have only to put $\theta = 0$, $z = a$, and we find

$$T = T_1 - Fa, \quad M = m,$$

and in the common case where $\theta = 90$ we have for any point of the crank arm,

$$T = m; \quad M = Fz - T_1.$$

These results refer to the side next the bearing A; on the other side T_1 must be changed into T_2. It must further be remembered that they refer exclusively to the set of forces perpendicular to the plane of the crank; the set of forces in that plane produce a shear F' and a moment M' perpendicular to those just considered, so that the resultant bending moment is $\sqrt{M^2 + M'^2}$.

The crank arm, however, is usually of rectangular section and the components M, M' must then be considered separately. The method of compounding a twisting moment with a bending moment will be explained in Chapter XVII.

187. *Spiral Springs* may be flat or conical, but the simplest and most important case is that in which the spring consists of a strip of metal, usually of rectangular or circular section, coiled into a cylinder of radius r, the pitch angle (θ) of the spiral being uniform. The length of the spring (x_0) measured along the axis of the cylinder is given by the formula

$$x_0 = l \sin\theta,$$

and therefore can only vary sensibly by variation of the pitch angle. The ends of the strip are bent to meet the axis and are inclined to each other at an angle ϕ given by the equation

$$r\phi = l . \cos\theta,$$

each complete convolution of the spring increasing the angle by 2π. The angle θ will for the present be supposed small.

Such a spring may be used in two distinct ways.

(1) One end being held fast the other may be attached to a spindle occupying the axis of the spring, and by a couple (M) applied to it the spring is turned through an angle $\phi - \phi_0$. The action here is one of simple bending, the bending moment being M and the elastic energy (U) being given by

$$U = \tfrac{1}{2}M(\phi - \phi_0) = \frac{M^2 l}{2EI} = \frac{EI}{2l}(\phi - \phi_0)^2,$$

equations which give for the angle turned through

$$\phi - \phi_0 = \frac{2U}{M} = \frac{Ml}{EI}.$$

An example of this kind occurs in the spring of the balance of a chronometer.

(2) Much more important, however, is the case so common in practice in which the spring is altered in length from x_0 to x by the action of a force P applied along the axis. Each section of the strip is now subject to the action of a twisting moment $T = Pr$, while the corresponding elastic energy is

$$U = \tfrac{1}{2}P(x - x_0).$$

The value of U is found from the formula given in Art. 185 (3), and the radius between $x - x_0$ and P is thus determined.

(3) The action on a spiral spring is not exactly one of pure bending or pure torsion as just supposed unless the pitch angle θ be exceedingly small. In the first case the bending moment is $M.\cos\theta$ and there is in addition a twisting moment $M.\sin\theta$; while in the second case the twisting moment is $Pr.\cos\theta$ which is accompanied by a bending moment $Pr\sin\theta$. It will be shown hereafter that the elastic energy due to the combination of bending and twisting is the sum of the values of U due to each taken alone. The total value of U can therefore be readily obtained by summation: by use of which the preceding formulæ for $\phi - \phi_0$ and $x - x_0$ will still apply. If θ be less than 15°, however, the correction is of little importance. For values of co-efficients, see Ch. XVIII.

When the spiral is flat as in the main spring or balance spring of a watch the action is one of simple bending as in (1), and the same formulæ apply with slight modification. The conical springs employed in the buffers of railway carriages and for other purposes act by torsion as in (2), but the calculation is somewhat more complex.

SECTION III.—SHEARING IN GIRDERS. JOINTS.

188. *Web of a Beam of I Section.*—Torsion is one of the few cases in practice where a simple distorting stress occurs alone and not in combination with other kinds of stress. It generally happens that a normal stress is combined with it; such, for example, is the case in the web of a beam of I section, to which we next proceed to direct our attention. Taking a transverse section, the normal stress at a point distant y from the neutral axis is given by the formula

$$\frac{p}{y}=\frac{M}{I},$$

and is therefore the same for the same values of M and I, whether the web be thin or thick, while it will be shown presently that the tangential stress is greater the thinner the web, and becomes the most important element when the web is thin.

Let us suppose, for simplicity, the flanges equal, and also that the beam is supported at the ends and loaded in the centre with a weight W.

As we have previously seen, the flanges will sustain the greater portion of the bending moment, the web carrying only a small portion of it, 1/7th, if the area of the web equals the area of each flange. For simplicity, let us imagine the flanges to take the whole of the bending. Let K_1 and K_2 (Fig. 144) be two transverse sections of the beam at distances x_1 and x_2 from the centre of the beam, $2a$ being the span of the beam the bending moment at the first section,

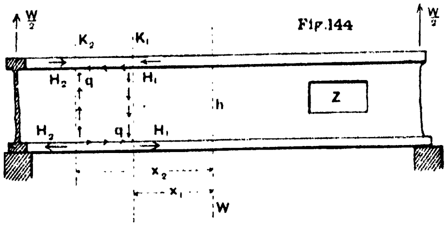


Fig. 144

$$M_1=\tfrac{1}{2}W(a-x_1)\text{ and at the second }M_2=\tfrac{1}{2}W(a-x_2).$$

Now, supposing the flanges to take the whole of the bending, the stress H produced on the flanges is given by the formula

$$Hh=M.\quad\text{Thus at }K_1\text{ we have }H_1=\frac{W(a-x_1)}{2h},$$

$$\text{and at }K_2\text{ we have }H_2=\frac{W(a-x_2)}{2h},$$

and similar forces on the bottom flange only reversed in direction. There will thus be a resultant force H_1-H_2 tending to push the portion K_1K_2 of the flange to the left,

$$H_1 - H_2 = \frac{W(x_2 - x_1)}{2h}.$$

This force is balanced by the resistance of the web to shearing along the line of junction with the flange.

Since $H_1 - H_2$ is proportional to the length of K_1K_2, the shearing force per unit of length of web $= W/2h$. If we suppose t to be the thickness of the web, the intensity of the shearing stress will be

$$q = \frac{W}{2ht}.$$

Thus, considering the portion of the web between the sections K_1 and K_2 apart by itself, we see that on the upper and lower horizontal edges of it, where it joins the flanges, it is subject to a shearing stress of intensity q, to balance which there must act on the vertical sides KK a shearing stress of equal intensity. Now, the shearing force for the vertical sections KK is $\frac{1}{2}W$. Supposing the web to be of rectangular section and of height h, then, assuming the whole of the shearing force to be borne by the web, the mean intensity of the shearing stress on the vertical sections is $W/2ht$. Therefore the assumption that the flanges take the whole of the bending moment is equivalent to supposing the web to take all the shearing. Assuming this, we see that the shearing stress, taken as uniformly distributed over the vertical section, will be accompanied by an equal shearing stress on any horizontal section. When considered alone, the effect of these shearing stresses on planes at right angles to one another is to produce tensile and compressive stresses on the web in directions making an angle of 45° with the horizontal and vertical planes; and thus the web may be superseded by an indefinite number of diagonal bars inclined at an angle of 45°, thus forming a lattice girder.

If the web is designed so as to be strong enough only to withstand the shearing stress, replacing q by f the co-efficient of strength against shearing f, we find

$$t = \frac{W}{2hf}.$$

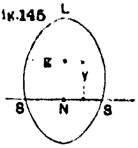
The influence of the normal stress due to bending will be considered in a subsequent chapter. Its effect is greatly to increase the strain on the web (see Art 207), which besides will in most cases exhibit weakness on account of the compressive stress in one of the diagonal directions. If the distance between the flanges is great, the web will be liable to yield by buckling or lateral flexure (see page 340). To prevent this, the web must be stiffened by angle irons rivetted on it. But the girder would then be made heavy, and it is therefore more economical to make large girders with openwork diagonal bracing.

We have in this investigation supposed the beam loaded in the middle, so that the shearing force is uniform throughout the length of each half, and the problem was thus simplified. But the same principles apply if the load be distributed in any manner. The shearing force will then vary from section to section along the beam.

189. *Distribution of Shearing Stress on the Section of a Beam.*—The foregoing preliminary investigation will give some idea of the effect of a shear on the web of a flanged beam; let us now consider the question more generally.

Taking a section of any type, let a line be traced cutting off from the whole area A any given portion. The line may be curved, but in the first instance assume it straight and parallel to the neutral axis SS (Fig. 145). Divide the area into strips of breadth b and thickness Δy, as in Art. 154, then the normal pressure on the portion cut off is

Fig.145



$$H = \Sigma b \,.\, \Delta y \,.\, p = \frac{M}{I}\mu,$$

the second form being obtained by substituting for p from the bending moment formula and writing

$$\mu = \Sigma b y \Delta y$$

for the moment of the portion cut off about the neutral axis SS, a quantity which can be directly calculated by summation or deduced when the position of the centre of gravity of the portion is known. Assuming the transverse sections K_1, K_2 as in the preceding article at a distance $x_2 - x_1$, which, however, we will now suppose to be the unity, let ΔM be the difference of the corresponding bending moment, then

$$H_1 - H_2 = \Delta M \,.\, \frac{\mu}{I}.$$

But referring to Art. 29, p. 55, it will be seen that if F be the shearing force $\Delta M = F(x_2 - x_1)$, and, as before, $H_1 - H_2$ is balanced by a corresponding shearing stress called into play over the horizontal base of the prismatic portion intercepted between the sections. If then S be the total shear on the base,

$$S = F \,.\, \frac{\mu}{I},$$

a formula which is equally true if the base be curved, or even if the portion is wholly enclosed in the solid mass of the beam.

If the mean shearing stress F/A on the transverse section be q_0 and on the base be q, the formula may be written, replacing I by nAh^2,

$$q = q_0 \frac{\mu}{nsh^2},$$

where s is the periphery of the base whether straight or curved. By the principle of Art. 183 the shear at any point of the base is also the shear on the transverse section in a direction normal to the base. Let us now consider various cases.

(1) Returning to the case of the I section, let y_1 be the distance of the base of the flange (Fig. 144) from the neutral axis and y the ordinate of some other point in the transverse section of the web, then $s=t$ and

$$\mu - \mu_1 = (y_1 - y)t \cdot \frac{y_1 + y}{2} = \tfrac{1}{2}(y_1^2 - y^2)t,$$

hence by substitution $$q = q_1 + q_0 \cdot \frac{y_1^2 - y^2}{2nh^2},$$

a formula which gives the shear at any point of the transverse section distant y from the neutral axis in terms of q_1 the shear immediately below the flange. The value of q_1 can be found from the formula, μ_1 being a given quantity. When the web is very thin, q_0 is relatively very small, and the shear on the web is approximately the same at all points; but otherwise, in addition to the shear on the web as a whole, there is a local shear represented by the ordinates of a parabolic arc, the chord of which is the depth of the web. The extreme case is that of a rectangular section when $q_1 = 0$, $h = 2y_1$, $n = \frac{1}{12}$, then

$$q = \frac{3}{2}q_0\left(1 - \frac{y^2}{y_1^2}\right).$$

At the neutral axis where $y=0$ the stress which is then a maximum is $1\frac{1}{2}$ times the mean.

(2) Consider a tube of circular section, mean radius a, thickness of metal t, under the action of a shear F, producing on the section a shearing stress the mean intensity of which is

$$q_0 = \frac{F}{2\pi at}.$$

In Fig. 146 draw the radii OP, OP' inclined at an angle θ to the vertical cutting off the arc PP. Then in the general formula given above

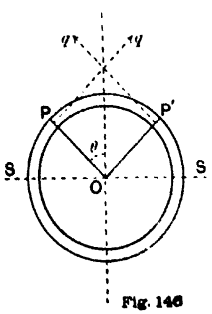


Fig. 146

$$\mu = 2\int_0^\theta a\cos\theta \,.\, at \,.\, d\theta = 2a^2t \,.\, \sin\theta,$$

$$s = 2t,\quad n = \tfrac{1}{8},\quad h = 2a,$$

$$\therefore\ q = q_0 \cdot \frac{2a^2t \,.\, \sin\theta}{a^2t} = 2q_0 \,.\, \sin\theta.$$

This gives the resultant shearing stress at any point P, which, as explained in Art. 185, is necessarily in the direction of the tangent. The maximum value occurs at the neutral axis SS and is double the mean.

(3) Taking a section of any type, consider the portion cut off by the

neutral axis SS in Fig. 145, and let $SS = b_0$ be the breadth of the beam there, then the mean shearing stress on the transverse section at points lying on the neutral axis is

$$q = q_0 \cdot \frac{\mu}{n \cdot b_0 h^2},$$

where μ is the moment of the area SLS about the neutral axis. For example, in a rectangular section

$$\mu = \frac{1}{2} b_0 h \cdot \frac{h}{4}, \quad n = \frac{1}{12},$$

and consequently $q = \frac{3}{2} q_0$ as already found. Similarly in an elliptic or circular section $q = \frac{4}{3} q_0$.

This formula is commonly accepted as giving the ratio of maximum to mean stress on the section; but this statement must be understood with very considerable qualifications, for all that is actually determined is the mean stress at points along the neutral axis; the maximum is generally greater, and sometimes very much greater, as will be seen from our next example.

(4) Take a square bar and imagine it bent and sheared by forces parallel to a diagonal. Through the centre O of the section draw two straight lines ON, OM perpendicular to the sides, cutting off one-fourth of the whole area. Then if h be the side

$$\mu = \frac{h^2}{4} \cdot \frac{h}{2\sqrt{2}}, \quad s = ON + OM = h,$$

$$\therefore \; q = q_0 \cdot \frac{3}{2\sqrt{2}}$$

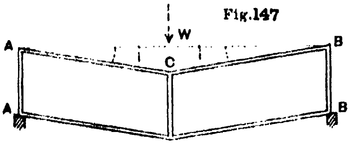
is the mean stress along ON perpendicular to ON, and along OM perpendicular to OM. If we suppose the stress at O equal to the mean in each direction it must be the resultant of two forces at right angles, each equal to q and will therefore be $\frac{3}{2} q_0$, the same as when sheared by forces parallel to a face. By the method employed in (2) it will be found that the mean stress at points along the neutral axis is q_0, but the extremities of the neutral axis being the corners of the square the stress there must be zero as pointed out in Art. 185, and this explains why a larger result is obtained by the present method, which will be found useful as a check in cases where the breadth of the section at the neutral axis is much greater than elsewhere. Since the direction of the stress at points lying on the edges of the square is necessarily along the edges, as explained in the article cited, there is little reason to suppose any considerable variation along the lines ON, OM.

In a circular section it will be found that the two methods give the same result.

The exact distribution of the stress in these cases, as in torsion, can only be found by considering the manner in which the section is warped by the action of the forces. If it be forcibly prevented from distortion, as in a rivet tightly fitting its hole, the shear is uniformly distributed, but otherwise it is not so, and in a loose fitting pin or a key may provisionally be taken as given by the methods just described. In certain cases more exact results are given by M. St. Venant's calculations, which will be further noticed in a later chapter.

190. *Deflection due to Shearing.*—A certain part of the deflection of a beam is due to the distortion of its central parts. Returning to the beam of I section, loaded in the middle, suppose the flanges hinged at the centres, and let vertical stiffening pieces AA, BB, CC, be rigidly connected to the web, but hinged to the flanges, then distortion of the web takes place, as shown in a very exaggerated way in the figure (Fig. 147), causing a deflection δ of the beam such that

Fig.147



$$\frac{\delta}{\frac{1}{2}l} = i = \frac{q}{C} = \frac{W}{2htC},$$

where C as before is the co-efficient of rigidity, and q the shearing stress is expressed as before.

$$\therefore \ \delta = \frac{Wl}{4htC} = \frac{ql}{2C}.$$

For wrought iron take $q = 9{,}000$ for the working load, and $C = 9{,}000{,}000$, then

$$\delta = \frac{l}{2{,}000},$$

which is above half the working deflection due to bending in ordinary cases.

This calculation, however, exaggerates the deflection due to shearing even in a beam of I section, for the web cannot in general be so thin as to give a stress of 9,000 lbs. per square inch, and the effect is much less for a uniformly distributed load. Nevertheless in beams of this class the deflection due to shearing is a considerable part of the whole, the more so as in rivetted girders the union of the parts seldom renders them completely rigid. This is the principal reason why large girders show a considerably smaller modulus of elasticity when the deflection is calculated in the usual way than solid bars.

In a section of any type of area A loaded and supported in the manner described the mean shear on any section will evidently be

$$q_0 = \frac{W}{2A},$$

and the deflection due to it

$$\delta_0 = \frac{q_0 l}{2C} = \frac{Wl}{4AC}.$$

The corresponding deflection (d) due to bending is found by the formula on page 324, which, on writing as usual nAh^2 for I, becomes.

$$d = \frac{Wl^3}{48nEAh^2}.$$

The ratio of these two is therefore

$$\frac{\delta_0}{d} = 12n \cdot \frac{E}{C} \cdot \frac{h^2}{l^2}.$$

To obtain the actual value of the ratio of the deflection due to shearing and bending, the result here found requires multiplication by a factor the value of which can be calculated approximately as explained in Chapter XVII. This factor for a rectangular section is 1·2, whence taking E/C equal to 2·5 and $n = \frac{1}{12}$ we find

$$\frac{\delta}{d} = \frac{3h^2}{l^2},$$

which agrees with Professor Pearson's estimate of the average value of this fraction for sections of various types. Thus if the ratio of depth to span be one-tenth, the correction due to shearing is 3 per cent.

In the case of a tube considered on page 368 the factor is 2 and $n = \frac{1}{8}$, giving a ratio $2\frac{1}{2}$ times as great: the correction when the depth is one-tenth the span being $7\frac{1}{2}$ per cent.

191. *Effects of Insufficient Resistance to Shearing.*—If the central part of a beam be cut away as shown at Z in Fig. 144, the strength of the beam will be diminished and its deflection increased. This will be true even if there be only a narrow longitudinal slot at the neutral surface, but the weakening is greater the more material is cut away, the condition of the beam in an extreme case becoming that of an N girder (Art. 25) without diagonal bracing. Imperfect union of the central parts will have the same effect in a less degree: thus if two beams be laid one upon another and bolted together the strength of the compound beam will be less than that of a solid beam of the same depth. Wooden ships not unfrequently exhibit weakness due to this cause, and to counteract it diagonal riders of iron are introduced to take part of the shearing force.

Theoretical considerations would lead us to conclude that in timber beams the deflection due to shearing is relatively much increased by the flexibility of the transverse sections, the modulus of rigidity being relatively small in most kinds of wood. This conclusion, however, does not as yet appear to have been experimentally verified.

The ordinary formula for resistance to bending cannot be applied in such cases without risk of serious error, and the same remark applies with still greater force to the formula of Art. 189, which gives the distribution of shearing stress which will be determined mainly by the relative resistance to shearing of the parts of the section.

192. *Economy of Material in Girders.*—It has been shown already in Art. 159 that a certain ratio of depth to span must be best as regards economy of material, and a calculation will now be given which will illustrate this point.

Let us suppose that in order to give sufficient stiffness and stability under the action of lateral forces the mean sectional area C of the web of a flanged girder should be proportional to the shearing force on the section multiplied by the rth power of the depth h, and let A be the area of each flange, then the total area S is $2A+C$ and the moment of resistance to bending approximately,

$$M=fh\{\tfrac{1}{2}S-\tfrac{1}{3}C\}=fh\{\tfrac{1}{2}S-\tfrac{1}{3}ch^r\},$$

where c is a co-efficient. Writing this equation

$$\tfrac{1}{2}S=\frac{M}{fh}+\tfrac{1}{3}ch^r,$$

it will be seen that for a given value of M, S is least when

$$M=\tfrac{1}{2}\cdot\frac{r}{r+1}\cdot f\cdot Sh\ ;\quad C=\frac{3S}{2(r+1)}.$$

In a girder with openwork web $S=C(r+1)$, but the value of M is the same.

Assume now $F=f'\cdot C$, where F is the shear on the section and f' is a co-efficient much less than the resistance to shearing, on account of various additional straining actions (Art. 188) which have to be considered then by substitution,

$$M=\tfrac{1}{3}r\frac{f}{f'}\cdot Fh.$$

On replacing M by μFL, where L is the span and μ a co-efficient connecting the shear and the bend, the best ratio (N) of span to depth will be determined. If the load be uniformly distributed

$$N=\frac{4r}{3}\cdot\frac{f}{f'}.$$

It is probable that in most cases $r=2$ nearly, but that the value of

f/f' will vary according to the type of girder from 3 to 4 for a continuous web. For an openwork web the formula is slightly modified.

The limiting span of a girder of uniform section is readily found, proceeding as in Ex. 13, page 319, to be

$$L = \frac{4r}{r+1} \cdot \frac{\lambda}{\lambda'}.$$

The weight of a smaller girder of the same type is found as in Ch. IV.

193. *Joints and Fastenings.*—Among the most important cases of shearing are those which occur in joints and fastenings of all kinds. Such questions are generally very complex, considered as purely theoretical problems, and the direct results of experience are always required at every step to interpret and confirm theoretical conclusions.

When two pieces butt against each other the pressure is transmitted by contact only, and fastenings are therefore required not for transmission of stress but merely to retain the pieces in their relative positions. With tension it is otherwise; it is still necessary to have surfaces which press against one another, and these can only be obtained by the introduction of fastenings which transmit stress laterally, and are therefore subject to shearing and bending. The parts of a joint should be so proportioned as to be of equal strength. One of the simplest examples is that of a pin joint connecting two bars in tension as in a suspension chain with bar links. Fig. 1 (Plate VIII.) shows a pair of bars of rectangular section connected together by links C and D united as shown by pins passing through eyes at their extremities. In suspension chains there are generally four or five bars placed side by side, but the principle is the same in any case. The pull on the chain is balanced by the resistance to shearing of the pins, which have besides to resist bending. Let d be the diameter of the pins, b the breadth, t the thickness of one of the bars, t' the thickness, b' the breadth of the links which for equality of strength, that is to say, of sectional area, will be connected by the equation

$$2b't' = bt.$$

Let f be the co-efficient of strength for tension, then $\frac{4}{5}f$ (Art. 230) will be the co-efficient for shearing, whence remembering that the maximum shearing stress exceeds the mean in the ratio 4 : 3 as shown above

$$P = btf = 2\pi\frac{d^2}{4} \cdot \frac{3}{5}f = \frac{3\pi}{10}fd^2.$$

According to this estimate the area for shearing should be five-thirds the area for tension, but the true ratio is probably not so great: the calculation supposes that the sides of the pin are subject to normal stress alone, whereas the tangential stress due to friction must be considerable.

Besides the strength of iron such as is used for pins is greater than that of plates. As the calculation applies only to stress within the elastic limit, it is impossible to test it by experiment. In practice the areas are made nearly equal when nothing else is considered except resistance to shearing. When, however, such a joint is actually pulled asunder it frequently gives way in quite a different manner before shearing commences. Imagine a cylinder pressed down into a semicircular hollow which it very exactly fits, and let the material be elastic and soft compared with the cylinder, then, reasoning as in Art. 115, page 241, it appears that the stress between the surfaces will be given by the equation

$$p = p_0 \, . \cos \theta,$$

and if P be the pressing force, l the length,

$$p_0 \, . \, \tfrac{1}{4}\pi dl = P \quad \text{or} \quad p_0 = \frac{4P}{\pi dl}.$$

If the pin fits the eye exactly the pressure will follow this law so long as the tension is small. As the tension increases, however, the pressure becomes more uniformly distributed over the semi-cylinder, because the eye-hole tends to contract laterally as the links of a chain of rings would do under tension. The other extreme supposition would be to suppose it uniformly distributed, then

$$p_0 \, . \, dl = P \quad \text{or} \quad p_0 = \frac{P}{dl}.$$

The actual pressure will be intermediate between these two values. If p_0 be too great the metal crushes under the pressure. The theoretical limit to p_0 will be considered hereafter; for the present it will be sufficient to say that the experiments of Sir C. Fox* have shown that the curved area should be at least equal to the sectional area under tension, that is to say we ought to have

$$\tfrac{1}{2}\pi dl = bt = \tfrac{1}{10} 3\pi d^2.$$

To satisfy these conditions we must have for the ordinary case where the thickness of the eye is the same as that of the rest of the bar

$$d = \tfrac{2}{3}b : t = \tfrac{2}{5}b \text{ approximately.}$$

The first of these gives the diameter of pin recommended by Sir C. Fox and other authorities; the second gives the greatest thickness of link for which this diameter gives sufficient resistance to shearing, but the thickness in actual examples of suspension links is generally considerably less. The pin has also to resist bending, but of small amount in the present example. The sides and end of the eye are subject to tension, but it is not uniformly distributed, the question being similar to that of a thick hollow cylinder under internal fluid pressure. The mode in which the eye crushes and then fractures transversely by tension, is shown in Plate VIII., and further described in Chapter XVIII.

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* Proceedings of the Royal Society, vol. xiv., p. 139.

In rivetted joints the question is further complicated by the friction between the plates united by the rivets. On the subject of joints and fastenings the reader is referred to Prof. W. C. Unwin's *Machine Design*.

EXAMPLES.

1. Find the diameter of a shaft for a twisting moment of 1,000 inch tons; stress allowed being 3½ tons per square inch. *Ans.* Diameter = 11·34″.

2. From the result of the previous question deduce the diameter of a shaft to transmit 5,000 H.P. at 70 revolutions per minute. Maximum twisting moment = $\frac{9}{8}$ the mean. *Ans.* 16·37″.

3. The angle of torsion of a shaft is not to exceed 1° for each 10 feet of length. What must be the diameter for a twisting moment of 100 inch-tons modulus of transverse elasticity, 10,500,000?

Compare the result with the diameter determined from consideration of strength, taking a co-efficient of 3¼ tons. *Ans.* Diameter determined from consideration of stiffness = 6·2″. Diameter from consideration of strength = 5·2″.

4. Show that the resilience of a twisted shaft is proportional to its weight.

$$\textit{Ans.}\ \text{Resilience}\ \tfrac{1}{2}Ti = \frac{f^2}{C} \times \frac{\text{Volume}}{4}.$$

5. Compare the strengths of a solid wrought-iron shaft and hollow steel shaft of the same external diameter assuming the internal diameter of the hollow shaft half the external, and the co-efficient for steel 1½ times that for iron. *Ans.* 32/45

6. The external diameter of a hollow shaft is double the internal. Compare its resistance to twisting with that of a solid shaft of the same weight and material.

$$\textit{Ans.}\ \text{Strength is greater in the ratio } \frac{5\sqrt{3}}{6} = 1\cdot443.$$

7. A pillar, whose sectional area is 1½ square feet, is loaded with two tons. Find in lbs. per square inch the intensity of the tangential stress on a plane inclined at 15° to the axis of the pillar. *Ans.* tangential stress = 5·18 lbs.

8. In a single rivetted lap joint, the pitch of the rivets being three diameters or six times the thickness of the plates, find, 1st, the mean stress on the reduced area; 2nd, the shearing stress on the rivets; and, 3rd, the mean direct stress between rivet and plate: the tension of the joint being 4 tons per square inch of the original area, and the friction between the two surfaces of the plate in contact neglected.

Ans. Mean tension on reduced area - = 6 tons.
Shearing stress on rivet - - = 7·6 tons.
Mean direct stress $\frac{4 \times \text{pitch} \times \text{thickness}}{\text{diameter} \times \text{thickness}} = 12$ tons per sq. in.

9. In a beam of I section with flanges and web which may be considered as rectangles, the thickness of each flange is one sixth the outside depth of the beam, and the breadth twice the thickness. The thickness of the web is half that of the flanges: find the ratio of maximum to mean shearing stress on the section. *Ans.* $\frac{18}{7}$.

10. In the last question find the fraction of the whole shearing force which is taken by the web. *Ans.* 80 per cent.

11. Find the moment of resistance and angle of torsion of an iron bar 1 inch square, 5 feet long, assuming $f = 3\tfrac{1}{4}$, $C = 5{,}000$ in. tons. *Ans.* $T = \cdot677$ inch tons. $i = 3°$.

12. Find a formula for the resilience, under torsion, per cubic inch of a bar of rectangular section.

$$\textit{Ans.}\ \frac{5f^2}{3C}\cdot\frac{1+\beta^2}{\{3+1\cdot8\beta\}^2}$$

13. Show that the weight in lbs. of a shaft to transmit a given horse power at a given number of revolutions is

$$W = 21{,}000 \cdot \frac{K \cdot HP}{N\lambda} \cdot \frac{l}{d},$$

the value of λ being given as in Ch. XVIII., the proper co-efficient of resistance to shearing being used. The rest of the notation is explained on page 356.

The distance to which power can be transmitted by shafting with a given loss by friction is given by Ex. 18, p. 262, when the angle of torsion is immaterial, but in practice is generally limited by the necessity of having sufficient stiffness. The bending and twisting of shafts is considered in Chapters XVII., XVIII.

14. If a bar of square section be sheared diagonally show that the mean shearing stress on the neutral surface is equal to the mean shearing stress on the section. Also find where the mean shearing stress on a longitudinal section parallel to the neutral surface is a maximum and the ratio of maximum to mean. *Ans.* At a distance from the neutral surface equal to one-eighth the depth. Ratio = 1·125.

Note.—If the shear on the transverse section in a direction perpendicular to the neutral axis be assumed uniform at points lying on a line parallel to the neutral axis, the maximum shear will be $1{\cdot}125\sqrt{2}$, or about 1·6 times the mean at points lying on the edges of the section.

15. A crank arm of rectangular section 6 in. × 12 in. is acted on by a twisting moment of 300 inch-tons, find the stress produced at (1) the middle of the long side and (2) the middle of the short side in tons per square inch. *Ans.* $q_1 = 2{\cdot}78$; $q_2 = 2{\cdot}25$.

16. In the last question, suppose the section further to be acted on by bending moments of 100 inch-tons in the plane of the crank, and 150 inch-tons about the shorter axis, find the normal stress produced at the points mentioned.

CHAPTER XVI.

IMPACT AND VIBRATION.

194. *Preliminary Remarks. General Equation of Impact.* Hitherto the forces applied to the body or structure under consideration have been imagined to have been originally very small, and to have increased gradually to their actual amount. This is seldom exactly the case in practice, while it frequently happens that the load is applied all at once, or that it has a certain velocity at the instant it first comes in contact with the body. Such cases may all be included under the head of IMPACT, and will form the subject of the present chapter.

When a body in motion comes into contact with a second body against which it strikes, a mutual action takes place between them which consists of a pair of equal and opposite forces, one acting on the striking body, the motion of which it changes, the other on the body struck, which it in general moves against some given resistance. Certain changes of figure and dimensions, or, in other words, **strains** are likewise produced in both bodies, in consequence of the stress applied to them.

The simplest case is where the impact is direct and the resistance to motion has some definite value, as, for example, where a pile is driven by the action of a falling weight. Here let R be the resistance which the pile offers to be driven, that is to say, the load which, resting steadily on the pile, would just cause it to commence to sink; let W be the falling weight, h the height from which it falls, x the space through which the pile sinks in consequence of the blow; then the mutual action between the pile and the weight at the instant of impact consists of a pair of equal and opposite forces R. The whole height through which the weight falls is $h+x$, and the space through which the resistance is overcome is x; hence, equating energy exerted and work done, we have

$$W(h+x)=Rx.$$

This equation shows that any resistance, however great, can be overcome by any weight, however small; and also, that the force of the blow, as measured by the space the pile is driven, is proportional to its energy. We have however assumed that the whole energy of the blow is employed in driving the pile, whereas some of it will be expended in producing vibrations and in damaging the head of the pile and the bottom of the weight. As the pile is driven deeper, the resistance to being driven increases, and at length becomes equal to the crushing stress of the material: the pile then sinks no farther, the whole of the energy of the blow being wasted in crushing.

This last is also the case of impact of a flying shot against a soft plastic substance, which exerts during deformation a definite force uniform or variable which brings the weight to rest in a certain space. Suppose V the velocity of the shot, x the space, and R the mean resistance which the substance offers, then the kinetic energy of the shot is $WV^2/2g$, while the work done is Rx, equating which

$$W.\frac{V^2}{2g}=Rx.$$

Here the whole energy of the blow is spent in producing changes of figure in the body struck; but if the striking body had been soft, and the body which it struck hard and immovable, the energy of the blow would have been employed in producing change in the shape of the striking body. Thus we may write down as the general equation of impact

Energy of blow = Work done in overcoming the resistance to movement of the body struck
+ Work done in the internal changes in the striking body
+ Work done in internal changes in the body struck.

Which of these three terms is the most important will depend on the relative magnitude of the resistance to movement, and the crushing stress of the materials of the two bodies. If either body have a sensible motion after impact, the corresponding kinetic energy must be taken account of in writing down the equation, as will be seen farther on.

195. *Augmentation of Stress by Impact in Perfectly Elastic Material.*—We now proceed to apply the equation to the case which most immediately concerns us, namely, that of impact on perfectly elastic material, including in this the effect of a load which is applied all at once.

We will suppose a structure or piece of material of any kind resting on immovable supports, and struck by a body harder than itself, so that we may neglect all changes produced in the striking body. Generally in both bodies there will also be produced vibrations, of the nature of those constituting sound, which absorb a certain amount of energy, but this we shall neglect. The whole energy of the blow then is supposed expended in straining the structure, or piece of material, struck by the blow.

Now the effect of impact is to produce a mutual action S, which represents a force applied to the structure at some definite point. In consequence of this the structure suffers deformation, and the point of application moves through a space x. The resistance to deformation is proportional to x, because the limit of elasticity is not exceeded; it therefore commences by being zero, and increases gradually till the velocity of the striking body is wholly destroyed. The mean value of the resistance is therefore one-half its maximum value. During the first part of the period occupied by the impact the mutual action S is greater than the resistance, and during the second part less, as will be explained fully presently; but, when the maximum strain has been produced, the mean value during the whole period must be exactly equal to the mean resistance, the weight and the structure being at rest. The state of rest is only momentary, for the strained structure will immediately, in virtue of its elasticity, commence to return to its original form; but for the moment, a strain has been produced, which is a measure of the effect of the blow, and which must not exceed the powers of endurance of the material.

Let now R be the maximum resistance, and let the blow consist in the falling of a weight W, through a height h above the point where it first comes in contact with the structure; then $h+x$ is the whole height fallen through, and it follows from what has been said that

$$W(h+x)=\tfrac{1}{2}Rx.$$

The resistance R may also be described as the "equivalent steady load," being the load which, if gradually applied at the point of impact, would produce the same stress and strain which the structure actually experiences. We most conveniently compare it with W by supposing that we know the deflection δ which the structure would experience if the striking weight W were applied as a steady load at the point of impact: we then have, since the limits of elasticity are not exceeded,

$$\frac{x}{\delta}=\frac{R}{W}.$$

Substituting the value of x we get

$$\frac{R^2}{W^2}=\frac{2R}{W}+\frac{2h}{\delta}.$$

Let the height h be n times the deflection δ, then solving the quadratic, the positive root of which alone concerns us,

$$R = W(1 + \sqrt{2n+1}),$$

an equation which shows how the effect of a load is multiplied by impact.

196. *Sudden Application of a Load.*—A particular case is when $h = 0$, then $R = 2W$. So that if a load W is suddenly applied to a perfectly elastic body, from rest, not as a blow, it will produce a pressure just twice the weight. This case is so important that we will consider a special example in detail.

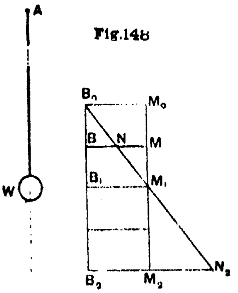


Fig. 148

Let a long elastic string be secured at A. If a gradually increasing weight be applied the string will stretch, and the weight descend. Let the load required to produce any given extension be represented by the ordinates of the sloping line B_0NN_2 (Fig. 148). Next, instead of applying a gradually increasing load, let a weight W represented by B_0M_0 be applied all at once to the unstretched string. The string will of course stretch, and the weight descend. When it has reached B (Fig. 146) the tension of the string pulling upwards, being represented by BN, will be less than W acting downwards. Moreover, in the descent B_0B an amount of energy has been exerted by the weight represented by the area of the rectangle B_0M_0MB. At the same time the work which has been done in stretching the string is represented by the area of the triangle B_0NB. The excess of energy exerted over work done has been employed in giving velocity to the descending weight, and is stored as kinetic energy in the weight.

On reaching B_1, the tension of the string is just equal to the weight, but the stretching does not cease here. The weight has now its greatest velocity, which corresponds to an amount of kinetic energy represented by the triangle $B_0M_0M_1$. Although any further extension of the string causes the upward pull of the string to be greater than the weight W, yet the weight will go on descending until the energy that it has exerted is equal to the work done in stretching the string; then the kinetic energy will be exhausted and the weight will be brought to rest. This will occur when the area of the triangle $B_0N_2B_2$ equals the area of the rectangle $B_0M_0M_2B_2$, that is when $B_2N_2 = 2B_2M_2$, or $B_0B_2 = 2B_0B_1$.

We thus see that the tension of the string produced by the sudden application of the load is twice that due to the same load steadily applied.

The string will not remain extended so much as B_0B_2, for now the upward pull of the string, exceeding the weight, will cause it to rise. On reaching B_1 it will have the same velocity upwards that it had on first reaching B_1 downwards. This will carry it to B_0, from which it will again fall, and so on. Practically the internal friction due to imperfect elasticity, and the resistance of the air, will soon absorb the energy and bring the weight to rest at B_1.

197. *Action of a Gust of Wind on a Vessel.* Another interesting example of the way in which the sudden application of a load augments its effect is furnished by the case of a vessel floating upright in the water and acted on by a sudden gust of wind, a question which, though not strictly belonging to this part of the subject, involves exactly the same principle.

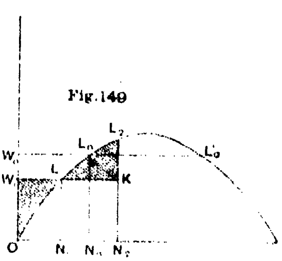


Fig. 149

First, suppose no wind pressure, but that a gradually increasing couple is applied to heel the vessel.

If along a horizontal line (Fig 149) angles of heel be marked off, such as ON, and for those points ordinates such as NL, are set up to represent on some convenient scale the magnitude of the couple required to produce that angle of heel, a curve OL will be obtained, which we have already (p. 184) called the curve of *Statical Stability* of the ship.

Now suppose a steady wind pressure to be gradually applied. It will produce on the masts and sails a definite moment, on account of which the ship will incline to a certain angle, such that the ordinate of the curve of stability corresponding to that angle will represent the moment of the wind pressure. So long as the wind is constant, she will remain inclined at that angle. Next suppose the same wind pressure to be suddenly applied all at once, as by a gust to the ship floating upright at rest. The ship will heel over, and until she is inclined to some extent the wind moment will be greater than the righting moment, and the excess will cause the ship to acquire an angular velocity. Accordingly, when she arrives at the angle of heel corresponding to the moment of wind pressure on the stability curve, she does not come to rest, but inclines farther, until the energy exerted by the wind pressure is all taken up in overcoming the righting moment through the angle of inclination. The work thus done is represented by the area of the curve of stability standing above the angle of heel reached.

Let OW_1 represent the magnitude of the wind moment. The ship will incline until the area OL_2N_2 = area OW_1KN_2, or area OW_1L_1 = area L_1L_2K; that is, if the moment of wind pressure remains undiminished as the ship heels, which will hardly be true in practice. Suppose the moment of wind pressure OW_0 to be such that the area OW_0L_0 = the area $L_0L_2L'_0$. In this case the sudden gust of wind will carry the ship to such an angle ON'_0 that she will not again return; and the smallest additional pressure of wind will capsize the ship, although that same wind pressure applied gradually would incline the ship to the angle ON_0 only.

198. *Impact at High Velocities. Effect of Inertia.*—Returning to the general case of impact against a perfectly elastic structure (Art. 195), let us now take the other extreme case in which the height through which the weight falls is great compared with the deflection δ due to the same weight gradually applied; then, since n is great, our equation becomes

$$R = W . \sqrt{2n} = W\sqrt{\frac{2h}{\delta}},$$

which may be written in either of the forms

$$R = \sqrt{\frac{2W}{\delta}} \cdot \sqrt{Wh}\ (1)\text{; or } x = \sqrt{2h\delta} \quad (2).$$

The first form shows that the stress produced by the impact is proportional to the square root of the energy of the blow, and the second, that the deflection occasioned by the fall of a given weight is proportional to the square root of the fall, or, what is the same thing, to the velocity of impact. These results are exact when the impact is horizontal, and the last has been verified by experiment. It is to be remembered that the limits of elasticity are supposed not to be exceeded; when a rail or carriage axle is tested by a falling weight, as is very commonly done, the energy of the blow is generally much in excess, and the piece of material suffers a great permanent set, the resistance is then approximately constant instead of increasing in proportion to the deflection. The effect of the blow is then more nearly directed proportional to its energy. It will be seen presently how small a blow matter is capable of sustaining without injury to its elasticity.

The effect of a blow, on a structure or piece of material as a whole, is diminished, on account of its inertia, by an amount which is greater the greater the velocity of impact, but which varies according to the relative mass and stiffness of its parts. In the act of yielding the parts of the body are set in motion, and the force required to do this is frequently greater than the crushing strength of the materials, so that a part of the energy of the blow is spent in local damage near the point of impact.

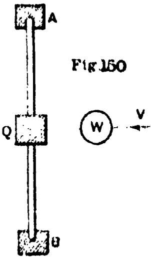


Fig 150

Figure 150 shows a narrow deep bar AB, the ends of which rest in recesses in the supports, which prevent them from moving horizontally, but do not otherwise fix them. The bar carries a weight Q in the centre, against which a second weight W moving horizontally strikes with velocity V. The bar being very flexible horizontally, the weight Q at the first instant of impact moves as it would do if free; that is, the two weights move onwards together with a common velocity v fixed by the consideration that the sum of the momenta of the two weights is the same before and after impact, so that

$$WV = (W+Q)v.$$

The energy of the two weights after impact is

$$(W+Q)\frac{v^2}{2g} = \frac{W^2}{W+Q}\cdot\frac{V^2}{2g},$$

showing that the energy of the blow has been diminished in the proportion $W : W+Q$. The loss is due to the expenditure of energy in damage to the weights.

If now, instead of a weight Q attached to the centre of a flexible bar, we suppose the bar less flexible and of weight Q, the effect of the blow is diminished by the same general cause, but not to the same extent: the diminution may be estimated by replacing Q in the preceding formula by kQ, where k is a fraction to be found approximately by calculation (Ex. 8, p. 399), or determined by experiment. In a series of elaborate experiments made by Hodgkinson on bars struck horizontally by a pendulum weight, it was found that k was $\frac{1}{2}$.

We are thus led to separate the energy of a blow into two parts:

$$E_1 = \frac{W^2}{W+kQ}\cdot\frac{V^2}{2g} : E_2 = \frac{k\,.\,WQ}{W+kQ}\cdot\frac{V^2}{2g}.$$

The first of these strains the structure or piece of material as a whole, and the second does local damage at the point of impact. Hence the great difference which exists between the effect of two blows of the same energy, one of which is delivered at a low, and the other at a *high velocity. At high velocities most of the energy is expended in local damage; at low velocities most is expended in straining the* structure as a whole.

If the body which is struck be in motion, instead of resting on immovable supports, as in Fig. 150, the energy of the blow will be diminished. This case has been considered in Ch. XI., p. 270, where it is shown that the energy of the collision is

$$E = \frac{WQ}{W+Q} \cdot \frac{V^2}{2g},$$

where V is the relative velocity of the bodies. Of this a part—represented, as before, by replacing Q by kQ—is spent in local damage and the rest in straining the structure as a whole.

The exceptional case where, as in the collision of billiard balls, the limit of elasticity is not exceeded at the point of impact, need not be here considered. The energy of local damage is, then, not wholly dissipated in internal changes: a part is recovered during the restitution of form which occurs in the second part of the process of impact, and increases the action on the structure as a whole. In ideal cases the whole may be thus recovered, but, in practice, a portion is always employed in producing local vibrations, and finally dissipated by internal friction.

199. *Impact when the Limits of Elasticity are not Exceeded. Resilience.*
The effect of impact on perfectly elastic material may also be dealt with by considering the amount of energy stored up in the body in consequence of the deformation which each of its elementary parts have suffered. We have already seen that when a piece of material is subjected to a simple uniform longitudinal stress of intensity p, the amount of work U done by the stress is

$$U = \frac{p^2}{2E} \times \text{Volume}.$$

Let w be the weight of a unit of volume of the material, and W the weight of the body considered, then we may write our question

$$U = W . H$$

where H is a certain height given by

$$H = \frac{p^2}{2Ew},$$

and the whole elastic energy of the body may be measured by this height, which is the distance through which the body must fall to do an equivalent amount of work.

If for p we write f the elastic strength of the material, then we obtain what we have already called the Resilience of the body, and H becomes what we may call the "height due to the resilience," which, for each material, has a certain definite value, given in feet in Table II., Ch. XVIII., for various common materials.

Now in cases of impact where the limit of elasticity is not exceeded, the whole energy of the blow is spent in straining the material or structure, and hence that energy must not, in any case, exceed the resilience. Thus, on reference to the table, it will be seen that in ordinary wrought iron the height is given as 2 ft. 9 in., from whence it follows that in the most favourable case a piece of iron will not stand a blow of energy greater than that of its own weight falling through about 3 feet, without being strained beyond the elastic limit. If the parts of the body are subject to torsion, about 50 per cent. may be added to these numbers, but, on the other hand, they are subject to large deductions on account of the inequality of distribution of stress within the body. Only a portion of the body is subjected to the maximum stress, the rest is strained to a less degree, and consequently has absorbed a less amount of the energy of the blow. Thus, for example, a beam of circular section, even though it be of "uniform strength" (Art. 161), has only one-fourth the resilience of a stretched bar of the same weight, because it is only the particles on the upper and lower surfaces which are exposed to maximum stress, the central parts having their strength only partially developed.

We now draw two very general and important conclusions.

(1) When a body or structure is exposed to a blow exceeding that represented by its own weight falling through a very moderate height, a part, or the whole, is strained beyond the elastic limit.

(2) When a body or structure is not of uniform strength throughout the excess of material is a cause of weakness.

On reference to Table II., Ch. XVIII., it will be seen that an exception occurs to the first principle in the case of the hardest and strongest steel; but, as a rule, the property of ductility or plasticity is essential to resistance to impact. Bodies which do not possess it are generally brittle. In good ductile iron and soft steel the non-elastic part of the resistance to impact will be seen hereafter to be at least 1,000 times the elastic part, assuming both equally developed through all parts of the material. These remarks apply to a single blow; the effect of repetition will be considered hereafter.

As an example of the application of the second principle we may mention the bolts for armour plates invented by the late Sir W. Palliser. In these bolts the shank is turned down to the diameter of the base of the thread so as to be of equal strength throughout. (See Ex. 4, p. 305.)

200. *Free Vibrations of an Elastic Structure.*—If a structure be loaded within the limit of elasticity and the load be suddenly removed, the elastic forces being unbalanced set the structure in motion and vibration ensues. The vibrations are described as "free" being uninfluenced by any external cause and take place in times which depend only on the inertia of the structure and the intensity of the elastic forces, while their extent is arbitrary being fixed by the magnitude of the original deformation. In the absence of friction the total energy of the structure must remain constant, a principle expressed by the equation

Kinetic Energy + Elastic Energy = Constant.

The effect of friction is gradually to dissipate the energy so that the vibrations speedily die out unless kept up by external forces. This action, however, is for the present neglected.

The simplest kind of vibration is that in which the deformation is of such a character that the elastic energy can be expressed in terms of a single varying quantity which may be either linear as in the deflection of a beam (page 332), or angular as in torsion (page 361). In all such cases, as will be seen on reference to the pages cited, the elastic energy is cz^2 where z is the varying quantity and c a constant co-efficient. Also the different points of the structure have velocities which are in a fixed proportion to each other, and also to dz/dt the rate of change of the varying quantity in question. This rate of change may be described as the velocity of vibration and denoted by V. The kinetic energy will therefore be bV^2 where b is a second constant co-efficient, and the equation of energy becomes

$$bV^2 + cz^2 = \text{Constant}.$$

This kind of motion has already been studied in Art. 103, Chapter VIII., and on reference to page 204 it will be seen that the period of vibration (T_0) is given by

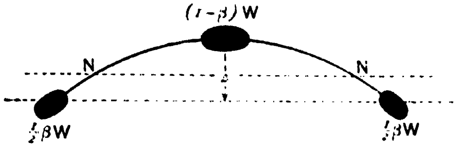
$$T_0 = 2\pi\sqrt{\frac{b}{c}},$$

a rule which includes both the simple examples there considered and applies to all cases.

Whatever kind of vibration is dealt with the process of determining T_0 is very similar, and the first example to be considered is that of the vibration of a loaded bar.

(1) In Fig. 151 a long flexible elastic bar is shown, to the middle

and ends of which weights are attached: the fraction $1-\beta$ of the whole weight W being placed in the middle, and the fraction $\frac{1}{2}\beta$ at each end. The bar is slightly bent into an elastic curve in a horizontal plane and then left to itself, being supposed resting on



a smooth table or suspended by vertical strings from two points NN called "nodes" which remain at rest during the motion, lying as they do in a line passing through the centre of gravity of the weights. The weight of the bar itself is supposed small enough to be neglected.

The bar at any instant will be bent into a curve which is the same as the deflection curve of a beam supported at the ends and loaded in the middle; hence if z be the versed sine of that curve

$$\text{Elastic Energy} = 24 \, . \, z^2 \frac{EI}{l^3} \quad \text{(p. 332).}$$

The weights βW, $(1-\beta)W$ are at distances $(1-\beta)z$ and βz respectively from NN, and their velocities are therefore $(1-\beta)V$ and βV respectively, where V or dz/dt is the relative velocity. Hence

$$\text{Kinetic Energy} = \frac{W\beta(1-\beta)^2 + W\beta^2(1-\beta)}{2g} V^2,$$

$$= \frac{W\beta(1-\beta)V^2}{2g}. \quad \text{(See also p. 270.)}$$

The equation of energy is therefore

$$W\beta(1-\beta)\frac{V^2}{2g} + \frac{24EI}{l^3} \, . \, z^2 = \text{Constant},$$

whence applying the general rule given above,

$$T_0 = 2\pi\sqrt{\frac{\beta(1-\beta)Wl^3}{g \, . \, 48EI}}.$$

The period is obviously unaffected by placing additional weight at the nodes for these points are at rest. Suppose then that of the total load W the fraction $(1-a)W$ is placed at the nodes and the remainder aW distributed as before, we shall then have

$$T_0 = 2\pi \, . \, \sqrt{\frac{\beta(1-\beta)}{48g}} \, . \, \sqrt{\frac{aWl^3}{EI}}.$$

It is often more convenient to consider the "frequency," that is, the number (N_0) of vibrations per second. The vibrations considered are *complete* including both a forward and a backward movement; thus in a reciprocating piece driven by a crank the frequency is the number of revolutions per second of the crank. On this understanding

$$N_0 = \frac{1}{T_0} = K\sqrt{\frac{EI}{Wl^3}},$$

where K is a numerical coefficient depending on the distribution of the load. The smallest value K can have is 12·5, which occurs when $\alpha = 1$, $\beta = \frac{1}{2}$, but if half the load be concentrated at the nodes the result is increased more than 40 per cent., becoming 17·7.

If the moment of inertia I_x vary with x the distance from one end of the section considered, then I must be understood to refer to the middle section and K must be divided by a numerical factor X, the square of which is given by the equation $\{a = \frac{1}{2}l\}$,

$$X^2 = 3\int_0^a \frac{x^2 I}{a^3 I_x} \cdot dx,$$

derived from the formula for the elastic energy given on page 331, in terms of the bending moment M which in this case varies as x. If I_x increases on going from the ends to the middle, this divisor is greater than unity and the value of K is diminished; if, for example, $I_x \propto x$ the divisor is $\sqrt{1 \cdot 5}$ or 1·21.

In Fig. 152 the bar is loaded at the ends and two intermediate points: the two halves are then bent in opposite directions and the bar left to itself. There are now three nodes NNN instead of two as in the preceding case. The time of vibration may be investigated as before, when it will be found that the same formula applies but with a greater value of K. Similarly there may be bending vibrations with four or more nodes, the vibration being quicker the greater the number.

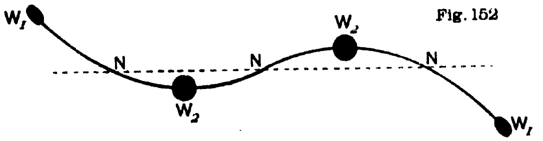


Fig. 152

If the weight be distributed continuously instead of being concentrated in given points, the formula for the frequency is still of the same form, but the calculation of K is more difficult. The case of a uniform bar has been thoroughly studied by writers on *Acoustics*, and in Lord Rayleigh's *Treatise on Sound* full details will be found. With no other load than its own weight the value of K is about

20·2 for two nodes, instead of 17·7 as found above in the case of a concentrated load, and for a greater number of nodes i about $2{\cdot}225\ \{2i-1\}^2$.

If the transverse section of the beam be of sensible magnitude a correction for "rotatory inertia" is necessary since these sections have a motion of rotation as the beam bends and unbends. The value of K is evidently diminished by this cause and in the case of a vessel or other larger girder-like structure the correction would be considerable.

The case of a vessel has of late attracted considerable attention and the value of the constant C in the formula,

$$N = C\sqrt{\frac{I}{Wl^3}}$$

has been determined experimentally by Herr Otto Schlick* who gives for

Torpedo Boat Destroyers, $C = 157{,}000$.
Large Mail Steamers, $C = 143{,}500$.
Merchant Steamers, $C = 128{,}000$.

These values are for *complete* vibrations per minute with two nodes, hence assuming $E = 10{,}000$ tons (see Art. 224) the corresponding values of K are about 26, 24, and 21 respectively. They increase with the fineness of lines of the vessel and, as might be expected, differ from the value (20·2) for a uniform rod, partly from the causes already pointed out, and partly from the influence of the water in which the vessel floats which virtually increases her inertia. Some further remarks on these points will be found in the Appendix.

(2) Let us next consider a weight W resting on an elastic platform or support of any kind, and let δ be the deflection which may be calculated by methods explained in previous chapters of this work, or if convenient may be found by observation. Let the weight and inertia of the structure itself be neglected for the present, then the equation of vibration will be as before

$$\frac{WV^2}{2g} + cz^2 = \text{Constant}.$$

Now cz^2 is the elastic energy of the structure, and therefore putting $z = \delta$

$$c\delta^2 = \tfrac{1}{2}W\delta.$$

Thus c is determined in terms of δ, and by the general rule already employed (p. 386),

$$T_0 = 2\pi\sqrt{\frac{\delta}{g}}$$

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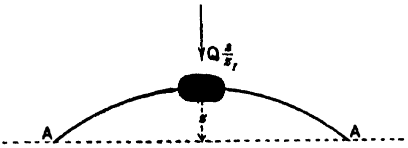
* *Transactions of the Institution of Naval Architects*, vol. XXXV., 1894.

showing that the period is the same as that of a simple pendulum (p. 204) of length δ reckoned in feet: a rule of very general application being true for example for a beam with ends either fixed or free, loaded with a weight placed at any point; or for a weight suspended by a spiral spring. The formula differs only in form from that previously obtained on page 387 for the case of a loaded bar, and may be deduced from it by putting $a = 1$ and replacing $\beta(1 - \beta)W$ by W, to represent the case where the ends are fixed by concentrating a very heavy load there.

(3) In similar structures the deflection due to a given load similarly placed varies inversely as the linear dimensions, and therefore the period of free vibration varies inversely and the frequency directly as the square root of the dimensions. An analogous rule applies to vessels, for in similar vessels EI/W varies as the length, and therefore EI/Wl^3 varies inversely as the square of the length. Hence in similar vessels the frequency of free vibration varies inversely as the length.

201. *Forced Vibrations.* If a structure be subject to a load of invariable amount the only vibrations which can occur are of the kind described in the last article as "free," and the periods are perfectly definite. But if the magnitude of the load be subject to a periodic change the deformations of the structure will also change, corresponding vibrations being set up which may be described as "forced." The period of such forced vibrations is that of the load, while their extent depends on the relation which that period bears to the period of free vibration. When the varying load has acted upon the structure for any considerable time these forced vibrations alone exist and in any case admit of being separately studied.

Fig. 153 shows a long flexible bar loosely fixed at the ends carrying a weight W concentrated in the middle. The weight is vibrating under the action of a force which goes through a periodic change being always proportional to the distance z of the vibrating weight from the central line. The maximum value of z being supposed z_1, and that of the force Q the actual force in any position will be Qz/z_1. We proceed to calculate z_1 supposing that we know the



period of free vibration (T_0) and the period of the varying load (T). It will be seen presently that if $T < T_0$, Q must act inwards and this case is indicated in the figure, whereas if it act outwards $T > T_0$.

Taking the first case

$$\text{Potential Energy} = \tfrac{1}{2} z^2 \left\{ \frac{48EI}{l^3} + \frac{Q}{z_1} \right\},$$

of which the first term is the elastic energy of the bent bar and the second represents energy derived from the force Q.

If now V be the velocity of the vibrating weight the equation of energy will be

$$\frac{WV^2}{2g} + \frac{1}{2}\left(\frac{48EI}{l^3} + \frac{Q}{z_1} \right) z^2 = \text{Constant}.$$

If Q were zero we should obtain the equation of free vibration from which it only differs in the co-efficient of z^2. The general rule (p. 386) already stated therefore gives

$$\frac{T_0^2}{T^2} = \frac{\dfrac{48EI}{l^3} + \dfrac{Q}{z_1}}{\dfrac{48EI}{l^3}} \qquad 1 + \frac{z_0}{z_1},$$

the quantity z_0 in the right hand equation being the deflection due to a steady load Q. Hence the extent of the vibration is determined by

$$z_1 = \frac{z_0}{\dfrac{T_0^2}{T^2} - 1}.$$

If $T > T_0$ z_1 becomes negative, the interpretation of which is that Q must then be taken as acting outwards instead of inwards. It then operates as a resistance to vibration, and thus by diminishing the intensity of the forces restoring equilibrium increases the period.

If $T = T_0$ the periods are said to be "synchronous." The effect of synchronism is that a force Q, however small, produces vibrations of indefinite extent: energy accumulating at each repetition of the force. This result is limited in practice by the effect of friction which absorbs the energy as fast as it is supplied.

On reference to Art. 103 it will be seen that the value of z is $z_1 \cos\theta$ where θ is the angle made with the central line by a uniformly rotating radius. Or if t be the time reckoned for convenience from the instant when $\theta = 90^\circ$,

$$z = z_1 \,.\, \sin 2\pi\frac{t}{T}\,; \quad S = Q \,.\, \sin 2\pi\frac{t}{T},$$

where S is the force needful to keep up the vibration. The effect of friction is to diminish the extent of vibration and to cause it to lag behind the variation of the force.

The results of this article are applicable directly to any case in which the inertia of the structure can be regarded as concentrated in a point to which the varying force is applied. When the inertia is distributed the value of z_0 will be reduced, but the general character of the results remains the same; there is always a certain critical speed or speeds at which by synchronism with some particular mode of vibration of the structure, excessive vibration is produced by a load which, if steady, would have no sensible effect. Any approach to these speeds must of course so far as possible be avoided.

If the load be originally resting quietly on the structure and then begin to fluctuate, the resulting vibration will in the first instance be a combination of the forced vibration of period T with a free vibration of period T_0; which will be represented by the equation

$$z = z_1\left\{\sin 2\pi\frac{t}{T} - \frac{T_0}{T}.\sin 2\pi\frac{t}{T_0}\right\},$$

the extent of the free vibration being determined by the consideration that dz/dt is zero when t is zero. When the periods are commensurable this represents vibrations of varying amplitude recurring in regular phases; but as before stated the free vibrations will generally be speedily extinguished by the effect of friction.

202. *Examples of Fluctuating Loads.*—Let us now consider some examples.

(1) The reciprocating parts of machines, especially steam engines, give rise to periodic forces the magnitude of which has already been investigated in Art. 144, p. 283, the period being a revolution. If N be the revolutions per second, it appears from the formulæ there given that the maximum value Q of the force arising from a reciprocating piece of weight W and stroke $2a$ is

$$Q = W\frac{a}{h} = W.\frac{4\pi^2}{g}N^2a.$$

To fix our ideas imagine the engine to stand upon a horizontal platform, the time of free vibration of which is T_0 corresponding to a frequency N_0. Then when the engine is working the extent of the vibration will be

$$z = \frac{z_0}{1 - \frac{N^2}{N_0^2}},$$

in which the quantity z_0 will be the deflection produced by Q when resting quietly upon the platform if the weight of platform and engine can be regarded as concentrated below the cylinder. The vibration becomes excessive when N approaches N_0.

In the case of a vessel the revolutions (N) of the engines in similar vessels at corresponding speeds vary inversely as the square roots of the lengths, while the frequency of free vibration as already pointed out varies inversely as the length, hence the ratio N/N_0 varies as the square root of the length. In small vessels the revolutions are not generally sufficient to produce vibration of sensible amount, but in large vessels vibration with two, three, and sometimes with four nodes occurs. The same is the case in torpedo boats on account of their excessive speed. According to Herr Schlick the revolutions must not approach the frequency of free vibration within 10 or 12 per cent.

Vibration due to this cause may in great measure be avoided by a proper system of balancing as has been explained in the article already cited. The example there considered is that of a locomotive in which the necessity for balancing arose at a very early stage, and its principles, therefore, have been long understood. For an approximately perfect balance, as there pointed out, the alternating couples must be considered as well as the alternating forces. In vessels the extent to which vibration is due to the reciprocating parts of the engines has only recently been recognized, and the subject of balancing has acquired great importance.

(2) When a vessel rolls in still water her period of unresisted rolling (page 207) is

$$T_0 = 2\pi \cdot \sqrt{\frac{r^2}{mg}},$$

where m is the metacentric height, and r the radius of gyration. Suppose now a weight Q, say of a number of men, be moved from the centre line to one side of a vessel at rest: the corresponding angle of heel (ϕ_0) will be given by the equation

$$Q \cdot \frac{B}{2} = Wm \tan \phi_0 = Wm\,\phi_0, \text{ nearly},$$

where B is the beam and W the displacement. Let now the men move backwards and forwards from port to starboard and back again, the period of the double movement being T. The result will be forced rolling of period T and extent ϕ_1 (suppose) which in the first instance will be accompanied by free rolling in period T_0. The free rolling, however, may be supposed to have been extinguished by hydraulic resistance. Assuming this, at any angle of heel ϕ there will be a couple due to the men given by

$$Q\frac{B}{2} \cdot \frac{\phi}{\phi_1} = Wm\frac{\phi_0}{\phi_1} \cdot \phi, \text{ nearly}.$$

If $T > T_0$ the men will always be moving outward when that side of the vessel is below the horizontal, and the moment due to them will diminish the righting couple so as to lengthen the natural period. If $T < T_0$ the converse will hold. The extent of the rolling will be approximately (p. 391),

$$\phi_1 = \frac{\phi_0}{1 - \frac{T_0^2}{T^2}}.$$

The motion before free rolling is extinguished will be represented by the equation already given on the pages cited: and the effect of hydraulic resistance on the forced rolling will be as already explained.

The artificial process here described represents closely the rolling of a vessel in a sea-way when broadside on to a series of uniform waves: and the same formulæ apply, ϕ_0 being now the maximum slope of the waves: but this subject being outside the limits of this treatise, the reader is referred for further information to Sir W. H. White's well-known treatise on *Naval Architecture.*

(3) As an example of a different character, take the case of the motion of an indicator piston under the action of the varying pressure of the steam and the longitudinal force of the spiral spring by means of which the steam pressure is indicated. If these forces exactly balanced each other the indication would be perfect, but in consequence of the inertia of the indicator piston a certain difference always exists.

In this case the indicator piston has a certain natural period of free vibration depending on the strength of the spring and the inertia of the piston. The corresponding frequency (N_0), can be determined by the method already described in Art. 200, p. 389. The pressure of the steam varies according to a complicated law, which for the moment we may suppose replaced by a simple variation of the kind already considered, the frequency being N, the revolutions per 1″ of the engine.

Then if p_0, p_1 be the excess of the actual and the indicated maximum pressures of the steam above their mean values,

$$p_1 = \frac{p_0}{1 - \frac{N^2}{N_0^2}}.$$

Thus the error of the indication consequent on the inertia of the indicator piston will be considerable unless the ratio N/N_0 be small. The result here obtained requires modifications noticed farther on in consequence of the complexity of the actual law of variation of the pressure of the steam, but the conclusion arrived at must be the same, and in fact experience shows that the ratio N/N_0 should not exceed one-tenth: the higher the speed the stiffer the spring must be in order to avoid undulations in the curve traced by the pencil.

202A. *Augmentation of Stress by Fluctuation.* As in the analogous case of impact the stress produced by a load of varying magnitude is much greater than if it were applied steadily. If as before Q be the maximum value and T the period, the equivalent steady load will be

$$R = \frac{Q}{1 - \frac{T_0^2}{T^2}},$$

in cases where the inertia of the structure can be considered as concentrated: Q being replaced by $k \cdot Q$ where k is a co-efficient when it is distributed.

In either case the equivalent steady load becomes indefinitely great when T approaches T_0: a theoretical result limited in practice by friction as already described. The actual stress produced by a small vibrating load may, however, be very great.

The formula just given determines the stress produced after a state of steady vibration has been reached; but the temporary augmentation before the free vibrations have been extinguished may be much greater. The ratio cannot be precisely stated but in cases where the periods T_0, T of free and forced vibration are commensurable it must generally be possible to have

$$z = z_1 \left\{ 1 + \frac{T_0}{T_1} \right\}$$

in the equation of combined vibration given on page 392, thus increasing the equivalent steady load to

$$R_1 = \frac{Q}{1 - \frac{T_0}{T_1}}.$$

On the other hand the effect of fluctuation is reduced by friction as already stated.

203. *Compound Vibrations.*—If a fluctuating load of given frequency N does not vary according to the simple harmonic law supposed in preceding articles, it may always be treated as made up of a series of periodic forces, each of which does follow that law. The first of these has the frequency N and the rest have the frequencies $2N$, $3N$, etc., in succession. For example, on reference to page 286 it will be seen that in addition to the primary periodic force due to the motion of a piston, which alone is considered in Art. 202 (1), there is a secondary periodic force,

$$S_2 = \frac{Q}{n} \cdot \cos 2\theta = \frac{Q}{n} \cdot \cos 4\pi Nt,$$

arising from the obliquity of the connecting rod. The maximum value of this secondary force is Q/n and its frequency $2N$. The other terms of the series are in this case very minute but they may be considerable, and each of the corresponding forces produces a forced vibration of its own which may be augmented by synchronism with some particular mode of vibration of the structure on which they act. For an exact balance each force must be neutralized, or the balance weight must follow the same law of vibration as the load. But as synchronism occurs at much lower speeds the secondary forces will often be of little importance.

Where the load increases suddenly to its maximum amount, as when steam is admitted to the cylinder of an indicator (page 394), free vibrations are superposed on the forced vibrations and are often very conspicuous.

203A. *Centrifugal Whirling of Shafts.*—When a shaft rotates, the centrifugal forces produced do not in general exactly balance one another: either because the shaft is not exactly straight or because the centres of gravity of the weights it carries do not precisely lie in the axis. The shaft therefore bends in a plane rotating along with it, and at certain speeds the bending may become indefinitely great, a condition of things known as "centrifugal whirling."

The simplest case is that of a shaft rotating on bearings at its ends, and carrying a weight W in the middle, the centre of gravity G of which is at a distance a from the axis of rotation C. The weight moves in a circle round C as a centre, the radius CG in the first instance being a, but subsequently increasing at revolutions N to $a+z$ where z is the deflection CO produced by the centrifugal force

$$Q = W \,.\, 4\pi^2N^2 \,.\, (a+z)/g.$$

The inertia of the shaft itself being supposed small enough to be neglected, the deflection z is $Ql^3/48EI$, and therefore

$$\frac{W}{g} \cdot 4\pi^2N^2 \,.\, (a+z) = \frac{48EI}{l^3} \cdot z.$$

If it were possible to have $a=0$ we should obtain a value of N for which the deflection z might be anything we please. If this be called N_0,

$$N_0 = \frac{1}{2\pi}\sqrt{\frac{g \,.\, 48EI}{Wl^3}} = 6{\cdot}25\sqrt{\frac{EI}{Wl^3}},$$

and expressing the preceding equation in terms of N_0,

$$a+z=\frac{a}{1-\dfrac{N^2}{N_0^2}}; \quad z=\frac{a}{\dfrac{N_0^2}{N^2}-1},$$

formulæ which determine CG the deviation of the centre of gravity and CO the bending of the shaft at any other speed N.

The formulæ show that, at speeds less than the critical speed N_0, z is a positive quantity which increases as the speed increases until $N=N_0$ when it becomes indefinitely great, however small the original deviation a, so that N_0 is a speed at which centrifugal whirling occurs. At greater speeds finite values are again obtained for both z and $a+z$, but they are both negative, z being numerically the greater. The interpretation of this is that the shaft is bent in the opposite direction, the centre of gravity G lying between O the geometrical axis and C the axis of rotation. They diminish as the speed increases so that G continually approaches the axis of rotation and coincides with it when the speed is infinite.

The conclusions here arrived at for the simple case where the weight W is concentrated at the centre of a shaft are also true in any other case, with a proper value of the coefficient in the formula for N_0. If we have a uniform shaft not otherwise loaded, the weight being uniformly distributed, the centrifugal forces are of much less intensity, and the speed of centrifugal whirling is much greater. If half the weight were concentrated in the middle and the other half at the ends where the centrifugal force is zero, the coefficient would obviously be increased in the ratio $\sqrt{2}:1$ or 41·4 per cent. As the deflection curve is not the same in the two cases this is not exact: the subject was first investigated by Rankine, whose results* show that the coefficient 6·25 in the formula should be replaced by 9, being an increase of 44 per cent. For a steel shaft d inches diameter, l inches long, this gives for revolutions per minute

$$N_0=4,860,000\frac{d}{l^2}.$$

In a shaft 1 inch diameter 4 ft. long the speed of centrifugal whirling is therefore about 2100 revolutions per minute. Much greater speeds are by no means unusual in modern practice, and in a certain type of steam turbine 35,000 revolutions per minute have been reached. Where the speed is excessive the important principle that, above the critical speed, the centre of gravity of the rotating masses tends to approach the axis of rotation should be borne in mind. A flexible shaft may for this reason be safer at certain speeds than one which in a state of rest is more rigid.

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* *Millwork and Machinery*, 1st edition, 1869, p. 549.

As in the corresponding case of plane vibration nodes may occur in a rotating shaft. Thus in the case just considered there may be a node in the middle of the shaft dividing it into equal parts which bend in opposite directions: the length of the shaft is then virtually halved, and a second critical speed four times as great as the first determined, which it is dangerous to approach. Similarly there may be three or more nodes and to each number corresponds a certain critical speed. So also for a different arrangement of bearings and for different weights attached the coefficient in the formula for the critical speed will be changed. The question therefore is one of much complexity of detail in each individual case. Where the weights are distributed exact results can only be obtained by the solution of a differential equation: though approximations can generally be found as indicated above. The subject has recently been investigated experimentally by Mr. (now Prof.) Dunkerly * who points out that the centrifugal righting couple introduced by a pulley mounted on a shaft anywhere but at the centre has the effect of increasing the critical speed.

Returning to the case considered in Art. 201, p. 390, of a weight attached to the centre of an elastic bar, and vibrating in one plane: the frequency of free vibration is determined by a formula identical with that found in the present article for the critical speed N_0, the extent (z_1) of the vibration being undetermined. Suppose now a second vibration in a plane at right angles to the first, and let the bar be of circular section: the frequency will be the same as before, and the extent (z_2) being undetermined may be taken at pleasure. If the two motions co-exist, the result, as is well known, will be that the centre of the moving weight will describe an ellipse which becomes a circle if $z_1 = z_2$. The geometrical axis of the bar lies in a plane rotating at revolutions N_0. Evidently then the centrifugal whirling of a shaft may be considered as a state of free vibration.

At other speeds the centrifugal forces may be resolved into two components at right angles which produce forced vibrations of frequency N in planes at right angles, centrifugal whirling being a consequence of synchronism between the forced and free vibration.

Theoretically, at any speed, free vibration, either plane or elliptic, with frequency N_0, due to some external cause, may occur in combination with the centifrugal vibration of frequency N which now takes place about a vibrating axis, but, as in previous cases, such free vibrations will speedily be extinguished by the effect of friction. So

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* *Phil. Trans.*, vol. 190, 1894.

also deflections due to the action of gravity or other permanent forces will not affect a centrifugal vibration about the displaced axis. The rotation of a bent shaft is not necessarily accompanied by the rotation along with it of its geometrical axis, as is well seen by the example of the flexible shafts often used on a small scale for special purposes.

EXAMPLES.

1. A hammer weighing 2 lbs. strikes a nail with a velocity of 15 feet per second and drives it $\frac{1}{8}$ inch, what is the mean pressure overcome by the nail? *Ans.* 673 lbs.

2. If the load on a stretched bar is suddenly reversed so as to produce compression, show that the stress will be trebled.*

Energy stored in stretched bar will on the release of the load be employed in compression, and in addition the load will be exerted through a distance = original extension + compression. The two together must be equal to the work done in compressing the bar.

Note. Such sudden reversal as is here supposed rarely if ever occurs in practice in a stretched or compressed piece, but it may occur in a bent piece, to which the same principle applies.

3. A load of 1000 lbs. falls through 1″ before commencing to stretch a suspending rod by which it is carried. If the sectional area of the rod is 2 sq. in., length 100″, and modulus of elasticity 30,000,000, find the stress produced.

Stress = 17,828 lbs. per sq. in.

4. A load of 5000 lbs. is carried by the rod of the preceding question, and an additional load of 2000 lbs. is suddenly applied; what is the stress produced?

Stress = 4500 lbs. per sq. in.

5. A beam will carry safely 1 ton with a deflection of 1 inch; from what height may a weight of 100 lbs. drop without injuring it, neglecting the effect of inertia? *Ans.* 10·2 inches.

6. The maximum stability of a vessel is 4000 foot-tons. The curve of stability is represented sufficiently approximately by a triangle, such that the angle of maximum stability is $1/n$ the angle of vanishing stability. Find the uniform moment which, applied suddenly to the ship upright and at rest, would just capsize her.

Ans. Capsizing moment $= \dfrac{4000\sqrt{n}}{1+\sqrt{n}}$

7. A crane is observed to deflect through 1 inch when a load of 1 ton is suspended from it. A load of 2 tons is lowered at the rate of 2 f.s. and then suddenly stopped; in what ratio is the stress on the parts of the crane augmented? *Ans.* 87 per cent.

8. A. vertical bar is supported as in Fig. 148, and struck horizontally. Assuming that the deflection curve of the bar is the same as if a horizontal force were steadily applied, compare the kinetic energy of the vibrating bar with the energy of an equal weight concentrated in the middle. *Ans.* $\frac{17}{35}$.

Note.—This result, obtained originally by Homersham Cox, agrees well with Hodgkinson's experimental result, showing that the energy of the secondary vibrations of the bar is relatively small.

9. A thin flat plate is stiffened by beams of uniform section. By an explosion a uniform pressure is suddenly applied over the whole surface: show that the resistance of the beams to impact is 60 per cent. greater than if the load were concentrated in the middle.*

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* Examples 2 and 9 are due to Prof. T. A. Hearson. Some good examples on impact will be found in Prof. Alexander's treatise on Applied Mechanics, part I.

CHAPTER XVII.

STRESS, STRAIN, AND ELASTICITY.

SECTION I.—STRESS.

204. *Ellipse of Stress.*—Stress consists, as we have said (Art. 147), in a mutual action between two parts, into which we imagine a body divided by an ideal section. If the section be plane, and if the stress be uniform, the intensity and direction of the stress at each point of the section are the same at all points of a given section, and, for a given point, depend only on the position of the plane. In a fluid the intensity is the same for all planes, and the direction is normal to the plane. In simple tension and compression the direction of the stress is the same for all planes, but its intensity varies, becoming zero for planes parallel to the stress. In a simple distorting stress (p. 352) the intensity is the same for all planes perpendicular to a third given plane, but the direction varies: on one pair of planes it is normal, on another tangential.

We now proceed to consider stress more generally, and we shall first examine the effect of combining together a pair of simple longitudinal stresses, the directions of which are at right angles and the intensities of which are given. Let the plane of the paper be parallel to the directions of the stresses, and let us consider a piece of material of thickness unity. If the stress be uniform, the size and shape of the piece are immaterial. Let us then imagine a rectangular block *ABCD* (Fig. 154) with sides perpendicular to the stresses p_1, p_2. On the faces *AB*, *CD* a stress, of intensity p_1, and of total amount $p_1 . AB$ will act; while on *BC* and *AD* there will be a stress of intensity p_2, and of total amount $p_2 . BC$. Divide now the rectangle by a diagonal plane *AC*; there will be a stress on that plane, which it is our object to determine in direction and magnitude. Let θ be the angle which the normal to the plane makes with the direction of p_1; by determining rightly the ratio of the sides of the rectangle this angle may be made what we please. Proceeding as in Art. 182, we find for the normal stress

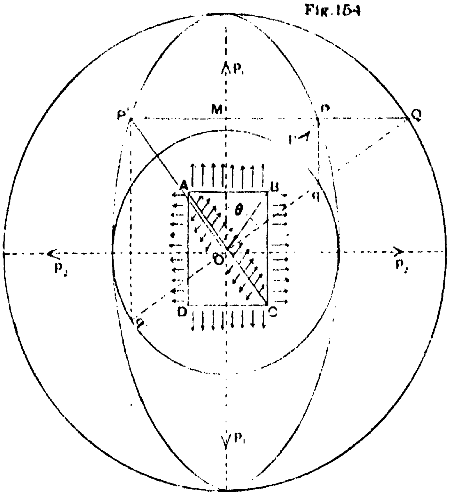
$$p_n = p_1 \cdot \cos^2\theta + p_2 \cdot \sin^2\theta,$$

and for the tangential stress

$$p_t = (p_1 - p_2) \sin\theta \cdot \cos\theta.$$

The resultant stress may be found in direction and magnitude by compounding these results, but it is better to proceed by a graphical construction. On the perpendicular set off OQ to represent p_1, and Oq to represent p_2; also draw the ordinates QM parallel to p_2, and qP parallel to p_1 to meet in P. Then

Fig. 154



$$OM = OQ \cdot \cos\theta = p_1 \cdot \frac{AB}{AC};$$

$$PM = Oq \cdot \sin\theta = p_2 \cdot \frac{BC}{AC}.$$

Whence it follows that the intensity of the stress on AC due to p_1 is represented by OM, and that due to p_2 by PM. If then we join OP we shall obtain the resultant stress on AC in direction and magnitude. It is easily seen that P lies on an ellipse of which p_1, p_2 are the semi-axes. This ellipse is called the Ellipse of Stress.

If the pair of stresses p_1, p_2 have opposite signs, then $Oq' = p_2$ must be set off on the opposite side of O, and OP' the radius vector of the ellipse lies on the other side of OM, but in other respects the construction is unaltered. When p_1, p_2, are equal the ellipse becomes a circle; if they have the same sign the stress is the same in all directions in magnitude and direction like fluid pressure; if they have opposite signs, as in the chapter on Torsion, the intensity is the same, but the angle of inclination $P'OQ$, called the "obliquity" of stress, is variable, being always twice QOM.

205. *Principal Stresses. Axes of Stress.*—We now propose to show that any state of stress in two dimensions (Art. 208) may always be reduced to a pair of simple stresses such as we have just considered.

For, drawing the same figure as in the last article, let us inquire the effect of replacing p_1, p_2 by other stresses of any magnitude and in any directions. Whatever they be, they evidently must have given tangential and normal components, of which, reasoning as in a former chapter, we know that the tangential must be equal and of opposite tendency.

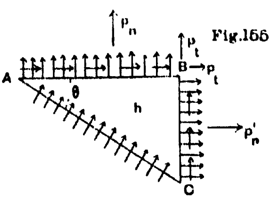


Fig.155

Let the equal tangential components be p_t and the normal components p_n and p'_n. Consider the equilibrium of the triangular portion ABC (Fig. 155), and let us determine under what conditions it is possible that the stress on AC should be a normal stress only, without any tangential component. Resolve parallel to BC; then, if p be that normal stress,

$$p \,.\, AC \,.\, \cos\theta = p_t \,.\, BC + p_n \,.\, AB:$$

or

$$p - p_n = p_t \,.\, \tan\theta.$$

Similarly, resolving parallel to AB,

$$p - p'_n = p_t \,.\, \cot\theta,$$

whence, subtracting one equation from the other,

$$p_n - p'_n = p_t \,.\, (\cot\theta - \tan\theta) = 2p_t \,.\, \cot 2\theta;$$

or

$$\tan 2\theta = \frac{2p_t}{p_n - p'_n}.$$

This equation always gives two values of θ at right angles, showing that two planes at right angles can always be found on which the stress is wholly normal. The magnitude of the stress on these planes is found by multiplying the equations together, when we get the quadratic

$$(p - p_n)(p - p'_n) = p_t^2,$$

the roots of which, p_1, p_2, are the stresses required. Having determined p_1, p_2, the ellipse of stress can now be constructed by the method of the last article.

Every state of stress in two dimensions then can always be represented by an ellipse, the semi-axes of which are called Principal Stresses, and their directions the Axes of Stress.

The particular case in which p'_n is zero is one of constant occurrence in practical applications. If q be the shearing stress, the equations may then be written

$$p_n \tan 2\theta = 2q \ (1); \quad p(p - p_n) = q^2 \ (2).$$

Of the roots of the quadratic the greater has the same sign as that of p_n, and the other the opposite. Also, we find by dividing the two equations for p by one another,

$$\tan^2\theta = \frac{p - p_n}{p} = \frac{q^2}{p^2},$$

from which it appears that of the two values of θ furnished by (1) the one less than 45° must correspond to the greater value of p. Hence the major principal stress is of the same kind as p_n, and inclined to it at an angle less than 45°.

206. *Varying Stress. Lines of Stress. Bending and Twisting of a Shaft.*—In proving the two very important propositions just given we have assumed (1) that the stress was uniform, throughout the region including the portion of matter we have been considering; (2) that gravity or any other force acting not on the bounding surface, but on each particle of the interior, may be neglected. It is however to be observed that by taking the portion of matter small enough, both these suppositions may be made, in general, as nearly true as we please: the first, because any change of stress must be continuous, and therefore becomes smaller the less the distance between the points we consider; the second, because any internal force is proportional to the volume, while any force on the boundary of a piece of material is proportional to the surface of the piece. Now the volume of a body varies as the cube, and the surface as the square of its linear dimensions, and it follows that the internal force vanishes in comparison with the stress on the boundary when the dimensions diminish indefinitely. Hence these propositions are still true as respects the state of stress at any given point of a body, even though the stress be variable, and notwithstanding the action of gravity. When, however, we consider the variation of stress from point to point, gravity must be considered. Thus, for example, in the case of a fluid the action of gravity does not prevent the pressure from being the same in all directions, but it does cause the pressure to vary from point to point.

When the stress varies from point to point, both the intensity and the direction may vary; thus, for example, in a twisted shaft the intensity of the stress at any point varies as the distance from the axis, and the direction of the stress varies according to the position of the point, the principal stresses making an angle of 45° with the axis of the cylinder. The axes of stress in this case always touch certain lines which give, at each point they pass through, the direction of the stress at that point. These lines are called Lines of Stress; in a simple distorting stress, or, in other cases where the principal stresses are of opposite signs, one is a Line of Thrust, the other a Line of Tension.

In a twisted shaft of elastic material the lines of stress are spirals traced on a cylinder passing through the point considered, the spirals being inclined at 45° to the axis. If the shaft be bent as well as twisted, the maximum normal stress at any point of the transverse section is given by the equation

$$p_n = \frac{M}{\frac{1}{4}\pi r^3} \quad \text{(Art. 155)},$$

where M is the bending moment and r the radius. The shearing stress at the external surface due to a twisting moment T is given by

$$q = \frac{T}{\frac{1}{2}\pi r^3} \quad \text{(Art. 184)}.$$

Combining these two together we get, by solving the quadratic for the principal stresses,

$$p = \frac{M \pm \sqrt{M^2 + T^2}}{\frac{1}{2}\pi r^3},$$

which gives the principal stresses at that point of the shaft where the stress is greatest. The maximum stress is the same as would be given by a simple twisting moment equal to $M + \sqrt{M^2 + T^2}$, which is sometimes, though improperly, called the simple equivalent twisting moment. The minor principal stress ought, however, also to be considered in calculations respecting strength, as will be seen hereafter.

The lines of stress here are spirals of variable pitch angle.

207. *Straining Actions on the Web of an I Beam.*—Let us now return to the case of an I beam with a thin web, in which the web resists nearly the whole of the shearing force F, and the flanges nearly the whole of the bending moment M. The intensity of the shearing stress q is approximately

$$q = \frac{F}{ht},$$

where h is the depth and t the thickness. The intensity of the normal stress at a point distant y from the neutral axis is

$$p_n = \frac{M}{I} \cdot y.$$

The principal stresses and axes of stress are given by the equations

$$p(p - p_n) = q^2 ; \quad \tan 2\theta = \frac{2q}{p_n}.$$

From this it appears that, even when the web is very thin so that it carries a very small fraction of the total bending moment, it cannot be treated as resisting shearing alone, and if it is so treated will be the most severely strained part of the beam. Let us, for example, suppose the flanges to be subject to a stress of 4 tons per sq. inch at a given section, and the web to a shearing stress also of 4 tons per sq. inch: then at points in the web near the flanges, say, for example, at a distance from the centre of three-fourths the half depth of the beam, the normal stress will be 3 tons per sq. inch. Putting these values in the formula, we get the quadratic equation

$$p(p - 3) = 16 ;$$

whence

$$p = 5{\cdot}77, \text{ or } -2{\cdot}77,$$

a result which shows that the web is much more severely strained than the flanges. The lines of stress are found from the equation for θ. The direct effect of any load resting on the upper flange must be provided for separately by vertical stiffening pieces.

208. *Remarks on Stress in General.*—We have hitherto been considering only the stress on planes at right angles to a certain primary plane, to which we have supposed the stress on every plane to be parallel. In most practical questions relating to strength of materials this is sufficient, since, though stress frequently exists on the primary plane, it is usually normal and of relatively small intensity. Thus, for example, in a steam boiler there is stress on the internal and external surface of the boiler due to the pressure of the steam and the atmosphere; but it is of small amount compared to the stress on planes perpendicular to the surface. We therefore content ourselves with a statement without demonstration of corresponding propositions in three dimensions.

(1) Any state of stress at a point within a solid may always be reduced to three simple stresses on planes at right angles.

(2) The resultant stress on any plane due to the action of three simple stresses at right angles to each other is always represented in direction and magnitude by the radius vector of an ellipsoid.

The first of these propositions may be regarded as the last step in a process of analysis, by which we reduce all external forces acting on a structure of any kind: *first*, into a set of forces acting on each piece of the structure; and *second*, into forces acting on each of the small elements of which we may imagine that piece composed; and *lastly*, into three forces at right angles acting upon the element, of which one in practical cases is usually small. All questions in Strength of Materials, then, ultimately resolve themselves into a consideration of the effects of forces so applied.

One method of conceiving the effect of three such forces is to imagine each separated into two parts, one of which is the same for all, being the mean value of the three: while the other is compressive for one and tensile for the two others, or *vice versa*. In isotropic matter (Art. 210) the first set produces change of volume only, and may be called the "volume-stress," or, as no other stress can exist in fluid bodies at rest, a "fluid" stress. The second is a distorting stress, consisting of three simple distorting stresses tending to produce distortion in the three principal planes.

EXAMPLES.

1. A tube, 12 inches mean diameter and ½ inch thick, is acted on by a thrust of 20 tons and a twisting moment of 25 foot-tons. Find the principal stresses and lines of stress.

Taking a small rectangular piece with one side in the transverse section, we find one face acted on by a normal stress of 1·06 tons per square inch due to the thrust, and a tangential stress of 2·66 tons due to the twisting. Substituting these values for p_n, p_t, and observing that the stress on the other face is wholly tangential, we find from the quadratic

Major principal stress = 3·24 (thrust);
Minor principal stress = 2·18 (tension).

Lines of stress are spirals, the lines of tension inclined at 50½° to the axis, and the lines of thrust at 39½°.

2. A rivet is under the action of a shearing stress of 4 tons per square inch, and a tensile stress, due to the contraction of the rivet in its hole, of 3 tons per square inch. Find the principal stresses.

Ans. Major principal stress = 5·8 tons (tension);
Minor principal stress = 2·77 tons (thrust).

3. The thrust of a screw is 20 tons; the shaft is subject to a twisting moment of 100 foot-tons, and, in addition, to a bending moment of 25 foot-tons, due to the weight of the shaft and its inertia when the vessel pitches. Find the maximum stress and compare it with what it would have been if the twisting moment had acted alone. Shaft 14 inches diameter.

Ans. Major principal stress = 2·9, Ratio = 1·32;
Minor principal stress = 1·6.

4. A half-inch bolt, of dimensions given in Ex. 6, p. 261, is screwed up to a tension of 1 ton per square inch of the gross sectional area. Assuming a co-efficient of friction of ·16, find the true maximum stress on the bolt while being screwed up. *Ans.* Principal stresses = 2 and ·35 tons.

5. It has been proposed to construct cylindrical boilers with seams placed diagonally instead of longitudinally and transversely. What is the object of this arrangement, and what is the theoretical gain of strength? *Ans.* Increase of strength = 26½ per cent.

6. A thick hollow cylinder is under the action of tangential stress applied uniformly all over its internal surface in directions perpendicular to its axis, the cylinder being

prevented from turning by a similar stress, applied at the external surface. Find the principal stresses and lines of stress. *Ans.* The principal stresses are equal and opposite, forming a simple distorting stress, of intensity varying inversely as the square of the distance from the centre. Lines of stress equiangular spirals of angle 45°.

7. In Ex. 9, page 375, suppose the beam so loaded that the maximum stress due to bending is 3 tons per square inch, and the total shearing force divided by the sectional area of the web 4 tons per square inch: find the principal stresses at points immediately below the flanges. *Ans.* Principal stresses $4\frac{1}{2}$ and 1·9 tons per square inch.

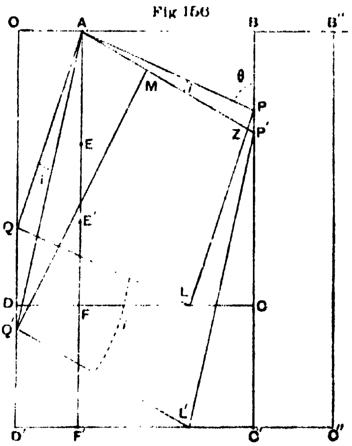
8. In any state of stress at a point in a body show that the sum of the normal stresses on three planes at right angles is the same however the planes be drawn.

SECTION II.—STRAIN.

209. *Simple Longitudinal Strain. Two Strains at Right Angles.*—We now go on to consider the changes of form and size which are produced by the action of stress. Such changes, it has already been said, are called Strains.

In uniform strain every set of particles lying in a straight line must still lie in a straight line, and two lines originally parallel must still be parallel. The lengths of all parallel lines are altered in a given ratio $1+e:1$, where e is a quantity, in practical cases very small, which measures the strain in the direction of the line considered. Two sets of parallel lines, however, will not in general remain at the same inclination to each other, nor will their length alter in the same ratio. Thus the sides of a cube remain plane, and opposite sides are parallel, but the parallelopiped is not generally rectangular, and its sides are not equal.

Fig 156



The simplest kind of strain is a simple longitudinal strain in which all lines parallel to a fixed plane in the body are unaltered in length, while all lines perpendicular to that plane remain so: that is to say a simple change of length, the breadth and thickness remaining unaltered.

Fig. 156 shows an extensible band *OBCD*, in which *OB* is fixed, while *CD* moves to *C'D'*, the breadth being in the first instance unaltered, and the length altered so that

$$CC' = e_1 . BC.$$

If any line AEF be traced in the band parallel to BC, the points EF will shift to $E'F'$ positions in the same line, such that

$$EE' = e_1 \,.\, AE : FF' = e_1 \,.\, AF.$$

$$E'F' = (1 + e_1)EF;$$

for since the strain is uniform the change of length of all parts of the band is the same. If, however, we draw a line QL inclined at an angle θ to BC, that line will shift to $Q'L'$, a position such that QL has not increased in so great a ratio, and is not inclined to BC at the same angle as before. We are about to determine the actual change of length and angular position of QL by finding that of a parallel AP drawn through A. It has been already remarked that parallel lines in uniform strain must suffer the same strain. Now AP shifts to AP' such that

$$PP' = e_1 \,.\, BP = e_1 \,.\, AP \,.\, \cos\theta.$$

If now the angle PAP' $(= i)$ be so small that i^2 may be neglected compared with i, and i compared with unity,

$$AZ = AP : P'Z = PP' \,.\, \cos\theta;$$

and therefore

$$AP' - AP = PP' \,.\, \cos\theta = e_1 \,.\, AP \,.\, \cos^2\theta.$$

Thus the strain (e) in the direction of AP is

$$e = e_1 \,.\, \cos^2\theta.$$

Also, it is clear that

$$i = \frac{PZ}{AP} = \frac{PP'}{AP} \,.\, \sin\theta = e_1 \,.\, \sin\theta \,.\, \cos\theta.$$

By these formulæ the changes of length and angular positions of all lines in the band are determined.

Next draw a line AQ perpendicular and equal to AP, and let AQ' be the position into which it moves in consequence of the strain; we find for e', the extension of AQ,

$$e' = e_1 \,.\, \sin^2\theta;$$

while the angle QAQ' is

$$i' = e_1 \,.\, \sin\theta \,.\, \cos\theta = i.$$

Imagine now the square AQL completed; this square, in consequence of the strain, will have its sides altered in length by the quantities e, e', and will have suffered a distortion given by

$$2i = 2e_1 \,.\, \sin\theta \,.\, \cos\theta.$$

In this way the effect of a simple longitudinal strain is completely determined, for we can calculate the changes taking place in any portion of the band we please.

Next suppose the band to suffer a second simple longitudinal strain e_2 in the direction of the breadth, and observe that, since the strains are very small, the effect of e_1, e_2 taken together must be the sum of those due to each taken separately; then we find for the change of length and position of any line AP,

$$e = e_1 \,.\, \cos^2\theta + e_2 \,.\, \sin^2\theta\,;$$
$$i = (e_1 - e_2)\sin\theta \,.\, \cos\theta,$$

results which may be applied as before to show the changes of dimension and the distortion of a square traced anywhere in the band.

We have here regarded the angle i as a measure of the distortion a square suffers in consequence of the strain. If, however, we drop $Q'M$ perpendicular to AP', we have

$$\angle AQ'M = 2i = \frac{AM}{AQ'}.$$

Now AM is the space through which the line LQ' has shifted parallel to itself in consequence of the strain, and we see therefore that the angle i also gives a measure of the magnitude of this shifting. By some writers this is called "sliding." It is also called "shearing strain."

If we compare the equations we have just obtained for strain with those previously obtained in Art. 204 for stress, we find them identical; and hence it appears that, so long at least as the strains are very small, all propositions respecting stress must also be true, *mutatis mutandis*, with respect to strain. Thus, for example, a simple distortion must be equivalent to a longitudinal extension accompanied by an equal longitudinal contraction; and, again, every state of strain can be reduced to three simple longitudinal strains at right angles to each other, and represented by an ellipsoid of strain. The simple strains are called Principal Strains, and their directions Axes of Strain. Strain, like stress, generally varies from point to point of the body: but the relations here proved still hold good at each point, and we have Lines of Strain just as we previously had Lines of Stress.

SECTION III.—CONNECTION BETWEEN STRESS AND STRAIN.

210. *Equations connecting Stress and Strain in Isotropic Matter.*—So far we have merely been stating certain conditions which stress must satisfy in order that each element of a body may be in equilibrium, and certain other conditions which strain must satisfy if the body is continuous. We now connect the two by considering the way in which stress produces strain, which differs according to the nature of the material.

We first consider perfectly elastic material (see Art. 147), and suppose that material to have the same elastic properties in all directions, in which case it is said to be isotropic. Metallic bodies are often not isotropic, as will be seen hereafter (Ch. XVIII.). Suppose a rectangular bar under the action of a simple longitudinal stress p_1, then there results (Art. 148) a longitudinal strain e_1 given by

$$p_1 = Ee_1,$$

where E is the corresponding modulus of elasticity. Accompanying the longitudinal extension we find a contraction of breadth that is a lateral strain of opposite sign, of magnitude $1/m^{\text{th}}$ the longitudinal strain where m is a co-efficient. The contraction in thickness will be equal, because the material is supposed isotropic. Hence the effect of the simple longitudinal stress p_1 is to produce three simple longitudinal strains at right angles,

$$e_1 = \frac{p_1}{E}; \quad e_2 = -\frac{p_1}{mE}; \quad e_3 = -\frac{p_1}{mE}.$$

Next remove p_1, and in its place suppose a simple stress p_2 applied in the direction of the breadth of the bar; we have by similar reasoning the three strains,

$$e_1 = -\frac{p_2}{mE}; \quad e_2 = \frac{p_2}{E}; \quad e_3 = -\frac{p_2}{mE}.$$

And similarly removing p_2 and replacing it by p_3 acting in the direction of the thickness,

$$e_1 = -\frac{p_3}{mE}; \quad e_2 = -\frac{p_3}{mE}; \quad e_3 = \frac{p_3}{E}.$$

These three sets of equations give the strains due to p_1, p_2, p_3, each acting alone; and we now conclude that if all three act together we must necessarily have

$$Ee_1 = p_1 - \frac{p_2 + p_3}{m},$$

with two other symmetrical equations.

Hence it appears that the effect of three principal stresses, and consequently of any state of stress whatever on isotropic matter, is to produce a strain, the axes of which coincide with the axes of stress, and in which the principal strains are connected with the principal stresses by the equations just written down.*

The product Ee_1 is the simple tensile stress which would produce the strain e_1, a quantity to which special importance is attached when e_1 is the greatest of the three principal strains and in consequence the maximum elongation in any direction at the point considered.

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* The form in which these equations are given is that employed by Grashof. For practical application it is more convenient than any other.

The value of Ee_1 in this case is frequently described as the "equivalent simple tensile stress."

211. *Elasticity of Form and Volume.*—The value of the constant m may be found directly by experiment, though with some difficulty, on account of the smallness of the lateral contraction which it measures; but it may also be found indirectly, by connecting it with the co-efficient employed in a former chapter to measure the elasticity of torsion. For if we subtract the second of the three equations just obtained from the first, we get

$$e_1 - e_2 = (p_1 - p_2)\frac{m+1}{mE},$$

or

$$p_1 - p_2 = \frac{m}{m+1} \cdot E(e_1 - e_2).$$

Now referring to Art. 204, we find

$$p_t = (p_1 - p_2)\sin\theta\,.\cos\theta,$$
$$2i = 2(e_1 - e_2)\sin\theta\,.\cos\theta,$$

where p_t is the tangential stress on a pair of planes inclined at angle θ to the axes, and $2i$ is the distortion of a square inclined at that angle to the axes of strain. Since now the axes of strain coincide with the axes of stress, we must have

$$\frac{p_t}{2i} = \frac{p_1 - p_2}{2(e_1 - e_2)} = \frac{1}{2}\frac{m}{m+1} \cdot E,$$

an equation which, compared with Art. 183, shows that the co-efficient of rigidity C must be

$$C = \frac{1}{2}\frac{m}{m+1} \cdot E.$$

Experiment shows that in metallic bodies C is generally about $\frac{3}{8}E$, whence it follows that m lies between 3 and 4. In the ordinary materials of construction the comparison cannot, however, be made with exactness, because such bodies are rarely exactly isotropic and homogeneous. The value of m for iron is supposed to be about $3\frac{1}{2}$.

Again, if we add together the three fundamental equations, we find

$$E(e_1 + e_2 + e_3) = \left(1 - \frac{2}{m}\right)(p_1 + p_2 + p_3).$$

Now the volume of a cube, the side of which is unity, becomes when strained $(1 + e_1)(1 + e_2)(1 + e_3)$, and therefore the volume strain is $e_1 + e_2 + e_3$ when the strains are very small. Hence, if we separate the stress into a fluid stress N and a distorting stress (Art. 209), we have

$$N = \frac{m}{3(m-2)} \cdot E \times \text{Volume Strain},$$

and the co-efficient

$$D = \frac{m}{3(m-2)} E$$

measures the elasticity of volume. The two constants C and D, which measure elasticity of distinctly different kinds, may be regarded as the fundamental elastic constants of an isotropic body. The ordinary Young's modulus E involves both kinds of elasticity.

212. *Modulus of Elasticity under various circumstances. Elasticity of Flexion.*—When the sides of a bar are free the ratio of the longitudinal stress on the longitudinal strain is the ordinary modulus of elasticity E; but the equations above given show that, when the sides of the bar are subject to stress, the modulus will have a different value. For example, let the bar be forcibly prevented from contracting, either in breadth or thickness, by the application of a suitable lateral tension, $p_2(=p_3)$, then e_2, e_3 are both zero, and

$$Ee_1 = p_1 - \frac{2p_2}{m}\,;\quad 0 = p_2 - \frac{p_1+p_2}{m},$$

whence we obtain for the magnitude of the necessary lateral stress

$$p_2 = \frac{p_1}{m-1},$$

and for the corresponding extension of the bar

$$Ee_1 = \frac{m^2-m-2}{m^2-m} \cdot p_1.$$

Hence the modulus of elasticity is now

$$A = \frac{m(m-1)}{(m+1)(m-2)} \cdot E.$$

This constant A is what Rankine called the direct elasticity of the substance: it is of course always greater than E. For $m=4$, $A=\frac{6}{5}E$: for $m=3$, $A=\frac{3}{2}E$.

If the bar be free to contract in thickness, but not in breadth, we have p_3 and e_2 zero, and the equations become

$$Ee_1 = p_1 - \frac{p_2}{m}\,;\quad 0 = p_2 - \frac{p_1}{m}\,;\quad Ee_3 = 0 - \frac{p_1+p_2}{m},$$

whence we find $$Ee_1 = p_1 \cdot \frac{m^2-1}{m^2},$$

so that the value of the modulus of elasticity is

$$\frac{m^2}{m^2-1} E.$$

In a similar way if p_2, p_3 have any given values the modulus can be found.

It will now be convenient to examine an important point already referred to in the theory of simple bending, that is to say the assumption (Art. 153) that the modulus of elasticity E was the same as in the case of simple tension, notwithstanding the lateral connection of the elementary bars, into which we imagined the whole beam split up. If these elementary bars were prevented from contracting freely, as they would do if separated from each other, the modulus could not be the same. In fact, however, there is nothing in their lateral connection which prevents them from doing so. Figure 157 shows, on a very exaggerated scale, the form assumed by a transverse section $ACBD$ originally rectangular, cutting a series of longitudinal sections orignally parallel to the plane of bending in the straight lines shown. Assuming the upper side stretched as in Fig. 122, page 306, these lines all radiate from a centre O' above the beam, which bends transversely, while the originally straight horizontal layers are cut in arcs of circle struck from the same centre. The upper side of the beam contracts and the lower side expands, and reasoning exactly in the same way as in Art. 153 when we derived the formula for the longitudinal curvature, we find a corresponding formula for the transverse curvature,

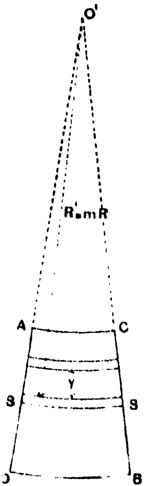

Fig. 157

$$p = m\frac{Ey}{R'},$$

whence it follows immediately that

$$R' = mR.$$

In order that this transverse curvature of the originally horizontal layers shall not be inconsistent with the reasoning by which the formula for bending is obtained, all that is necessary is that the deviation from a straight line shall be small as compared with the distance of the layer from the neutral axis. Let u be that deviation, then (see Art. 163) if b be the breadth and h the depth,

$$u = \frac{b^2}{8R'} = \frac{b^2}{8mR} = \frac{b^2 . p}{8mEy}.$$

Now the stress being within the elastic limit p/E is very small, for example take the case of wrought iron, for which p/E is not more than $\frac{1}{1200}$th, and suppose $m = 4$,

$$u = \frac{b^2}{30,400 . y_1} = \frac{b^2}{19,200h}.$$

It is obvious that u must be always very small compared with y, except very near the neutral axis, unless b be very large compared with h, and we conclude therefore that when a beam is bent *within* the limit of elasticity, the lateral connection of the parts cannot have any sensible influence on its resistance to bending, unless its breadth be great.

Experience shows, however, that a broad thin plate remains straight in the direction of the breadth when bent longitudinally, and cannot therefore be supposed free to expand or contract laterally except near the edges. In this case, then, there must be a normal stress (p') at every point of a longitudinal section parallel to the plane of bending and this stress must be proportional to the corresponding stress (p) on the transverse section being given by the equation,

$$p' = \frac{p}{m}.$$

Change of breadth being thus prevented the elasticity of flexion (p. 412) becomes

$$E'' = \frac{m^2}{m^2 - 1} \cdot E,$$

being from 7 to 12 per cent. greater than Young's modulus.

213. *Remarks on Shearing and Bending.*—When a beam is subjected to bending without shearing the only assumption made in the usual theory given on page 307 is that of complete freedom to expand and contract laterally; but in general there is also a shear on each section and in consequence tangential as well as normal stress at each point. Hence if two plane sections be taken before the beam is bent, those sections not only rotate about their neutral axes as in Fig. 122 on the page cited, but are also distorted and the consequences of this distortion will now be briefly considered.

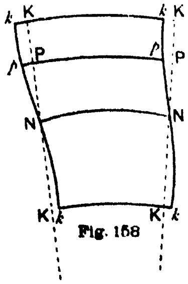

Fig. 158

(1) Fig. 158 shows a longitudinal section of a bent beam, the plane of bending being as before a plane of symmetry, and NN the geometrical axis as in Fig. 122. The dotted straight lines $KPNK$ as before show the positions of two transverse sections when bending alone exists, and simply rotate about axes through NN to meet in a centre of curvature not shown in the present figure. The full curves $kpNk$ show the intersections of the longitudinal section with the actual sections after distortion by the action of the shearing stress at each point. Let us now suppose the shearing force to be *constant*, that is, the same

on all sections, as when a beam is fixed horizontally at one end and loaded at the other with a given weight, then as in other analogous cases (pp. 299, 306) we may suppose the shearing stress and consequently the distortion the same at corresponding points of the two sections; that is to say, the two curves will be exactly alike and the deviation Pp from the straight line will be the same for both. Hence pp the actual length of a longitudinal layer of the beam is the same as PP the length which it would have had if the shearing stress had been absent. The actual form of the distorted section is very complex, no line in it remaining straight but in general becoming a curve of double curvature; it is clear, however, that the same reasoning applies to every pair of corresponding points and not merely to points lying in the central plane. Hence the changes of length of all the elementary bars into which the beam is analysed are the same as if there were no shearing, and reasoning as on page 307, we arrive at the same general equations,

$$\frac{p}{y}=\frac{M}{I}=\frac{E}{R'}$$

for the normal stress and the curvature as in simple bending. We conclude, therefore, that these equations must be true notwithstanding the distortion produced by shearing, provided only the shearing force be constant.

The truth of the simple reasoning here given is borne out by a complete investigation of the bending and shearing of a beam which—like the corresponding investigation for torsion—we owe to the late M. St. Venant.* This investigation, based on the supposition of complete freedom to expand and contract laterally (see last article), shows that the usual equations are exact when, and only when, the shearing force is constant.

In any case of continuous loading the shearing force is zero at sections of maximum moment, and there is consequently no distortion there, so that at such sections the equations still apply. Where the load is concentrated at one or more points, there will always be shearing and often of great magnitude, but in these cases if not absolutely constant it varies slowly in consequence of a relatively small continuous load between the sections at which the load is concentrated. Hence the equations may be regarded as substantially exact in most practical cases where it is necessary to determine the resistance of a beam or girder to bending. Some qualifications of this statement have already been given in preceding articles of this book (Art. 189), and it may be further added that when a section of maximum moment occurs in the neighbourhood of the ends of the beam or of a concentrated load additional strength may in some cases be required. The local stress due to the direct action of a concentrated load is frequently very considerable,* though its effect in weakening a solid beam is probably not in any proportion to its magnitude. At the ends of a beam additional strength is generally required for constructive reasons.

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* *History of Elasticity*, vol. ii., Part I., p. 58.

(2) The transverse curvature of the originally horizontal layers of a beam of rectangular section subject to bending and shearing has the effect of altering the distribution of the shearing stress, which is not uniform along lines parallel to the neutral axis but is less at the centre than at the outer surface. The mean along the neutral axis is 1½ times the mean over the whole section, but the actual value is less than this at the centre and greater at the outer surface.

This inequality of distribution laterally is in the first instance due to the elevation of the sides of the beam (Fig. 157) above the centre which is caused by the transverse curvature. So long as there is no shearing the curvature and therefore the elevation remains the same for all sections, but when the curvature changes the elevation also varies and produces a corresponding excess distortion at the sides. The whole action is very complex and cannot be reduced completely to calculation in any simple way, but some further remarks will be found in the Appendix. When the depth is not less than 2½ times the breadth this effect may be disregarded, but in a square section the difference is 6 per cent., and as the breadth increases becomes much greater.†

In other types of section as already stated there is often a large discrepancy between the mean and the true maximum, apart from the effect of transverse curvature. Complete results have been obtained for a circle and some other forms. These calculations of St. Venant, however, only apply to sections of a beam the outer surface of which is free from stress. The direct action of the pressure on the sides of a pin which is being sheared most probably tends to equalize the shear on the section, and the provisional method, already described, when properly checked, is perhaps the best approximation attainable.

214. *Thick Hollow Cylinder under Internal Pressure.*—The equations connecting stress and strain in combination with suitable equations expressing the continuity of the body and the equilibrium of each of its elements are theoretically sufficient to determine the distribution of stress within an elastic body exposed to given forces, and in particular to determine the parts of the body exposed to the greatest stress, and the magnitude of such stress. The most important cases hitherto worked out, in addition to those considered in preceding chapters, are the torsion of non-circular prisms and the action of internal fluid pressure on thick hollow cylinders and spheres. For M. St. Venant's investigations on torsion we must refer to Art. 185, page 360, and the authorities there cited. We shall only consider the comparatively simple case of a homogeneous cylinder.

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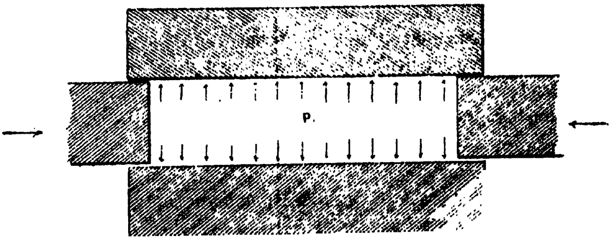
* *The Influence of Surface Loading on the Flexure of Beams.* By Prof. C. A. Carus-Wilson. *Proceedings of the Physical Society of London*, December, 1891.

† *History of Elasticity*, vol. ii., Part I., p. 68.

Fig. 159*a* shows a longitudinal section of a hollow cylinder open at the ends, which are flat: the cylinder contains fluid which is acted on by two plungers forced in by external pressure so as to produce an internal fluid pressure p_1. Fig. 159*b* shows the same cylinder in transverse section: imagine a cylindrical layer of thickness *t*, this thin cylinder will be acted on within and without by stress which symmetry shows must be normal; let these stresses be p and p' and the internal and external radii of the thin cylinder be r and r'. Now, if p' the external pressure had existed alone, a compressive stress q would have

Fig. 159a



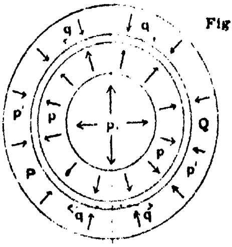


Fig. 159b

been produced on the material of the cylinder given by the equation (see Art. 150)

$$p'r' = qt;$$

and if the internal pressure had existed alone, we should have had a tensile stress given by

$$pr = qt;$$

hence when both exist together, we must have

$$p'r' - pr = qt,$$

where q is the stress on the material of the cylinder on a radial plane in the direction perpendicular to the radius reckoned positive when compressive. Clearly $t = r' - r$, and therefore proceeding to the limit we may write the equation

$$\frac{d}{dr}(pr) = q,$$

which is one relation between the principal stresses p, q at any point of the cylinder. We now require a second equation, to get which it is necessary to consider the way in which the cylinder yields under the application of the forces to which it is exposed. The simplest way to do this is to assume that the cylinder remains still a cylinder after the pressure has been applied: if so, it at once follows that points in a transverse section originally, remain so, or, in other words, that the longitudinal strain is the same at all points. It is not to be supposed that there is anything arbitrary about this assumption: no other, apparently, can be made if the ends of the cylinder are free, the pressure on the internal surface exactly uniform, and the cylinder be homogeneous and free from initial strain. For when this is the case, there is no reason why the cylinder should be in a different condition in one part of its length than in another. If the ends are not free, or if the pressure is great in the centre, the middle of the cylinder will bulge, but not otherwise.

It is also clear that the total pressure on a transverse section must be zero because the ends are free, and hence it is natural to suppose that it is also zero at every point of the transverse section, an assumption which we shall presently verify. For greater generality we in the first instance suppose it a constant quantity p_0.

The equations connecting stress and strain therefore become

$$Ee_1 = p - \frac{q + p_0}{m};$$

$$Ee_2 = q - \frac{p + p_0}{m};$$

$$Ee_3 = p_0 - \frac{p + q}{m},$$

where e_1, e_2, e_3 are the compressions in the direction of the radius, the direction perpendicular to the radius in the transverse section, and the direction of the length, respectively. Of these the last is constant, as just stated, and therefore

$$p+q=\text{constant}=2c_1$$

is the second equation connecting p, q. Substituting for q, we find

$$\frac{d}{dr}(pr)+p=2c_1\,;$$

or

$$r\frac{dp}{dr}+2p=2c_1.$$

Multiply by r and integrate, then

$$p=\frac{c_2}{r^2}+c_1,\ \text{and consequently}\ q=c_1-\frac{c_2}{r^2},$$

where c_2 is a constant of integration. The two constants, c_1, c_2, are now determined by consideration of the given pressure within and without the cylinder.

If n be the ratio of the external radius to the internal radius R, we have at the internal surface

$$\left.\begin{array}{l} p=p_1 \\ r=R \end{array}\right\}\ \therefore\ p_1=c_1+\frac{c_2}{R^2}\,;$$

and at the external surface

$$\left.\begin{array}{l} p=0 \\ r=nR \end{array}\right\}\ \therefore\ 0=c_1+\frac{c_2}{n^2R^2}\,;$$

from which two equations we get

$$c_1=-\frac{p_1}{n^2-1},\ \text{and}\ c_2=\frac{p_1n^2}{n^2-1}\cdot R^2.$$

Substituting these values in the equation for q,

$$q=-\frac{p_1}{n^2-1}\left\{1+n^2\cdot\frac{R^2}{r^2}\right\};$$

the negative sign in this formula indicates that the stress is tensile, as we might have anticipated. The formula shows that the stress decreases from $\frac{n^2+1}{n^2-1}\cdot p_1$ at the internal surface to $\frac{2p_1}{n^2-1}$ at the external surface. The mean stress is obtained from the equation (Art. 150)

$$q_0(nR-R)=p_1R\,;$$

hence the maximum stress is greater than the mean in the ratio $n^2+1 : n+1$, and it is clear that it can never be less than p_1. The minor principal stress at the internal surface is p_1 and (omitting p_0) the so-called simple equivalent tensile stress can be found.

Verification of Preceding Solution.—**The radial strain (e_1) and the hoop strain (e_2) are given by the above equations in terms of the stress. Now these changes of dimension are not independent, but are connected by a certain geometrical relation which it is necessary to examine in order to see whether it is satisfied by the values we have found.**

Returning to the diagram, suppose the internal radius of the elementary ring represented there to increase from r to s, and the external radius from r' to s'; then

$$2\pi s = 2\pi r(1+e_2),$$

$$2\pi s' = 2\pi r'\left(1+e_2+t\frac{de_2}{dr}\right),$$

$$\therefore\ s'-s=(r'-r)(1+e_2)+r't\frac{de}{dr};$$

or, since the thickness of the ring changes from t to $(1+e_1)t$,

$$1+e_1=1+e_2+r\frac{de_2}{dr},$$

$$e_1=\frac{d}{dr}(e_2r).$$

This relation must always hold good, in order that the rings after strain may fit one another, and should therefore be satisfied by our results. On trial it will be found that it is satisfied, and we conclude that the solution we have obtained satisfies all the conditions of the problem, and is therefore the true and only solution, subject to the conditions already explained. For further remarks on this question, see Appendix.

214A. *Strengthening of Cylinder by Rings. Effect of great Pressures.*—The stress within a thick hollow cylinder under internal fluid pressure may be equalized, and the cylinder thus strengthened by constructing it in rings, each shrunk on the next preceding in order of diameter. For a cylinder so constructed will be in tension at the outer surface and compression at the inner surface before the pressure is applied, and therefore after the pressure has been applied will be subjected to less tension at the inner and more tension at the outer surface than if it had been originally free from strain. It is theoretically possible to determine the diameters of the successive rings so that the pressure shall be uniform throughout. The principle is important, and frequently employed in the construction of heavy guns.

When the limit of elasticity is overpassed the formula fails, and the distribution of stress becomes different. If the pressure be imagined gradually to increase until the innermost layer of the cylinder begins to stretch beyond the limit, more of the pressure is transmitted into the interior of the cylinder, so that the stress becomes partially equalized. If the pressure increases still further, the tension of the innermost layer is little altered, and in soft materials longitudinal flow of the metal commences under the direct action of the fluid pressure. The internal diameter of the cylinder then increases perceptibly and permanently. This is well known to happen in the cylinders employed in the manufacture of lead piping, which are exposed to the severe pressure necessary to produce flow in the lead. The cylinder is not weakened but strengthened, having adapted itself to sustain the pressure. Cast-iron hydraulic press cylinders are often worked at the great pressure of 3 tons per sq. inch, a fact which may perhaps be explained by a similar equalization.

215. *Elastic Energy of a Solid.*—If a cube of side unity be under the action of normal stresses p_1, p_2, p_3 on its faces the elastic energy will evidently be

$$U=\tfrac{1}{2}p_1e_1+\tfrac{1}{2}p_2e_2+\tfrac{1}{2}p_3e_3,$$

e_1, e_2, e_3 being as before the strains given by the general equations connecting stress and strain (p. 410). In most cases one of the stresses p_3 will be small enough to be neglected, then substituting for e_1, e_2,

$$U=\frac{1}{2E}\left(p_1^2+p_2^2-\frac{2p_1p_2}{m}\right).$$

Suppose now these principal stresses p_1, p_2 are due to the action of normal stresses p_n, p_n' on oblique planes combined with a tangential stress q as on page 403, then on solving the quadratic given on the page cited and substituting for p_1, p_2,

$$U=\frac{1}{2E}\left\{p_n^2+p_n'^2-\frac{2p_np_n'}{m}+\frac{2(m+1)}{m}\cdot q^2\right\},$$

which, using the value of C the co-efficient of rigidity given on page 411 becomes

$$U=\frac{p_n^2+p_n'^2}{2E}-\frac{p_np_n'}{mE}+\frac{q^2}{2C}.$$

Thus the elastic energy per unit of volume at any point of a solid is, as might be expected, the sum of that due to the normal stress and the tangential stress taken separately. This important principle holds good for each particle, and therefore for the whole, of a beam subject to bending and shearing, a shaft subject to bending and twisting, as well as many other cases. As an example of the use of the formula take the case of the deflection of a beam due to shearing considered in Art. 190. The beam being supposed supported at the ends and loaded with a weight W, its deflection will be $2U/W$ and the part due to shearing will therefore be

$$\delta=\int\frac{q^2}{CW}\cdot dV,$$

where dV is an element of volume. Taking for simplicity the case of a uniform transverse section the formula may be written

$$\delta=\frac{q_0^2Al}{CW}\int\frac{q^2}{q_0^2}\cdot\frac{dA}{A}=\delta_0\cdot\int\frac{q^2}{q_0^2}\cdot\frac{dA}{A},$$

where dA is an elementary portion of the transverse section, l the span, and q_0 the mean stress W/A, as on page 368. The integral taken over the whole transverse section is a numerical factor by which

δ_0 must be multiplied to get the actual deflection due to shearing. Take for example a tube the shear at each point P of the annular section of which was found on page 368 (see Fig. 146).

$$q = 2q_0 . \sin\theta\,; \quad dA = at . d\theta\,; \quad A = 2\pi at\,;$$

$$\therefore \text{ Factor} = 4\int_0^{2\pi} \sin^2\theta . \frac{d\theta}{2\pi} = 2.$$

In this case the determination of the factor is simple but generally it can only be found approximately. In a rectangle the mean value of q at points distant y from the neutral axis is given in Art. 189 (1), page 368. The effect of lateral contraction and expansion (p. 416) being neglected this will be the actual value of q, and proceeding as before the factor will be found to be 1·2. This method may be applied without difficulty to an I section.

In sections of other types such as the circle or ellipse, it is first of all necessary to suppose, as in Ex. 14, page 376, that the stress in a direction perpendicular to the neutral axis is uniformly distributed, a supposition which as pointed out on page 369 cannot be considered legitimate. It is, however, one which is frequently made, and the consequent error is probably not in general considerable. It is further needful to find the horizontal component of the shearing stress. This can be done when necessary, but the calculation is not one of much practical value.

216. *Rigidity of Shafts.*—In a tube under torsion we have with the notation of Art. 185, p. 357, for the elastic energy per unit of length, remembering that qt is constant,

$$U = \frac{1}{2C}\int q^2 . t ds = \frac{q^2t^2}{2C} . \int\frac{ds}{t},$$

and since the angle of torsion multiplied by the twisting moment on the tube, that is by $2qAt$, is $2U$,

$$i = \frac{qt . l}{2AC} . \int\frac{ds}{t} = \frac{qp . l}{2AC} . \int\frac{ds}{p},$$

the right-hand equation applying to the case of similar tubes considered at length in the article cited; qp being constant for the same tube.

For different tubes of the same set the integral will have the same value, and therefore if all tubes are twisted through the same angle $q \propto A/p$, that is, at corresponding points q is proportional to the radius vector as already stated. Replacing as before (p. 359) qp at the outer surface by q_1r_1 or $2T/A$ we get a general formula for the angle of torsion of a nest of similar tubes

$$i = \frac{Tl}{A^2C}\int\frac{ds}{p}.$$

In the case of a rectangular tube the integral is easily evaluated, for evidently if the sides be b and c,

$$\int\frac{ds}{p}=4\left(\frac{b}{c}+\frac{c}{b}\right)=4\,.\,\frac{b^2+c^2}{A}=48\,\frac{I}{A^2},$$

where I is the polar moment of inertia showing that the general formula given on page 361 is applicable if $4\pi^2$ is replaced by the somewhat larger number 48. Hence, as might be expected, the tubes are somewhat less rigid than the solid shaft.

For an ellipse of semi-axes b and c it is easily shown that

$$\int\frac{ds}{p}=\pi\left(\frac{b}{c}+\frac{c}{b}\right)=\pi^2\,.\,\frac{b^2+c^2}{A}=4\pi^2\,.\,\frac{I}{A^2},$$

thus verifying the formula.

EXAMPLES.

1. When the sides of the bar are forcibly prevented from contracting, show that the necessary lateral stress is given by

$$p_2=Be,$$

where $B=\frac{mE}{m^2-m-2}$. This constant B is what Rankine called the "lateral" elasticity of the substance.

2. With the notation of the preceding question prove that

$$C=\frac{A-B}{2}.$$

3. In a certain quality of steel $E=30{,}000{,}000$; $C=11{,}500{,}000$; find the elasticity of volume and the values of A and B, assuming the material to be isotropic. *Ans.* $m=3\frac{2}{7}$; $D=25{,}555{,}000$.

4. The cylinder of an hydraulic accumulator is 9 inches diameter. What thickness of metal would be required for a pressure of 700 lbs. per square inch, the maximum tensile stress being limited to 2,100 lbs. per square inch? Also, find the tensile stress on the metal of the cylinder at the outer surface. *Ans.* Thickness = 1·84″; Stress = 1,000 lbs. per square inch.

5. If the cylinder in the last question were of wrought iron, proof resistance to simple tension 21,000 lbs. per square inch, at what pressure would the limit of elasticity be overpassed? $m=3{\cdot}5$. (See Art. 229.) *Ans.* 6,400.

6. Find the law of variation of the stress within a thick hollow sphere under internal fluid pressure. By a process exactly like that for the case of the cylinder (page 418) it is found that the equation of equilibrium is

$$\frac{d}{dr}(pr^2)=2qr.$$

The equation of continuity is the same as that for a cylinder (Art. 214), and the equations connecting stress and strain are now

$$Ee_1=p-\frac{2q}{m};$$

$$Ee_2=q-\frac{p+q}{m}.$$

We can now by elimination of q, reduction, and integration obtain

$$p=c_1+\frac{c_2}{r^3};$$

$$q=c_1-\frac{c_2}{2r^3};$$

the constants being found as in the cylinder.

7. The cylinder of an hydraulic press is 8 inches internal and 16 inches external diameter. If the pressure be 3 tons per sq. inch, find the principal stress at the internal and external circumference.

$$\textit{Ans.}\ \text{At inner circumference}\begin{cases}\text{Major Stress}=5\ \text{(Tension).}\\ \text{Minor Stress}=3\ \text{(Thrust).}\end{cases}$$

$$\text{At outer}\quad\text{,,}\quad\begin{cases}\text{Major Stress}=2\ \text{(Tension).}\\ \text{Minor Stress}=0.\end{cases}$$

8. In the last question find the "equivalent simple tensile stress" (p. 411), assuming $m=3{\cdot}5$. *Ans.* 5·86 and 2 tons.

9. In examples 15, 16, page 376, find the "equivalent simple tensile stress" at the points indicated, assuming as before $m=3{\cdot}5$.

CHAPTER XVIII.

MATERIALS STRAINED BEYOND THE ELASTIC LIMIT. STRENGTH OF MATERIALS.

217. *Plastic Bodies.*—If the stress and strain to which a piece of material is exposed exceed certain limits its elasticity becomes imperfect, and ultimately separation into parts takes place. We proceed to consider what these limits are in different materials under different circumstances: it is to this part of the subject alone that the title "Strength of Materials" is, strictly speaking, appropriate.

Reference has already been made (Art. 147) to a certain condition in which matter may exist, called the Plastic state, which may be regarded as the opposite of the Elastic state, which has been the subject of preceding chapters. In this condition the changes of size of a body are very small, as before; but if the stress be not the same in all directions the difference, if sufficiently great, produces continuous change of shape of almost any extent. Some materials are not plastic at all under any known forces, but many of the most important materials of construction are so, more or less, under great inequality of pressure.

Fig. 160

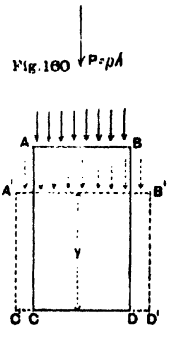


Fig. 160 shows a block of material which is being compressed by the action of a load P applied perfectly uniformly over the area AB. Let the intensity of the stress be p, then so long as p is small the compression is small and proportional to the stress; but when it reaches a certain limit the block becomes visibly shorter and thicker. This limit depends on the hardness of the material, and the value of p may be called the "co-efficient of hardness." In an actual experiment the friction of the surfaces between which the block is compressed holds the ends together, so that it bulges in the middle, as in Fig. 166, p. 435, which represents an experiment on a short cylinder of soft steel. In the ideal case the cross section remains uniform, changing throughout inversely as the height, as expressed by the equation

$$Ay = A_1 y_1,$$

where A is the area and y the height of the block.

In a truly plastic body p the intensity of the stress remains constant, and therefore the crushing load P varies as A, that is inversely as y. This is the same law as that of the compression of an elastic fluid when the compression curve is an hyperbola, and we therefore conclude (Art. 90) that the work done in crushing is

$$U = Py \,.\, \log_e r = pAy \log_e r = pV \log_e r,$$

where r is the ratio of compression and V the volume. Certain qualities of iron and soft steel will endure a compression of one-fourth or even of one-half the original height, and amounts of energy are thus absorbed which are enormous compared with the resilience of the metal. To illustrate this, suppose that plasticity begins as soon as the limit of elasticity f is overpassed, then for p we must write f, and by Art. 149 the resilience for a volume V is

$$\text{Resilience} = \frac{f^2}{2E} \,.\, V.$$

The ratio which the work just found bears to the resilience is therefore

$$\text{Ratio} = \frac{2E}{f} \,.\, \log_e r.$$

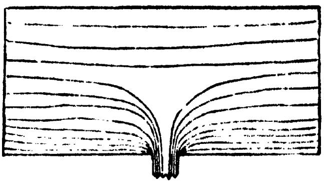
In wrought iron for a compression of one-fourth the height ($r = 1 \cdot 333$) this is about 800. The actual ratio must be much greater, because, as we shall see presently, the hardness of the material increases under stress.

If lateral pressure of sufficient magnitude be applied to the sides of the block, the longitudinal force being removed, the effect is elongation instead of compression, contraction of area instead of expansion. The magnitude of the lateral pressure is found by imagining a tension applied both longitudinally and laterally of equal intensity. Such a tension has no tendency to alter the form of the block, being analogous to fluid pressure, but it reduces the lateral pressure to zero, while it introduces a longitudinal tension of the same amount, which has the same value as the longitudinal compression of the preceding case. We see then that in every case a certain definite difference of pressure is required to produce change of shape in a plastic body, the direction of the change depending on the direction of the difference. The work done is found by the same formula as before, r meaning now the ratio of elongation.

In the process of drawing wire the lateral pressure is applied by the sides of the conical hole in the draw-plate, which are lubricated to reduce friction, and the force producing elongation in the wire is the sum of the tensile stress applied to draw the wire through the hole and the compressive stress on the sides. The work done is given by the same formula as before, p being now the sum in question.

218. *Flow of Solids.*—When a plastic body changes its form the process is exactly analogous to the flow of an incompressible fluid, which indeed may be regarded as a particular case. In the solid the distorting stress at each point at which the distortion is going on has a certain definite value which in the fluid is zero. The experimental proof of this is furnished by the experiments of M. Tresca, of which Fig. 161 shows an example. Twelve circular plates of lead are placed one upon another in a cylinder, which has a flat bottom with a small orifice at its centre. The pile of plates being forcibly compressed, the lead issues at the orifice in a jet, and the originally flat plates assume the forms shown in the figure. The lines of separation, indicating the position of particles of the metal originally in a transverse section, are quite analogous to the corresponding lines in the case of water issuing from a vessel through an orifice in the bottom. Tresca's experiments were very extensive, and showed that all non-rigid material flowed in the same way. Lead approaches the truly plastic condition; the difference of pressure necessary to make it flow being always about the same. Tresca ascribes to it the value of 400 kilogrammes per square centimetre, or about 5,700 lbs. per square inch;* but it is probably subject to considerable variations.

Fig. 161.



The manufacture of lead pipes, the drawing of wire, and all the processes of forging, rolling, etc., by which metals are manipulated in the arts, are examples of the Flow of Solids.

219. *Preliminary Remarks on Materials. Stretching of Wrought Iron and Steel.*—Materials employed in construction may roughly be divided into three classes. The first are capable of great changes of form without rupture, and, when possessing sufficient strength to resist the

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* The co-efficient employed by Tresca, and called by him the "co-efficient of fluidity," is half that used in the text. It is the magnitude of the distorting stress necessary to produce flow. See also note in Appendix.

necessary tension, may be drawn into wire. The last property is called ductility, and this word may be used to describe the class which we shall therefore call Ductile Materials. The second, being incapable of enduring any considerable change of this kind, may be described as Rigid Materials. The third are in many cases not homogeneous, but may be regarded as consisting of bundles of fibres laid side by side; they may therefore be described as Fibrous Materials; they are generally of organic origin.

We shall commence with the consideration of ductile materials, and more especially of

WROUGHT IRON AND STEEL.

Accurate experiments on the stretching of metal are difficult to make, the extensions being very small and the force required great. If levers are used to multiply the effect of a load or to magnify the extensions, errors are easily introduced. If the levers are dispensed with, a great length of rod is necessary and a heavy load, the manipulation of which involves difficulties. The experiment we select first for description was made by Hodgkinson on a rod of wrought iron ·517 inch diameter, 49 feet 2 inches long, loaded by weights placed in a scale pan* suspended from one end. The load applied was increased by equal increments of 5 cwts. or 2667·5 lbs. per square inch of the original sectional area of the bar; each application of the load being made gradually, and the whole load removed between each. At each application and removal the elongation was measured so as to test the increment of elongation, both temporary and permanent, occasioned by each load. If the rod were perfectly elastic the temporary increments should be equal and the permanent elongations (usually called "sets") zero.

The annexed table shows part of the results of this experiment, the first column giving the load, the second the total elongation, the third the successive increments of the elongation, the fourth the total permanent set.

On examining the table we see that, after some slight irregularities at the commencement due to the material not being perfectly homogeneous, the increments of elongation are nearly constant till we reach the eighth load of 21,340 lbs. per square inch, after which the increments show an increase at first moderate and subsequently very rapid. Further, the permanent set, which at the commencement is very minute and increases very slowly, at the same point shows a corresponding increase indicating that the observed increase is almost wholly due to a permanent elongation of the bar, the temporary increase following approximately the same law as before. Notwithstanding this the bar is not torn asunder till a much greater load is applied. The table shows the results up to a load of 32,000 lbs. per square inch, but rupture did not occur till a load of 53,000 lbs. was applied. The extension at the same time increased to nearly 21 inches, being more than forty times its amount at the elastic limit.

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* Being one of the best of its kind of old date this experiment has often been quoted. For the original description, see the *Report of the Commissioners appointed to enquire into the Application of Iron to Railway Structures.*

STRETCHING OF A WROUGHT-IRON ROD, 49 FEET 2 INCHES LONG.

Load.		Elongation in Inches.	Increment of Elongation.	Permanent Set.
2667·5 × 1	2667·5	·0485	·0485	
,, × 2	5335	·1095	·061	
,, × 3	8003	·1675	·058	·0015
,, × 4	10,670	·224	·0565	·002
,, × 5	13,338	·2805	·0565	·0027
,, × 6	16,005	·337	·0565	·003
,, × 7	18,673	·393	·056	·004
,, × 8	21,340	·452	·059	·0075
,, × 9	24,008	·5155	·0635	·0195
,, × 10	26,675	·598	·0825	·049
,, × 11	29,343	·760	·162	·1545
,, × 12	32,010	1·310	·550	·687

We conveniently represent the results graphically by setting off the elongations as abscissæ along a base line with corresponding ordinates to represent the stress, thus obtaining a curve of "stress and strain" (Fig. 162). The curve will be seen to be nearly straight up to a stress of 22,000 lbs. and then to bend sharply, becoming nearly straight in a different direction. A curve of permanent set

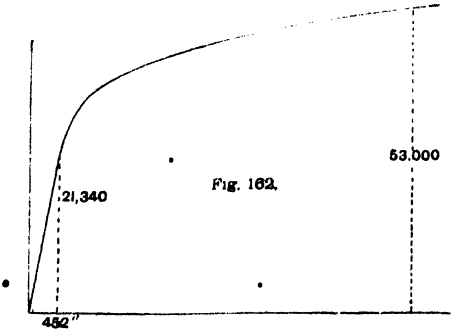

Fig. 162.

may also be constructed which is seen to follow the same general law.

Accompanying the increase of length of the bar we find a contraction of area; within the elastic limit, however, this is so small as to escape observation. Outside the limit it becomes visible, consisting in the first instance of a more or less uniform contraction at all or nearly all points, followed by a much greater contraction at one or sometimes two points where there happens to be some local weakness.* Within the elastic limit the density of the bar diminishes, but by an amount so small that the fact is rather known by reasoning than determined by experiment. Outside the limit there is a permanent diminution which is perceptible, though still very small.

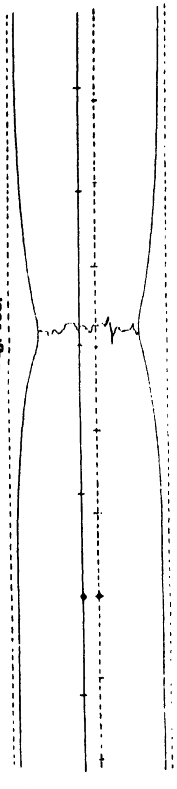

Fig. 163.

Thus beyond the elastic limit the bar draws out, changing its form like a plastic body without sensible change of volume. The bar finally tears asunder at the most contracted section, as shown by the annexed figure (Fig. 163) representing an experiment by Mr. Kirkaldy on a bar of iron 1 inch diameter, in which the contraction of area was 61 per cent., and the elongation 30 per cent., ultimate strength 58,000 lbs. per square inch of original area, 146,000 lbs. per square inch of fractured area. The contraction of section in good iron and soft steel is 50 or 60 per cent.

220. *Breaking-down Point.*—The foregoing experiment may, as far as it goes, be taken as a type of a multitude of such experiments which have been made on wrought iron and steel, which show that a tolerably well-defined limit exists, within which the extension is proportional to the pull and the sets are very small, but beyond this limit the process of stretching can only be completely studied by aid of a machine. A full description of various types of testing

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*On this point see *Preliminary Experiments on Steel by a Committee of Civil Engineers*, London, 1868. On account of the uncertainty of the amount of contraction at various points, the ultimate extension may sometimes be an imperfect measure of the ductility of the iron, even when the pieces are of the same length and sectional area.

machines will be found in Professor Unwin's treatise on *Testing*, from which we take such particulars as are necessary for our present purpose.

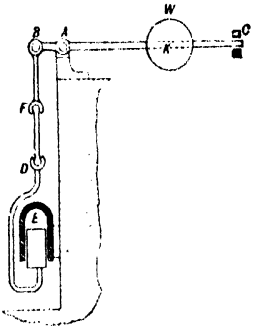

Fig. 164.

Fig. 164 is a diagram showing the essential parts of one of these machines: *BAK* is a lever balanced on knife edges at *A* and carrying a weight *W*, which can be moved along it by a screw; *E* is an hydraulic cylinder, the plunger of which is connected with the lower end of the test piece, *FD*. The short end *B* of the lever is connected with the upper end of *FD*, and the weight *W* is thus balanced by hydraulic pressure. When making the experiment water is pumped into the cylinder and a gradually increasing pull is thus applied to the test piece. This pull is measured by continuously moving the weight *W* by a screw, so as to keep the lever horizontal, stops *C* being provided to prevent it from moving far in either direction. The extensions are measured with great accuracy by a suitable apparatus, which not unfrequently automatically traces a curve of stress and strain.

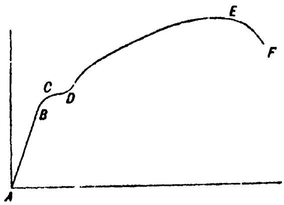

Fig. 165.

Fig. 165 shows roughly the form of curve obtained, the straight line *AB* representing the elastic part of the process already described. After passing the point *B* the curve bends away from the straight line, but the deviation is not large till a sharply defined point *C* is reached at which the curve is nearly horizontal, showing that a considerable stretch has occurred while the load remains nearly the same. The suddenness of this drawing out which is so characteristic of wrought iron and soft steel, is not distinctly perceived in the original way of making the experiment, because the load is not applied continuously. The point at which it occurs is described as the "yield-point," or "breaking-down point." The term "limit of stability," though in some respects preferable, is not so often used. When a bar is stretched in the workshop without recourse to delicate measurements this point may often be recognized by the falling-off scale and the obvious extension accompanied by lateral contraction which then occurs. It marks the end of the elastic stage and the commencement of the plastic stage in the process of stretching, and in the roughest class of experiments is the apparent "limit of elasticity," a term which may conveniently be applied when a closer specification is not necessary.

After passing *C* the stress goes on continuously increasing till a second point *E* is reached, where the curve is once more horizontal, and now if the stretching is carried still further it is found that to prevent the lever from resting on the lower stop, *W* must be moved gradually back again, showing a continuously diminishing stress till the bar tears asunder as already described. This part of the process for obvious reasons is not perceived when the experiment is made in the absence of a machine. The interpretation appears to be that as far as the point *E* the contraction of area is taking place throughout the whole length of the test piece, while the part *EF* of the curve represents the stretching which takes place after local contraction has begun.

221. *Real and Apparent Tensile Strength.*—The ordinates of a curve of stress and strain as usually plotted represent the total pull divided by the original sectional area of the test piece, and the greatest ordinate at *E* gives the ultimate strength as usually reckoned. It is clear that this is not the real tenacity of the material, for the sectional area has diminished considerably during stretching, and to meet this difficulty the area at fracture was formerly often employed as a divisor instead of the original area, the result obtained being called the "real tensile strength." This, however, gives much too large a value, for the real stress must, at least approximately, be the actual pull at any point divided by the actual area at that point. For this reason the contraction of area is now employed by most authorities exclusively as a measure of the ductility of the material without reference to its tenacity.

Nevertheless the actual stress on the contracted area is much greater than the apparent, and hence it follows that if the form of the piece be such as partly or wholly to prevent contraction the apparent strength will be increased. For example, if two pieces of the same bar be taken and one turned down to a certain diameter, while in the other narrow grooves are cut so as to reduce the diameter to the same amount at the bottom of the grooves, the strength of the grooved piece will be found to be much greater than that of the piece the diameter of which has been reduced throughout, and this can only be explained by observing that the length of the reduced part of the grooved bar is insufficient to permit contraction to any considerable extent. This is a point to be noticed in considering experimental results.* The form of the specimen tested may have much influence. Further, since the limit of elasticity is the point at which flow commences, and since the flow is due to difference of stress, it follows that the same causes must raise the limit of elasticity, and thus we are led to the conclusion that there are two elements constituting strength in a material, first, tenacity, and, secondly, rigidity. In some materials, such as these we are now considering, the tenacity is much greater than the rigidity, and in them the limit of elasticity will depend on the rigidity and will have different positions according to the way the stress is applied. It will lie much higher, and the apparent strength will be much greater when lateral stress is applied to prevent contraction.

222. *Increase of Hardness by Stress beyond the Elastic Limit.*—In clay and other completely plastic bodies a certain definite difference of pressure is sufficient to produce flow: in iron, copper, and probably other metals, however, as we have just seen, this is not the case, the metal acquiring increased rigidity in the fact of yielding to the pressure. Thus the effect of stress exceeding the elastic limit is always to raise the limit, whether the stress be a simple stensile stress or whether it be accompanied by lateral pressure. All processes of hammering, cold rolling, wire drawing, and simple stretching have this effect. If a bar be stretched by a load exceeding the elastic limit and then removed, on re-application of a gradually increasing load we do not find a fresh drawing out to commence at the original elastic limit, but at or near the load originally applied.† If the load be further increased drawing out recommences. Hence, whenever iron is mechanically "treated" in any way which exposes it to stress beyond the elastic limit, contraction is prevented and the apparent strength is increased; for example, iron wire is stronger than the rod from which it is drawn; when an iron rod is stretched to breaking, the pieces are stronger than the original rod. It is not certain that the real strength of materials is always increased by such treatment; perhaps in some cases the contrary, for we know that the modulus of elasticity and specific gravity are somewhat diminished.‡ On the other hand there are cases in which the increase of strength is greater than can be accounted for in

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* See *Experiments on Wrought Iron and Steel*, by Mr. Kirkaldy, p. 74. 1st edition. Glasgow, 1862.

† Styffe, *On Iron and Steel*, p. 68.

‡ The raising of the limit of elasticity by mechanical treatment of various kinds has long been known: in the case of simple stretching the effect appears to have been first noticed by Thälen in a paper, a translation of which will be found in the *Philosophical Magazine* for September, 1865.

this way. On annealing the iron it is found to have resumed its original properties, a circumstance which indicates that the increased rigidity is due to a condition of constraint which is removed by heating the metal till it has assumed a completely plastic condition. This process of hardening and annealing may be repeated a number of times without altering the yield-point, and it has recently been suggested that hardening by application of stress is analogous to the hardening of steel by heating and sudden cooling, and may be due to a similar change of molecular arrangement.*

In considering the effect of impact, the diminution of ductility occasioned by the application of stress beyond the elastic limit is a most important fact to be taken into account (see Art. 232). Working iron or steel hot has generally the effect of increasing both its strength and its ductility.

223. *Compression of Ductile Material.* In a perfectly elastic material compression is simply the reverse of tension, the same changes of dimension being produced by the same stress, but in the reverse direction. Also in a plastic body a given difference of stress produces flow, whether the stress be tensile or compressive; hence in ductile metals we should expect to find the modulus of elasticity and the limit of elasticity nearly the same in compression as in tension. These conclusions are borne out by experiment. In the case of wrought iron and

EXPERIMENT BY SIR W. FAIRBAIRN ON A BLOCK ·72 INCH DIAMETER OF SOFT BESSEMER STEEL.

TOTAL LOAD. $=P$.	HEIGHT OF BLOCK. $=u$.	CRUSHING STRESS. $p=\frac{Pu}{Au_1}$.
0	·997	0
16·7	·92	37·8
20·1	·865	42·9
23·3	·797	45·9
26·3	·731	47·4
29·5	·672	48·9
32·6	·613	49·4
35·8	·574	50·6
39·3	·535	51·9
41·0	·505	50·8

REMARKS.—The apparent ultimate tensile strength of this steel was 36 tons, its limit of elasticity 22 tons per square inch. Modulus of elasticity 30,300,000 lbs. Ratio of contraction ·41.

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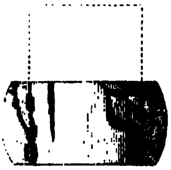
* Effect of Repeated Straining and Heating. W. C. UNWIN, *Proceedings of the Royal Society*, Vol. lvii.

steel, experiments on the direct compression of a bar are more difficult to carry out than experiments in tension, the bars are necessarily of limited length, and must be enclosed in a trough to prevent lateral bending; minute accuracy is therefore hardly attainable. A considerable number have, however, been made, from which it appears that the modulus of elasticity and the limit of elasticity are nearly the same in the two cases.*

The metal yields beyond the limit by a process of flow of the same character as in tension, but expanding laterally instead of contracting. This is especially seen in experiments made by the late Sir W. Fairbairn in 1867, and somewhat earlier by Mr. Berkeley, on the compression of short blocks of steel. In both, the blocks were pieces of round bars, of height somewhat greater than the diameter, and the results were very similar.

The annexed table gives the results of one of Sir W. Fairbairn's experiments. Column 1 gives the actual load laid on; column 2 the corresponding height of the block, both given directly by the experiments; column 3 is calculated by dividing the product of load and height by the original sectional area and height, and represents the crushing stress per square inch of the mean sectional area. If the block did not bulge in the centre (Fig. 166) this would be the actual crushing stress, which, however, must in fact be less. The table shows that after a compression of about one-third, the crushing stress remains nearly constant at about 50 tons per square inch. The experiment terminated at a compression of one-half. This kind of steel then is perfectly elastic up to 22 tons per square inch, is partially plastic between 22 and 50, and behaves as a plastic body under a difference of stress of 50 tons per square inch. The point at which the material becomes perfectly plastic may be described as the "limit of plasticity," it probably corresponds to the point where the load is a maximum and local contraction begins (p. 423) in a stretched bar.

Fig. 166.



The compression of iron blocks has been less thoroughly studied than that of steel, but it is known that the results are similar although the strength and the ultimate ratio of compression are much less. Set becomes sensible at about 10 tons per square inch, and the ultimate strength is from 40,000 to 50,000 lbs. per square inch if lateral flexure be prevented.

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* Perhaps the best set of experiments are those made by the "Committee of Civil Engineers." See their Preliminary Report already cited, pp. 7-13.

The bulging which occurs when a short block of ductile material is compressed is due to the instability of a cylindrical flow of the metal, and would probably occur even if there were no friction between the block and the compressing surfaces. Fracture occurs by lateral tearing asunder along longitudinal cracks when the height is small. When of greater height the block crushes by lateral bending. In wrought iron the ratio of length to diameter at which lateral bending commences is about 3 and the corresponding crushing stress is 36,000 lbs. per sq. inch, remaining independent of the length until the ratio reaches about one-third of the values given on page 342, after which the length begins to influence the crushing load, as described in the chapter cited. In tubular struts this limit is about 15.

224. *Bending within and beyond the Elastic Limit.*—Since wrought iron and steel are nearly perfectly elastic when the stress applied is not too great, it follows that the formulæ already obtained for the moment of resistance to bending and deflection of a bar must be true for these materials so long as the stress does not exceed the elastic limit determined by tension experiments of the kind just described.

(1) Very careful experiments were made by M. Styffe* on the deflection of bars of small size, 4 feet long, which fully confirm this conclusion: the value of the modulus of elasticity deduced from the observed deflection by the formula given on page 324 of this work closely agreeing with the value found by stretching the same bar. When smaller values are obtained by experiments on bending it is now recognized that this is due to the effect of shearing discussed on page 370, which, when neglected, may reduce the apparent value of the modulus by 20 per cent. or more. Some recent experiments by Messrs. Read and Stanbury, described in a paper which will be further referred to presently, give a modulus (apparent) of about 10,000 tons for beams of channel and Z section. In the case of a broad thin plate the modulus in bending should be greater than that in tension or compression (p. 414), but this theoretical conclusion appears as yet not to have been verified.

In built-up beams the modulus, as might be expected, is still further reduced: thus Rankine in his *Civil Engineering* states that the value for large girders is on the average 17,500,000 lbs. or about 8000 tons. In the paper just referred to the authors make a very interesting comparison between the observed deflection of a vessel and the result of a careful determination by graphical integration of the differential equation of the deflection curve. Two different vessels tested in this way gave nearly the same value for the modulus which was found to be about 10,000 tons.*

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* See Styffe, *On Iron and Steel*, already cited.

(2) Again, it has been repeatedly explained in the earlier part of this book that the lateral connection of the several layers into which we imagine a beam divided has no influence on the stress produced by bending so long as the limit of elasticity is not exceeded. But when the limit is passed, the connection between those layers which are most stretched and compressed with those layers which have not yet lost their elasticity prevents their contraction and expansion, and so raises the limit of elasticity in accordance with the general principle explained in Art. 221. Thus, the limit of elasticity lies higher, and the apparent elastic strength is greater in bending than in tension. In Fairbairn's experiment quoted above the same steel was tested in tension, compression, and bending. The elastic limit in bending was 30 tons, in tension 22 tons. The magnitude of the difference will depend on the form of transverse section, and on the ductility of the material. According to Mr. Barlow it may reach 50 per cent. in a rectangular section.† The case of cast iron will be referred to further on.

(3) As soon as the elastic limit is passed, the stress, at points near the surface, no longer varies as the distance from the neutral axis. It does not increase so fast because the extension or compression is not accompanied by a proportionate increase of stress. Hence a partial equalization of stress is produced, and the maximum stress for a given moment of resistance is reduced. To illustrate this it may be interesting to make a calculation of the effect of equalization by supposing, that under a bending moment very slowly and steadily applied beyond the elastic limit, the metal behaves like a truly plastic material throughout the transverse section, so that the stress is uniform. Referring to the formula on page 309, we have

$$\Sigma pybt = M,$$

in which we must now, instead of assuming that p varies as y, suppose p a constant. Then

$$M = 2p \,.\, A\bar{y}$$

where A is the area of the part of the section which lies on either side of the neutral axis and $\bar{y}$ the distance of its centre of gravity from that axis. For the same value of the modulus this gives a moment of resistance in a rectangular section 50 per cent. greater than if the material had been elastic. How far any apparent increase of strength due to equalization or lateral connection may be regarded in practice is

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* *On the Relation between Stress and Strain in Vessels*, by T. C. Read and G. Stanbury. *Transactions of the Institute of Naval Architects* for 1894, Vol. xxxv., p. 372.

† *Phil. Trans.*, 1855-57.

uncertain. A failure of elasticity must have taken place at certain points in order that there may be any increase at all, and in cases where the load is frequently reversed the bar must be weakened. (See Art. 230.)

CAST IRON AND OTHER RIGID MATERIALS.

225. *Stretching of Cast Iron.*—The phenomena attending rupture by tension of cast iron are essentially different from those described above for the case of ductile metals. This will be sufficiently shown by an experiment, also made by Hodgkinson, on a bar of this material 50 feet long, 1·159 inch diameter. The experiment was made in the same way as that already described on the wrought-iron rod,* and the results are shown in the annexed table. The first four loads were applied as before, by increments of 5 cwt., here equivalent to 531 lbs. per square inch; the whole load, after measurement of the elongation, being completely removed, and the permanent set measured. After the fourth load the increment was 10 cwt., and this was carried on till the bar broke at a stress of

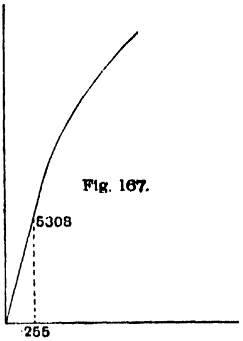

Fig. 167.

STRETCHING OF A CAST-IRON BAR, 50 FEET LONG, 1·159 INCH DIAMETER.

LOAD IN LBS. PER SQUARE INCH.	ELONGATION IN INCHES.	INCREMENT OF ELONGATION.	PERMANENT SET.
1. 531	·024	·024 × 2 = ·048	Perceptible.
2. 1,062	·0495	·0255 × 2 = ·051	·0015
3. 1,592	·0735	·024 × 2 = ·048	·002
4. 2,123	·09828	·0247 × 2 = ·0514	·0045
5. 3,185	·1485	·0503	·0105
6. 4,246	·200	·0515	·0155
7. 5,308	·255	·055	·022
8. 6,370	·313	·058	·028
9. 7,431	·374	·061	·037
10. 8,493	·435	·061	·046
11. 9,554	·504	·069	·056
12. 10,616	·572	·068	·067
13. 11,678	·648	·076	·0795
14. 12,739	·728	·080	·095
15. 13,801	·816	·088	·1115
16. 14,863	·912	·096	·132
17. 15,924	1·000	·088	—

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* *Report of Commissioners on the Application to Railway Structures*, p. 51.

16,000 lbs. per square inch. The third column as before shows the increments of elongation, which, after a stress of 5,308 lbs. per square inch, or $\frac{1}{3}$ the breaking load, has been reached, show a gradual increase till actual rupture occurs. The results of the experiments are graphically exhibited in the annexed diagram (Fig. 167) of stress, strain, and permanent set. The form of the curve is different from that of wrought iron, showing no point of maximum curvature, because in this material the bar does not draw out.

Hodgkinson experimented on a large variety of different kinds of iron, and expressed his results by a formula, which may be written

$$p = Ee(1 - ke),$$

where, as before (Art. 148), p is the stress, e the extension per unit of length, E the ordinary modulus of elasticity, and k a constant. The term ke here expresses the defect of elasticity of the bar. From the results of his experiments we find the average values

$$E = 14,000,000 ; \quad k = 209.$$

Cast iron, however, is a material of variable quality, and the value of these constants may have a considerable range. Up to one third the breaking load it may be regarded as approximately perfectly elastic, but the limit is by some authorities placed much higher.

226. *Crushing of Rigid Materials.*—In the ductile metals the effects of compression are nearly the reverse of those of extension, as has been sufficiently shown in previous articles, but in cast iron this is by no means the case. Hodgkinson experimented in this question with great care and accuracy, testing pieces of iron of exactly the same quality under compression and tension to enable a comparison to be made. The bars were enclosed in a frame and tested by direct compression. Hodgkinson expressed his results by a formula, which may be written

$$p = Ee(1 - ke),$$

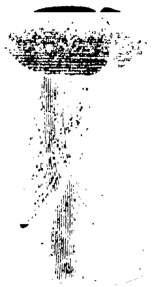
the symbols having the same meanings as before, and the values may be taken as

$$E = 13,000,000 ; \quad k = 40.$$

The smaller value of k indicates that the elasticity under compression is much less imperfect under the same stress. Short cylinders of the metal were also crushed, and the crushing load found to be five times the tensile strength or more.

It thus appears that in compression cast iron is six times stronger than in tension, and this is true not merely of the ultimate resistance but in great measure also of the elastic resistance, for the elasticity of the metal is not sensibly impaired until one third the crushing load is reached.

Fig. 168.



The manner in which crushing occurs is shown in the accompanying figure; instead of bulging out like a ductile metal, oblique fracture takes place on a plane inclined at 45° or rather less to the axis, being (approximately) the plane on which the shearing stress is a maximum (Fig. 168).

Great resistance to compression, as compared with tension, and sudden fracture by shearing obliquely or by splitting longitudinally are characteristics of all non-ductile materials, of which cast iron may be taken as a type. They are, in fact, materials the tenacity of which is much less than the rigidity.

In rigid materials crushing takes place not only by oblique shearing but also by longitudinal cracks. UNWIN (*Testing of Materials*, p. 419) finds that the mode of crushing and the resistance to crushing are much influenced by the material on which the specimen rests. When bedded on a soft material, the lateral flow of this material supplies by friction a transverse force on the base of the specimen, in consequence of which it crushes by longitudinal cracks at a smaller load than if the bed were hard, in which case oblique shearing occurs. This is a highly interesting observation, but it would be premature to say that all cases of crushing by longitudinal cracks can be explained in this way.

227. *Breaking of Cast-Iron Beams.*—When a cast-iron bar is bent till the tensile stress at the stretched surface exceeds one-third the tensile strength of the material, the defective elasticity of the metal causes a partial equalization of stress on the transverse section as in the case of wrought iron. Besides this, the elasticity being much more perfect under compression than under tension, the equalization is greater on the stretched side than on the compressed side, and the neutral axis moves towards the compressed side of the beam. For both these reasons the moment of resistance to bending is greater for a given maximum tensile stress than it would be if the material were perfectly elastic. Thus it follows that if the co-efficient in the ordinary formula for bending be assumed equal to the tensile strength of the material, the calculated moment of resistance will be less than the actual moment of rupture of the beam by an amount which is greater for a rectangular section than for an I section. The value of the co-efficient in the formula which corresponds to the actual breaking weight is known as the "modulus of rupture" or the "bending strength" of the material, a quantity greater than the simple tensile strength in a ratio which varies according to the type of section. A very complete set of experiments on the breaking of cast-iron bars was made in 1888-9 by Professor Bach, who shows that the ratio ranges from 1·45 in an I section and 1·75 in a rectangular section with side vertical to 2·1 in a circular or H section, and 2·35 in a square section with diagonal vertical: these numbers naturally being slightly different in different qualities of iron. The experimental result is always greater the more material is concentrated in the neighbourhood of the neutral axis, and this circumstance renders it almost certain that the increased apparent strength of cast iron in bending is simply due to the causes above mentioned and not, as has often been supposed, to any influence of curvature on the strength of the metal. Two examples (10, 11, page 462) which are given at the end of this chapter will serve to show how great an effect is produced by equalization combined with a moderate shift of the neutral axis.

SHEARING AND TORSION. COMPOUND STRENGTH.

228. *Shearing and Torsion.*—We now pass on to cases where the ultimate particles of the material are subject not to a simple longitudinal stress, but to stress of a more complex character. The simplest case is that of a simple distorting stress where the stress consists of a pair of shearing stresses (Fig. 140) on planes at right angles, or what is the same thing (Art. 183) of a pair of equal and opposite longitudinal stresses (Fig. 141) on planes at right angles. Examples of this kind of stress occur in shearing, punching, and twisting. Experiments on shearing are subject to many difficulties and are often not conducted in such a way as to satisfy the conditions necessary for uniformity of distribution of stress on the section. Moreover they necessarily give the ultimate resistance only without reference to the limit of elasticity. The whole process of shearing and punching is very complex, being at the commencement of the operation usually accompanied by a flow of the metal similar to that already referred to. Thus, when a hole is punched in a thick plate the punch sinks deep into the plate before the actual punching takes place, the metal being displaced by lateral flow, and the piece ultimately punched out being of less height than the thickness of the plate.*

Separation takes place in the first instance by the formation of fine cracks inclined at 45° to the plane of shearing. In soft materials the surfaces slide past each other and separate, but in harder materials there is a strong tendency to the formation of an oblique fracture. In wrought iron and steel the ultimate resistance to shearing, though varying considerably, may be taken as about three-fourths the ultimate resistance to tension of the same material. The question of a theoretical connection between the elastic strengths in the two cases is considered further on.

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* On this subject see M. Tresca's paper cited above, and two articles in the Journal of the Franklin Institute.

Experiments on torsion are not numerous, and many of those which exist are not experiments on simple twisting, but on a combination of bending and twisting. Such experiments would be of great value if accompanied by corresponding experiments on simple twisting and bending made on similar pieces of material. It is known, however, that in the ductile metals the elastic resistance to torsion is less than the resistance to tension. A series of experiments on torsion made by Prof. Thurston give some interesting results.* Curves are drawn, the abscissæ of which represent angles and the ordinates twisting moments, and the form of these curves shows that in some cases defective homogeneity causes a great deficiency in the elasticity at small angles of torsion. In general, however, the curves closely resemble the ordinary curve of stress and strain, already given for a stretched bar, being nearly straight up to a certain point and then curving towards the axis.

The formula for the angle of torsion of shafts given on page 361 has been tested by Bauschinger, in the case of square and circular sections by comparison with experiments made by him in 1878 on 13 pair of test pieces of iron and steel of various degrees of hardness, the mean result of the whole agreeing well with the formula. Some pieces of cast iron of rectangular and elliptic section showed, as might be expected, a less perfect agreement.

In twisting, as in bending, after passing the elastic limit, the stress at each point of the section, instead of varying as the distance from the centre, as it must do in perfectly elastic material, varies much more slowly so as to become partially equalized. Hence the twisting moment corresponding to a given maximum stress is greater than it would be if the elasticity were perfect. In the case where the equalization is perfect it is easy to show that the twisting moment is increased in the proportion 4 : 3, a result first given in 1849 by Prof. J. Thomson. The curves given by Thurston show that in many cases an approximately constant twisting moment was reached indicating that nearly complete equalization must have existed.

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* See Paper on *Materials of Machine Construction*, read before the American Society of Civil Engineers, 1874. No diameters are given, except for the woods, so that the stress corresponding to the limit of elasticity cannot be found.

Similar conclusions may be drawn from experiments made by Mr. Appleby[*] and by Messrs. Hayward and Platt.[†] The resistance to torsion was found to be greater than the resistance to a simple shear, just as in the corresponding case of bending and tension; and it may be added that in hollow shafts the difference has been shown to be less than in solid shafts, a case which corresponds to that of an I section in bending. The remarks already made on bending apply here also, and the case of cast iron will be further considered presently.

The modulus of elasticity of torsion is connected with Young's modulus by the equation (p. 411)

$$C = \frac{1}{2} \cdot \frac{m}{m+1} \cdot E = \frac{E}{2(1+\mu)}$$

when the material is isotropic, m being a number, the reciprocal μ of which is commonly known as Poisson's Ratio. The value of μ can evidently be determined by this equation when the moduli C and E have been found by torsion and tension experiments. But it is also possible to determine μ by observing directly the lateral contraction of a stretched piece or the expansion of a compressed piece and comparing it with the extension or contraction of length. A long series of experiments of this kind were made by Bauschinger in 1878,[‡] and their results show that by either method nearly the same value of μ is obtained for wrought iron and steel, the average being about ·3. Cast iron is more variable and in some qualities μ is less than ·2 for small stresses, but increasing with the stress, a phenomenon still more apparent in sandstone.

Similar experiments on a great variety of metals and alloys have recently been carried out by Mr. C. E. Stromeyer.[§]

229. *Connection between Coefficients of Strength.*—A simple distorting stress is included in the more general case of three simple longitudinal stresses of any magnitudes acting on planes at right angles. To this, indeed, all cases of stress can be reduced, and if we knew the powers of resistance of a material to three such stresses simultaneously, all questions relating to strength of materials could (at least theoretically) at once be answered. Unfortunately, experiments fitted to decide the question have not hitherto been made, and in consequence hypotheses have explicitly or implicitly been resorted to.

First, it is often tacitly supposed that the powers of resistance of a material to a simple longitudinal stress are unaffected by the existence of a lateral stress. For example, if a material bears 10 tons per square inch under a simple stretching force, it is assumed that when formed into a cylindrical boiler shell and exposed to internal fluid pressure it would also bear 10 tons on the square inch if the shell were homogeneous and free from joints, notwithstanding the fact that the material is exposed to stress (Art. 150) tending to tear it transversely as well as longitudinally. It is, however, certain that this cannot be the case, at any rate as regards the elastic strength. In ductile materials, the limit of elasticity of which depends to so great an extent on rigidity, any lateral force must raise or lower the elastic limit according as it acts in the same direction as the longitudinal stress or in the opposite direction.

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[*] *Proceedings of the Institution of Civil Engineers.* Vol. 74, p. 258.

[†] *Ibid.* Vol. 90, p. 382.

[‡] *Civil Ingenieur* for 1879, p. 81.

[§] *Proceedings of the Royal Society* for April, 1894.

Secondly, it may be supposed that the maximum elongation or contraction of a material in a given direction must be a certain definite quantity, irrespective of any elongation or contraction in any other direction. This hypothesis leads to results which in many cases are much more probable than the preceding, and is in common use by Continental writers; we shall therefore give some examples.

Let us take a piece of wrought iron and imagine that when exposed to a simple stretching force its limit of elasticity corresponds to a stress of 10 tons per square inch, accompanied by an elongation of $\frac{1}{1200}$th of its length. The second theory asserts that the maximum admissible elongation is still $\frac{1}{1200}$th, even though the sides of the bar be acted on by any force, the effect of which will be that quite a different longitudinal stress will be required to produce that elongation.

The relations between stress and strain are expressed by the equations (Art. 211), of which one is

$$Ee_1 = p_1 - \frac{p_2 + p_3}{m}.$$

The first theory supposes that p_1 can never exceed 10 tons, and the second that e_1 can never exceed $\frac{1}{1200}$th (or Ee_1 10 tons), whatever p_2, p_3 are. In the case of a thin pipe with closed ends under internal fluid pressure $p_3 = 0$ (nearly), $p_2 = \frac{1}{2}p_1$ (Art. 150); thus assuming $m = 4$ we have on the second or elongation theory

$$10 = p_1 - \frac{p_1}{8}, \quad \text{or,} \quad p_1 = 11{\cdot}43,$$

so that the material will bear under these circumstances a stress of 11·43 tons per square inch as safely as it bears 10 tons under simple tension, and this value, therefore, may be assumed for the co-efficient in the formula which gives the corresponding internal pressure. In like manner in the case of a thin sphere the material will bear a stress of 13¼ tons per square inch, being an increase of 30 per cent. If, on the other hand, we imagine the lateral stress compressive, then the maximum stress is reduced to 8·89 tons in the first case and 8 in the second.

On either theory the resistance to a simple distorting stress may be found in terms of the resistance to simple tension, for such a stress consists (p. 349) of a pair of equal and opposite simple stresses of equal intensity. In the first case the resistances to tension and shearing ought to be equal, in the second, since, writing $p_2 = -p_1$, we find

$$Ee_1 = p_1 + \frac{p_1}{m}; \text{ or, } p_1 = \frac{m}{m+1} Ee_1,$$

it follows that the resistance to shearing is $m/(m+1)$ or about four-fifths the resistance to tension, a result on the whole borne out by experience. It should be remarked that the theory only professes to give a connection between the elastic resistances in the two cases, the equations only holding good for perfectly elastic material, which, moreover, must be supposed isotropic. The ultimate resistance to torsion of a cast-iron shaft of square section is 10 per cent. greater than its resistance to tension, which is no doubt due to the same causes as in the case of bending, since in a hollow shaft of circular section it is only 80 per cent. of the tensile strength, and in a solid round shaft about the same.

Again, rigid materials on this theory are imagined to give way to longitudinal compression, when the lateral expansion produced by the compression is the same as would be produced by a simple tensile stress; from which it appears that the elastic resistance to compression should be from three to four times the elastic resistance to tension, as may easily be supposed to be the case.

Next suppose the three principal stresses to be equal and tensile, forming a tensile volume-stress p the sole effect of which is to produce an increase of volume. Evidently we have

$$p\left(1 - \frac{2}{m}\right) = Ee,$$

or if $m = 4$, $p = 2Ee$. If this stress be increased till rupture occurs, the limiting value of p is the real tenacity of the material, and, according to the hypothesis we are considering, should be a definite multiple of (say double) the simple tensile stress. No experiments on stress of this kind appear as yet to have been made on solids.

The elongation theory is employed by some writers of great authority on the subject of elasticity as confidently as if it were a statement of observed facts, and has been greatly developed in connection with the elastic properties of matter which is not isotropic. An addition to the *tenacity* of a material, consequent on the application of a lateral tension, can, however, hardly be considered as intrinsically probable, and such direct experimental evidence as exists is against the supposition.*

A third theory, more easily conceivable *a priori*, is to suppose that each material is capable of enduring, without injury to its elasticity, a certain definite change of volume and a certain definite change of shape. We thus have two co-efficients of elastic strength analogous to the two fundamental constants which express the other elastic properties of isotropic matter. On this hypothesis, however, if the resistance to a simple distorting stress in any plane be independent of the existence of any other kind of stress whether fluid or otherwise, as in fact is the case before the limit is reached, it would follow that this resistance must be one-half the resistance to a longitudinal stress.

Until a complete experimental investigation has been made no method can be completely satisfactory; but, in the absence of the necessary experimental data, the elongation theory may be provisionally assumed, at least in cases where it leads to a smaller result than the supposition of a given limiting stress. It is applied by first finding the principal stresses as in Ch. XVII., and then deducing the principal strains as just now explained. The greatest of these strains multiplied by E may be described as the "equivalent simple tensile stress," and should not exceed the limit prescribed by the strength of the material.

REPETITION AND IMPACT.

230. *Wohler's Experiments on Fluctuating Stress.* In bodies which satisfy the definition of perfect elasticity a load within the elastic limit produces no permanent change, unless perhaps some thermodynamic effect, and it follows from this that after removal the body is completely uninjured, so that the load may be repeated indefinitely. Experience confirms this conclusion. The balance spring of a watch bends and unbends more than a million times a week for years together, and the parts of a machine if originally sufficiently strong, remain so to all appearance for an indefinite time. But, if the load be beyond the elastic limit, permanent changes are produced, and there is every reason to believe that a slow deterioration of strength is ultimately destructive. The most definite information on this point is furnished by the experiments of M. Wöhler† published in 1870. Bars were loaded in various ways and the load wholly or partially removed; the process was repeated till the bar broke: the number of repetitions necessary

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*See a paper by M. Wehage, an abstract of which is given in the *Proceedings of the Institution of Civil Engineers*, vol. 95, p. 410. Some experiments by Bach bearing on the question will be referred to in the Appendix.

†*Die Festkigeits Versuche.* Berlin, 1870.

for this purpose being counted was found to depend, first, on the maximum stress, and secondly on the fluctuation of stress.

First suppose the stress alternately tensile and compressive of equal intensity. Wöhler tried this both in bending and twisting. Fig. 169 represents a round bar *DE*, with one end enlarged and fitted into a socket in a revolving shaft *S*. At the free end *E* a load *P* was applied, which produced at *D*, the point of maximum bending, a stress of intensity found by the usual formula. The shaft being set in motion the piece of material was bent alternately backwards and forwards once in each revolution. A number of exactly similar pieces being tried successively with gradually diminishing loads, the revolutions necessary to produce fracture were found to increase as shown by the annexed table for the case of wrought iron, which gives the revolutions necessary for fracture at a given stress.

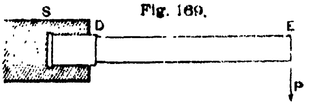

Fig. 169.

ALTERNATE BENDING OF A BAR OF AXLE IRON FURNISHED BY THE PHŒNIX COMPANY IN 1857.

STRESS IN LBS. IN SQ. INCH.	REVOLUTIONS.	REMARKS
33,300	56,430	The last of these pieces was unbroken after more than 132 million revolutions. The ultimate tensile strength of this iron was 47,000 lbs. per square inch and the elongation about 20 per cent.
31,200	99,000	
29,100	183,145	
27,000	479,490	
25,000	908,800	
23,000	3,632,588	
20,800	4,918,000	
18,700	19,187,000	
16,600		

It is already very large at 18,700 lbs. per square inch, and at 16,600 the piece cannot be broken at all. We may therefore place the resistance to alternate bending of this kind of iron at about 17,000 lbs. per square inch, while for cast steel of various qualities it was found to range from 25,000 to 30,000, and for copper 10,400.

Similar experiments were made by Wöhler with a different apparatus on alternate twisting. They were less extensive, but led to the important conclusion that the strength of the qualities of steel for which they were tried was four-fifths that of the same steel under alternate bending. From this it is inferred that the proof resistance to shearing is four-fifths the proof resistance to tension, as required by a theory of strength already referred to. (See Art. 229.)

Next, suppose the stress to fluctuate within given limits. It had already been shown by Prof. J. Thomson, in a paper published in 1848,* that twisting or bending a bar beyond its elastic limit in one direction must increase its powers of resistance to a second strain in the same direction, and diminish it to a strain in the opposite direction. Accordingly, we find that when a bar is strained in one direction only its powers of resistance to unlimited repetition are greatly increased. Wöhler made very extensive experiments on stretching, bending, and twisting of pieces of iron and steel to a given maximum stress, the load being wholly or partially removed at each repetition. The number of repetitions necessary for fracture was found to vary, not only according to the magnitude of the maximum stress, but also according to the fluctuation. It was greater when the load was only partly removed than when it was wholly removed. Some results are given in the annexed table, which shows the limits between which the stress varied when fracture was just not produced by unlimited repetition.

RESISTANCE TO UNLIMITED REPETITION OF BENDING.

NATURE OF FLUCTUATION.	FLUCTUATION OF STRESS.	
	IRON.	STEEL.
Alternating, - - -	+17,000 ; −17,000	+29,000 ; −29,000
Load wholly removed, -	31,000 ; 0	50,000 ; 0
Load partially removed, -	45,000 ; 25,000	83,000 ; 36,500

REMARK. The ultimate tensile strength of the iron was 47,000, and of the steel, 106,000.

Any greater fluctuations with the given maximum stress, or any greater maximum stress with the given fluctuation, produced fracture. Experiments on stretching and twisting led to similar results, and it should be especially noticed that in cases of unlimited repetition the resistance to stretching is the same as the resistance to bending, but the resistance to twisting less. In the case of cast iron the resistance to stretching with complete removal of load was found to be 10,400, but no experiments on bending or twisting were made.

Thus it appears that the ultimate strength of a material is very different according to the fluctuation in the load to which it is exposed; the same iron which will bear only 17,000 lbs. per square inch when bent alternately backwards and forwards, will bear 31,000 when bent in one direction only, and 45,000 when the stress varies between 25,000 and 45,000. Several formulæ have been devised to represent the results of the experiments, of which one will now be given.* Let p_0 be the ultimate tensile strength of a material and Δ the fluctuation, then the actual ultimate strength under unlimited repetition will be

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* *Cambridge and Dublin Mathematical Journal.*

$$p = \tfrac{1}{2}\Delta + \sqrt{p_0(p_0 - \tfrac{3}{2}\Delta)}.$$

When $\Delta = 2p$ we get the case of alternative stress with which we commenced, where $p = \frac{1}{3}p_0$, and when $\Delta = p$ we have the case of repeated stress in one direction with complete removal at each repetition. The formula gives nearly the same results as the experiments in the extreme cases, and may be expected to be approximately correct in intermediate cases.

231. *Influence of Repetition on the Elastic Limit* The remarkable results obtained by Wöhler, as described in the last article, have since been verified by further researches, amongst which may be especially mentioned those made by M. Bauschinger and Sir B. Baker.† In all such experiments we find the same regular increase in the number of repetitions necessary to produce fracture with a given maximum stress and a given fluctuation, and the limiting values below which a piece of given material does not break, however many the repetitions are, is much the same. The true interpretation of the experimental results is still uncertain. The gradual deterioration of strength which takes place is so far as is known confined to the section where fracture occurs even in stretching where all sections are exposed to the same stress. The character of the fracture indicates that minute cracks are produced which gradually become large flaws as the repetition continues. These facts show that it is not absolutely necessary to suppose an actual diminution of strength of the material itself, such as is commonly described by the term "fatigue." The crack may conceivably be initially produced by vibrations in the elastic solid of the nature of those producing sound, for such vibrations, though representing a small amount of energy, are capable of considerably augmenting the stress due to an external load.

Another exemplification of Thomson's principle (p. 448) is furnished by the not less important conclusion arrived at by M. Bauschinger that stress in one direction beyond the elastic limit lowers the elastic limit for stress in the opposite direction. Thus if a bar of wrought iron be stretched beyond its natural elastic limit of 10 tons per square inch, the elastic limit in compression becomes less than its original value of 10 tons, and by alternating tension and compression it is possible thus to lower the limit in both directions, and when thus lowered by continued alternate stress the limit is found to agree with the value obtained by Wöhler for the resistance to alternate stress. Evidently the two results must be closely connected, though the nature of the connection is as yet obscure, and difficult to conceive if we reject entirely the idea of "fatigue."

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* *Elements of Machine Design*, by Prof. W. C. Unwin, p. 25. A modified form of this equation, replacing the co-efficient $\frac{3}{2}$ by a quantity k slightly different in different materials, is given by the same writer in his work on *Testing*, p. 391.

† *Report of the British Association* for 1887.

Whatever the true explanation, the resistance to alternate tension and compression is of great importance as fixing for a given material a minimum value of its elastic strength. If we attempt to define the elastic limit of a material merely with reference to the degree in which the material possesses the properties of an ideal elastic solid, we find that it may change to any extent by repetition. This applies not only to the "breaking-down point" (p. 431), which does not always exist even in ductile materials, but also to the lower limit where stress is more or less approximately proportional to strain. It would seem that the natural condition of matter is one of imperfect elasticity, though under small loads the deviations may be very minute, and that a nearly perfectly elastic state under considerable loads is generally a constrained condition, due to the mode in which the material has been treated during manufacture or otherwise.

The resistance to unlimited repetition of alternate tension and compression, either direct or in bending, of various materials is given in the annexed table.

RESISTANCE TO ALTERNATE STRESS.

MATERIAL.	LIMIT STRESS, TONS PER SQ. INCH.	MATERIAL.	LIMIT STRESS, TONS PER SQ. INCH.
Plate Iron	7	Soft Steel	9½
		Cast Steel	13½
Bar Iron	8	Copper	4¾

So far as the term "elastic strength" is admissible at all this might be described as the *real* elastic strength of the material. (See page 457.)

232. *Impact.*—In Wöhler's experiments the load was applied without shock. In cases of impact also there is reason to believe that within the limit of elasticity a material will bear unlimited repetition. Thus in Hodgkinson's experiments on beams struck by a pendulum weight, it was found that if the blow produced less than one-third the ultimate deflection, the beam would sustain more than 4,000 blows without apparent injury, a plate of lead being introduced to prevent local damage.

In most cases of impact, however, the elastic limit is exceeded, and the destructive effect of repetition is then much greater than when the load is gradually applied. In the ductile metals the resistance to impact is at first very great, as has already been sufficiently explained; but every time the limit of elasticity is overpassed the hardness of the metal is increased, so as to make it less able to resist the second blow. This may be illustrated by a diagram in which OQQ' is a curve of stress and strain, Q the original elastic-limit, $Q'N$ the stress produced by the first blow, so that the area $OQQ'N$ represents the energy of that blow. The effect of the blow is to raise the limit from the stress QM to the stress $Q'N$ nearly. Hence the curve of stress and strain now becomes OVS, where V is the new limit, and the material will only bear a blow the energy of which is the triangle OVK, without the original stress $Q'N$ being exceeded. Thus by constant repetition of blows, which originally only produced a stress not much exceeding the elastic limit, a much greater stress may be produced. It is believed that this is in the main the explanation of the destructive effect of repeated blows and continuous severe vibration: pieces of material exposed to which are found to have a short life. As already remarked, however, vibrations representing a small amount of energy may considerably augment the maximum stress on a piece of material.

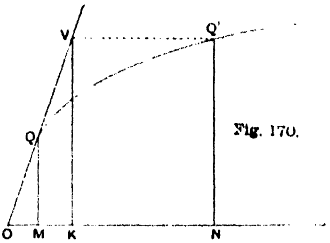

Fig. 170.

Co-efficients of Strength and Factors of Safety.

233. *Factors of Safety and Co-efficients of Working Strength.*—Before we can apply theoretical formulæ to the determination of the dimensions of actual structures and machines, it is necessary to know the value of the co-efficients of strength to be used, and this is always a matter which requires great care and attention to the circumstances under which certain dimensions are found to be sufficient by long practical experience. In the first instance it depends on the ultimate strength of the material, and may be expressed by dividing that quantity by a Factor of Safety. But the ultimate strength varies as we have seen, and the word "factor of safety" is used with various meanings.

The primary meaning of the expression is the divisor necessary to provide a margin of strength for unknown contingencies such as the following:

(1) The ultimate strength of a piece of material is uncertain, for two pieces of material of the same description and manufacture are not always equally strong. The liability to variation is much greater in some materials than others, for example, in cast iron than in wrought iron. The strength of stone varies so much that, in carrying out any important work, experiments are frequently made on the stone to be employed in it.

(2) The piece of material may be subject to corrosion or other influence, which in course of time diminishes its strength.

(3) Errors of workmanship are unavoidable, and in some instances may greatly increase the stress to which the material is exposed. This, for example, is the case in pillars, the factor of safety for which must always be greater than for other parts of a structure.

(4) The magnitude of the load and its mode of application is generally more or less uncertain. This, however, may be provided for by assuming a maximum load.

The factor required to provide for contingencies such as these may be called the "real" factor of safety, but by an addition to its value it may be made to provide against contingencies which can if necessary be exactly foreseen and calculated. Assuming all the forces acting on a structure to be known it is possible to find the stress on each part of it, but the calculation may be too complex to be often used, or its result may be known approximately under similar circumstances. Hence it often happens that the dimensions of a piece are determined by a formula involving only part of the straining forces which act on it, and the rest are provided for by an increased factor of safety. Thus the real stress on the metal of a screw bolt, when the effect of screwing up is taken into account, may be double the total tension per square inch of the gross sectional area. If that bolt be used for a cylinder cover exposed to steam pressure the total tension will be much greater than that due to the pressure of the steam. These two circumstances taken together may be taken into account by the use of a factor of safety three or four times greater than the real one. Such cases are common in practice, but the factor to be used must then be determined by comparisons with good examples under similar circumstances.

Again, it is necessary that a piece should be stiff enough as well as strong enough, and when formulæ for strength are used in such cases it is often necessary to employ very large and very arbitrary factors of safety. Here, however, the difficulty arises from an erroneous method of calculation.

234. *Values of Co-efficients.*—In parts of machines subject to alternating straining actions we know by Wöhler's experiments that the ultimate strength is somewhat less than the elastic strength under simple tension, being for wrought iron and soft steel about one-third the ultimate tensile strength. The load on such parts will rarely be applied without shock, the effect of which cannot precisely be determined. In ordinary cases it will be sufficient to treat this case as if the load were suddenly applied by using a further divisor of 2. We thus obtain the working strength by using a total factor of safety of 6. For wrought iron this gives a co-efficient of 4 tons, or 9,000 lbs. per square inch, which is known by experience to give sufficient strength where all the straining actions are taken into account. For timber the usual factor is 10. The co-efficient for shearing and torsion is to be taken provisionally as four-fifths that for tension and bending, that is for wrought iron 3¼ tons per square inch; but from the incompleteness of experimental data it is not certain that this value is not too large.

In structures the fluctuation of the straining actions is in general much less, and the ultimate strength by Wöhler's experiments is much greater. Yet the working strength employed is not very different. In the first place it is rarely permissible to approach the elastic limit from the danger of a permanent deformation. In the second place, the whole of the straining actions on each piece of the structure, especially the effect of imperfect joints, are rarely included in calculations. For example, the friction of pin joints may, under unfavourable circumstances, add 60 per cent. to the maximum stress on the links of a suspension chain (Ex. 4, p. 462). Hence the working strength for wrought iron rarely exceeds 4½ or 5 tons per square inch. In reckoning the load Rankine recommended that the "dead" load should be divided by 2 and added to the "live" load in order to obtain the effective live load. More recently the importance of Wöhler's experiments has been recognized, and it has been proposed to find the ultimate strength of each piece under the maximum stress and fluctuation of stress to which it is subject, and divided by a constant factor of safety. Some rule based on this principle is now very generally adopted.

In the case of marine boilers another important step has been taken by the employment of a fixed *margin* of safety instead of a factor. Thus, instead of taking the working pressure as (say) one-fifth the bursting pressure, whether that pressure be low or high, it is taken as (say) 200 lbs. less, giving a much smaller factor at high pressure than low. It is on this principle that the co-efficient in the usual formula given in Ch. XII. (p. 302) is taken so large, even reaching 20,000 lbs.

per square inch. In designing cylindrical boilers for the navy a margin of 90 lbs. below proof appears to be allowed. This gives a co-efficient of 12,500 lbs. per square inch for the working pressure of 150 lbs., the proof strength of steel being taken as 12 tons and the efficiency of the joints ·75.

235. *Fibrous Materials. Ropes.*—Fibrous materials are those which may be regarded as made up of fibres, usually of organic origin, more or less closely united by cohesion or interlacing. The relative movements of the fibres are hindered by forces of the nature of friction, which are much less than the molecular forces to which the tenacity of a homogeneous solid body is due. Hence the strength and stiffness of a piece of material are much less than those of the fibres of which it is made up.

In most kinds of woods the fibres are arranged longitudinally, and the material is therefore especially characterized by its low resistance to divisions into parts longitudinally. Thus the resistance to longitudinal shearing of fir timber is only 600 lbs. per square inch, whereas its tenacity is about 20 times this amount, approaching that of cast iron. So, again, crushing takes place by longitudinal splitting under a stress little more than half the tenacity. Further, the condition of the material greatly influences the lateral cohesion of the fibres and thus affects its strength and elasticity. In timber which has been artificially dried the elasticity is nearly perfect up to the breaking point, whereas in the green state the elasticity is imperfect and the strength greatly reduced. Hence the importance of seasoning timber so as to be moderately dry.

The ordinary formulæ, however, will apply in all cases where the stress is a simple longitudinal stress, the direction of which is that of the fibres; that is to say, in tension, compression, and ordinary cases of bending. They will only fail when the bending is accompanied by crushing and shearing of considerable intensity, as when short pieces are acted on by transverse forces. (See Appendix.)

In cloth and similar materials two sets of fibres at right angles are united by interlacing. Resistance to tension is thus obtained with almost complete flexibility.

In ropes of all kinds the fibres are ranged in spiral curves in the process of manufacture, and their tension then produces lateral pressure, the friction arising from which is sufficient for union. The strength of a rope, though very great compared with its weight, is only one-third that of the yarn of which it is spun, and on a similar principle the strength of large cables is less than that of the smaller ropes called "hawsers" of which they are made up. The strength of a rope is usually expressed by the formula

$$T = \frac{C^2}{k},$$

where C is the girth of the rope in inches, T the tension in tons, and k a constant. The old rule in the navy was to take $k = 5$ to obtain the breaking weight of a rope, but the table now employed gives $k = 3·3$, that is, a strength 50 per cent. greater. In small ropes k may be even less. The safe working load is not more than one-sixth the breaking load. In iron wire ropes $k = 1$, or for ropes above 6 inches girth somewhat more. The strength of wire ropes is more than doubled by the employment of steel. The safe working load may be taken as one-fifth their breaking load.

236. *Tables of Strength.* For a detailed account of the properties of the materials of construction the reader is referred to Professor Unwin's excellent treatise.* A convenient summary of the older experimental results is given in Rankine's *Useful Rules and Tables.* It will be here sufficient to give a few examples.

Table I. gives the weight and working strength of various materials.

TABLE I.—WEIGHT AND WORKING STRENGTH.

MATERIAL.	WORKING STRENGTH				WEIGHT IN POUNDS.	
	Tons per Square Inch		Equivalent in Feet of Material.		Per Cubic Foot	Per Yard of Prism 1 Inch Square.
	T.	C.	T.	C.		
METALS.						
Cast Iron,	1·5	4·5	1070	3210	450	9·4
Wrought Iron,	4·5	4·5	3020	3020	480	10
Soft Steel,	6	6	4030	4030	480	10
Steel Wire,	13		8800	—	480	10
Brass,	2½		1400		570	12
Copper Wire,	4		2340		550	11·5
Aluminium,	2½		5050		160	3·3
WOODS.						
Deal,	·5	·3	4480	2700	36	·75
Oak,	·75	·45	5040	3020	48	1
STONE, ETC.						
Granite,		·3		576	165	3·5
Brickwork,		·06		160	120	2·5
ROPES.						
Hemp,	—		2700		—	
Iron Wire,	—		2600			—
Steel Wire,	—		6000		—	—

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* *Testing of the Materials of Construction.* Longmans, 1888.

In estimating the working strength it is supposed that the load is always in one direction, and allowance is made for a moderate fluctuation. A larger value may sometimes be used when the load is perfectly steady and gradually applied, while for an alternating load it would be too large for safety. The first two columns give the safe load per sq. inch in tension or compression, and the second two the equivalent length in feet of a bar or column of the material. It is on this last quantity which is denoted by λ in Arts. 40, 41, p. 81, and elsewhere, that the limiting dimensions of a structure depend. It will be observed that weight for weight timber is stronger than wrought iron, but on the other hand the joints of the timber structure are, with certain exceptions, much weaker than in the case of iron, so that the comparison is not actually so favourable. For springs, see next page.

Table II. gives a few examples of the elastic properties of materials used in construction.

TABLE II. ELASTICITY.

MATERIAL.	ELASTIC STRENGTH. (Tons per Square Inch.) T.	C.	S.	ELASTICITY. (Tons and Inches.) E.	C.	RESILIENCE UNDER TENSION. Foot-Pounds per Cubic Foot.	Equivalent Height. (Feet and Inches.)
Cast Iron, - -	3	9		8500	3400	171	4½"
Wrought Iron, -	10	10	8	12500	4900	1290	2' 8"
Mild Steel, - -	15	15	12	13000	5200	2800	5' 10"
Hard Steel, - -	25	25	20	14000	5200	7200	15'
Copper (Rolled), -	5·4	5·4	4·3	7600	2800	620	1' 1½"
Fir, - - -	1·5			700	---	518	14' 5"
Oak, - - -	2			700		920	19' 2"

Very various estimates may be made of the elastic strength of a material according to the delicacy of the methods of measurement adopted, and the treatment to which the material has been subjected. Cast iron is an example of a material which can hardly be said to have an elastic limit, and at the limit given of 3 tons per sq. inch the elongation in many cases may be as much as 10 per cent. greater than if it were perfectly elastic. Materials which possess a yield-point are far more nearly perfectly elastic up to a limit in its immediate neighbourhood. In ordinary testing therefore the yield-point itself, or some point not far below it, determined from a diagram or by some conventional rule, is adopted as the limit. Thus 85 per cent. of the stress at the yield-point or otherwise 45 per cent. of the ultimate strength are limits within which the failure of elasticity is very small, especially when the bar has been stretched before. This would give in bar iron a limit of 12 or 13 tons per sq. inch. Yet there can be no doubt that when accurate measurements are made on a bar which has not been stretched before, there is a perceptible failure at a much smaller stress. In the long rod of wrought iron, tested by Hodgkinson, the extensions were sixty times as great as in an ordinary test-piece, and the limit obtained was about 10 tons, as already described. With a sufficiently delicate measuring apparatus the same result has been reached with a test-piece of ordinary length. In the softest kinds of steel the limit thus defined is little higher, in plates lower, and in the softer metals generally appears to be about 40 per cent. of the ultimate strength. In the hardest and strongest kinds of steel the material is elastic nearly up to the point of rupture. The resilience* under simple tension or compression of steel springs when reckoned in feet of material is probably nearly 400, which, as previously explained, is reduced in bending to one-third, or in torsion to about two thirds of this amount. This corresponds to an elastic strength in tension of 125 tons per sq. inch. The working strength of steel springs bent in one direction only may be taken as 50,000 to 60,000 lbs. per sq. inch, values which are sometimes greatly exceeded.† In torsion the co-efficient to be employed is smaller as already explained, probably about 40,000. The strongest steel pianoforte wire has a strength of 150 tons per sq. inch and a resilience of 570 feet. The specimen of copper chosen is taken from a diagram given by Professor Unwin as a normal example of rolled copper: the limit is diminished one-half by annealing, a circumstance which shows the artificial character of the elastic state in the ductile metals. It would seem that such metals in their natural condition are always slightly plastic.

Co-efficients of elasticity are difficult to estimate with exactness, though the difficulties are not so great as in the case of the limit of elasticity. The values given in the table are only common examples in round numbers. From these the resilience is calculated as already fully explained.

The greatest stress which a material will bear without damaging it in any way is commonly described as the *proof* stress. Rankine defined it as the stress which might be applied twice or more times without producing an *increased* permanent set. A natural extension of this definition is furnished by the resistance to alternate stress given in the table on page 448.

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* See a paper by Mr. Lewis, *Van Nostrands Magazine*, May, 1885

† *Taschenbuch vom Hütte*, 1892; Abtheilung I., pp. 309, 372.

Table III. gives examples of the ultimate strength and ductility of materials.

By the ultimate strength is understood the maximum load divided by the original sectional area of the test-piece. As previously explained, in the stretching of a ductile metal to fracture the actual breaking load is a smaller quantity, and the difference is sometimes considerable: it is however seldom given in tables of strength. The table gives in the first three columns the ultimate resistance to stretching (T), compression (C), and shearing (S).

TABLE III.—ULTIMATE STRENGTH AND DUCTILITY.

MATERIAL.	Ultimate Strength.			Yield-Point.	Elongation per Cent.	Work done in Stretching Inch-Tons per Cubic Inch.	
	T.	C.	S.			To Yield-Point.	To Fracture.
Iron Bars,	25	22	19	15	20	$8·65 \times 10^{-3}$	4·33
Iron Plates,	22	19	17	14	10	7·54 „	1·8
Soft Steel (·15 to ·3 per cent. of Carbon),	30		23	18	25	12·5 „	6·5
Medium Steel (·3 to ·5 per cent. of Carbon),	35			22	15	18·6 „	4·6
Hard Steel (·5 to ·75 per cent. of Carbon),	45			30	8	34·6 „	3·2
Cast Iron,	7½	45		–	–		$6·7 \times 10^{-3}$
Lead,	1·1						
Sheet Copper,	13½						
Cast Copper,	8½						
Fir,	5½		·27				
Oak,	5½		1	–			

The yield-point in the ductile metals is given in the next column; it is from ·5 to ·7 of the ultimate strength, being generally about ·6. Ductility is commonly measured by the total elongation up to fracture reckoned as a percentage of the original length of the test-piece. To render the measure definite, the length of the test-piece shall be some given multiple of the diameter, generally 8 or 10. The ultimate strength and ductility of steel vary according to the amount of carbon it contains in such a way that the sum of the two remains nearly constant, being about 53 in the example given in the table. In steel compressed in a fluid state by the Whitworth process the constant sum would be about one-third greater.

The total amount of work done in stretching till fracture occurs may be separated into two parts, one before and the other beyond the yield-point. The first is calculated approximately by supposing the elasticity perfect up to the yield-point, and furnishes a rough maximum estimate of the resilience. The second may be obtained graphically from a diagram of stress and strain, or calculated by a formula given by Professor Kennedy, based on the supposition that the curve is a parabola. If x is the fractional elongation, p the yield stress, f the ultimate strength,

$$\text{Work done} = x \, . \, \frac{p + 2f}{3} .$$

The last two columns of the table give the two parts of the work, the first part being a small fraction the numbers are multiplied by 10^8 to avoid fractions. The whole is only a small part of the total resistance to impact of a cubic inch of the material, because only a part of the test-piece is fully stretched. If we consider the crushing of a small block, the result is many times greater; in the example given in the table on page 434, the total work done is about 32 inch-tons per cubic inch.

The value given for cast iron is obtained by integration from Hodgkinson's equation of the curve of stress and strain. In many kinds of cast iron the work done in fracture would be 2 or 3 times greater, but in any case is a small fraction.

ADDENDA.

237. *Principle of Similitude.*—When geometrically similar test-pieces of similar material are stretched till fracture occurs, the percentage of elongation is the same, the pull is proportional to the sectional area and the work done to the volume of the piece This law, which has been proved by the experiments of M. Barba, is merely a particular case of a general principle which applies to all similar and similarly loaded pieces, whether or not the limit of elasticity has been over-passed. To produce similar deformations, whether in stretching, bending, crushing, or in any other way, the load must be proportional to the sectional area and the work done to the volume of the pieces. It has been verified by experiments on the crushing of stone by M. Bauschinger, and a number of other examples will be found in a small treatise by Professor Kick.* Any deviations from this law should be due to differences of material and mode of manufacture between small pieces and large ones. In framing semi-empirical formulæ for cases in which exact formulæ are unattainable this law should be borne in mind. Thus in the case of pillars the formulæ proposed by Hodgkinson do not satisfy the law, and should be rejected in favour of some formula, such as Gordon's, which does satisfy it.

238. *Expansion and Contraction.*—We conclude this division of our work by giving some explanation of the effect of changes of temperature, a subject too important to pass by unnoticed.

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* *Das Gesetz der Proportionalen Widerstande*, Leipzig, 1885.

When a homogeneous body, free from initial strain, is uniformly heated throughout its whole mass, it undergoes a change of linear dimensions which is the same in every direction, being given by the equation

$$e = \frac{t}{K},$$

where t is the rise of temperature and K a quantity which in a given material is roughly approximately constant for a moderate rise of temperature, being for degrees Fahrenheit given by the annexed table.

LINEAR EXPANSION OF METALS.

METAL.	K	$\frac{K}{E}$	σ	$\frac{100K}{\sigma E}$
Wrought Iron, - -	147,500	11°·3	7	1·26
Cast Iron, - - - -	162,000	20°	6·5	·77
Copper, - - - -	104,500	16°	35	·18
Brass, - - - -	95,400	25°		-

If the change of dimension is forcibly resisted the stress produced is the same as, reversed, would be required to produce that change in the absence of a change of temperature. Thus if a heated bar be prevented from contracting as it cools through $t°$, the tensile stress

$$p = \frac{E}{K} \cdot t$$

will be produced. The change of temperature corresponding to 1 ton per sq. inch is here K/E, a quantity given in degrees Fahrenheit in the third column of the table.

If the body be unequally heated internal stress will generally be produced, even if there be no external resistance to deformation. In exceptional cases, however, there may be no internal stress, and of these one of the most simple will now be considered.

Let us suppose that heat is flowing steadily through a flat plate of thickness y inches in consequence of a difference of temperature $t°$ of its faces. The quantity of heat which flows per sq. ft. per minute is

$$F = \sigma \cdot \frac{t}{y},$$

where σ is a co-efficient of conductivity given in the fourth column of the table. The outer surface of the plate being hotter than the inner expands more, and if there be no external resistance, the plate bends without internal stress into a spherical form. If R be the radius, reasoning as in Art. 153, p. 307,

$$\text{Curvature} = \frac{1}{R} = \frac{e}{y} = \frac{t}{Ky} = \frac{F}{\sigma K}.$$

Hence the stress produced if curvature be forcibly prevented is for a given flow proportional to $E/\sigma K$, a quantity which, multiplied by 100, is given in the last column of the table.

In designing a structure or machine the possible effects of expansion or contraction by changes of temperature have always to be carefully considered.

The importance of the stress produced by unequal heating of boiler plates was pointed out by Mr. Yarrow in a paper read before the Institution of Naval Architects in 1891.*

EXAMPLES

1. Show that the modulus of rupture (p. 441) of a material is 18 times the load which will break a bar of the material 1 inch square and 1 foot long; the bar being supported at the ends and the load applied at the centre.

2. A balcony, 6 feet long and four feet broad, is supported by a pair of cast-iron beams fixed in the wall at one end. The beams are of rectangular section, 2 inches broad, and depth near the wall 4 inches What load per square foot will the balcony bear, the stress on the iron being limited to 1 ton per square inch? Also, how should the depth vary for uniform strength along the length of the beam?

Ans. Equating the greatest bending moment to the maximum moment of resistance to bending we find the load which the balcony will bear

$$= 41{\cdot}5 \text{ lbs. per square foot.}$$

As to the depth of the beam: for uniform strength $\frac{M}{I}y$ must be constant from which we find that the depth at any point of the beam must be proportional to the distance from the outer end of the beam; so that the lower side of the beam should be a sloping plane.

3. A paddle shaft is worked by a pair of engines with cranks at right angles. Supposing the steam pressure constant, and the resistance of each wheel equal and uniform, and obliquity of connecting-rod neglected: compare the co-efficients of strength to be used in calculating the diameter of the paddle and intermediate shafts.

Ans. The uniform moment of resistance of the paddle wheel = ½ the mean turning moment of the two engines. The twisting moment of the paddle shaft, when either crank is on the dead centre, = ½ maximum twisting moment of one engine. At the same instant this is the same twisting moment on the intermediate shaft. When the other crank is on the dead centre the twisting moment on intermediate shaft is the same in magnitude, but reversed in direction, and when the two cranks make angles of 45° with the dead centres the twisting of the paddle shaft = ½ the maximum combined twisting moment of the two engines, that is $\sqrt{2}$ times its amount when either crank is on the dead centre; but the twist is in the same direction always. Therefore on the paddle shafts the stress alternates between x and $x\sqrt{2}$, and on the intermediate shaft between x and $-x$.

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* *Transactions of the Institution of Naval Architects*, 1891.

Hence applying the formula

$$p=\frac{1}{2}p+\sqrt{p_0(p_0-\frac{3}{2}p)},$$

we have for paddle shaft,

$$p=\cdot414x;\ p=1\cdot414x;\ \therefore\ \bar{p}=\cdot292p;$$

substituting, we obtain

$$p=\cdot9p_0.$$

For intermediate shaft, $p=2x$; $p=x$; $p=2p$; and $p=\frac{1}{3}p_0$.

If the stress on the paddle shaft alternates to zero, by the wheels rolling out of the water, or by the stopping of the engine, then $p=\cdot6p_0$.

4. A suspension chain is constructed with bar links united by pin joints; the diameter of the pins is two-thirds the breadth of the link (p. 374). If the bridge vibrate show that the maximum stress on the links may be increased by deviation (p. 240) due to friction of pins in the ratio $1+2f:1$, where f is the co-efficient of friction.

5. Find the work done in crushing the block of steel, particulars of which are given in the table on page 434. *Ans.* 31·7 inch tons per cubic inch.

6. If similar armour plates with similar backing are struck by similar shot with a *given* velocity, show that for the same penetration the diameter of the shot must be proportional to the thickness of the armour plate.

7. If the breaking load of a beam 4 inches deep, 2 inches broad, and 3 feet span, be 1 ton: find the breaking load of a beam 9 inches deep, $4\frac{1}{2}$ inches broad, 11 feet 3 inches span.

8. Find the weight of steel springs necessary to provide one-quarter of a horse power for one hour, assuming that the springs operate by bending, and allowing for safety a margin of 25 per cent. of the total resilience. (See page 457.) *Ans.* $2\frac{1}{4}$ tons.

9. Heat is flowing through a circular plate of wrought iron $\frac{3}{4}$ths of an inch thick at the rate of a 100 thermal units per sq. foot per minute. The plate when cold is flat; find the amount of bulging, assuming the curvature to be unresisted.

10. In a cast-iron beam of rectangular section, the "bending strength" with side vertical is found to be $1\frac{7}{8}$ times the simple tensile strength (p. 441). Assuming that the stress is completely equalized on each side of the neutral axis: find the shift of the neutral axis. *Ans.* Shift $=\frac{1}{8}$ depth.

11. In the last question, suppose the diagonal vertical instead of the side and the bending strength 2·35 times the simple tensile strength: find the shift of the neutral axis. *Ans.* Shift $=\cdot044\times$ depth.

PLATE VIII.

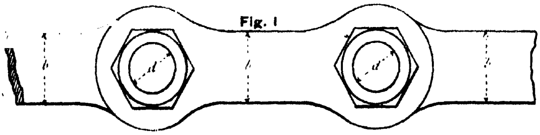


Fig. 1

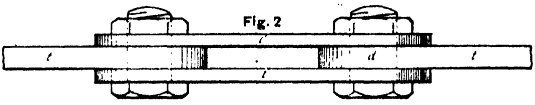


Fig. 2

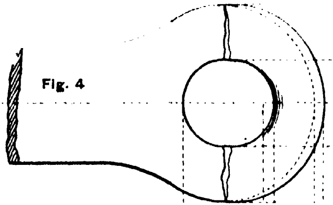


Fig. 4

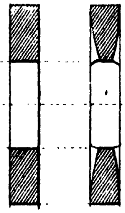


Fig. 3

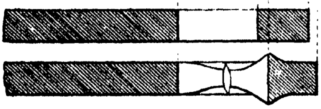
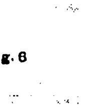


Fig. 5

Fig. 6



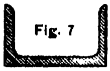


Fig. 7

Fig. 8

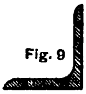


Fig. 9

Fig. 11

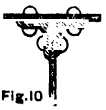


Fig. 10

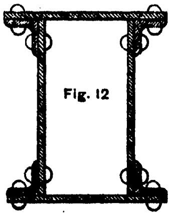


Fig. 12

Fig. 13

Fig. 14

To face page 463.

DESCRIPTION OF PLATE VIII.

To illustrate various questions considered in Chapters XII. and XV., Plate VIII. has been drawn.

Figs. 1, 2 represent the pin joint connecting two bars in tension, discussed in Art. 193, p. 374. Figs. 3, 4, 5 show the way in which the joint yields when the pins are too small. In Fig. 4 the original dimensions of the eye and eyehole are shown by dotted lines, while the full lines show what they become after yielding. Fig. 3 gives transverse sections of the eye before and after failure, showing the thinning out due to lateral contraction during stretching beyond the elastic limit. After this contraction has reached a certain limit the metal tears asunder, as shown in Fig. 4. The longitudinal section (Fig. 5) shows the corresponding spreading out at the top of the hole due to compression beyond the elastic limit. This lateral expansion is partially prevented in riveted joints, and (p. 432) this may be the reason why direct stress in them is of less importance. The failure of pin joints in this way furnishes a good example of the "flow of solids."

The remaining figures of this plate are intended to give some idea of the manner in which iron girders are constructed. Figs. 6, 7, 8, 9 are transverse sections of "H iron," "channel iron," "tee iron," and "angle iron"; these are rolled in one piece, and, in combination with plates form the materials from which large girders are built up. For small beams such as floor joists H iron or tee iron of the requisite depth and sectional area may be used. Figs. 10, 12 are sections of two of the commonest forms of built-up girders. In the first the web is a single plate to which angle irons are riveted, to form the flanges, further strength being obtained by an additional covering plate. The second is similar, but the web consists of a pair of plates, a form known as a "box-beam." Fig. 11 is commonly used in shipbuilding as a deck beam or otherwise: a "bulb iron" here forms the web and lower flange, while the upper flange is formed by a pair of angle irons as before. Figs. 13, 14 give examples of girders of more complex construction employed where greater strength is necessary: one flange only is shown in section in each case.

PART V.

TRANSMISSION AND CONVERSION OF ENERGY BY FLUIDS.

PART V.—TRANSMISSION AND CONVERSION OF ENERGY BY FLUIDS.

239. *Introductory Remarks.*—We now return to the subject of Machines, with the object of studying those machines in which fluids are employed as links in a kinematic chain for the purpose of transmitting energy, or as a means by which energy is supplied, stored, or converted.

A fluid is a body in which change of form is produced by the action of any distorting stress, however small, if sufficient time is allowed. In a perfect fluid a sensible change would be produced by a stress of sensible magnitude in an indefinitely short time, but in all actual fluids a time is required which is inversely as the stress—that is, the stress is proportional to the rate of change. This property of fluids is called Viscosity, and is measured by a co-efficient, as will be seen hereafter. The viscosity of a fluid varies greatly in different fluids, and, in the same fluid, is dependent on the temperature. At high temperatures it is much less than at low temperatures. The viscosity of water is exceedingly small.

Fluids are either liquid or gaseous. In liquids the changes of volume are in general small, and no diminution of pressure on the bounding surface will cause their volume to increase beyond a certain limit. Gases, on the other hand, expand indefinitely as the external pressure diminishes.

Liquids are employed in machines either as a simple link in a kinematic chain transmitting energy from some source independent of the liquid, or as a medium by means of which the force of gravity exerts energy. Such machines are called Hydraulic Machines, the fluid employed being in most cases water. On the other hand, gases in general serve as the means by which that form of energy which we call Heat is converted into mechanical energy, capable of being utilized for any required purpose. They may, however, also be employed for the storage and transmission of energy.

The motions of fluids may be studied in two different ways. In the first the Principles of Work and Momentum are applied to the whole mass of fluid under consideration, or to portions which, though small, are yet of visible magnitude; but no attempt is made to conceive, much less to determine, the movements of the smallest particles of which the fluid may be imagined to be made up. This method may be described as the experimental theory, and, as applied to water, forms that part of the subject which is called "Hydraulics." It is based directly on experiment, and requires continual recourse to experiment, just as is the case in questions relating to the friction of solids. Nevertheless, being continually verified by the large-scale experiments of the hydraulic engineer, its results, as far as they go, are as certain as those of any purely experimental subject. On the other hand, an analytical theory has been constructed by means of which the motions of fluids are determined directly from the laws of motion, without reference to experience. This theory is usually called Hydrodynamics in treatises on mechanics. In the cases in which it is applicable it completely determines the motion of all particles of the fluid, and not merely that of the fluid as a whole.

The first two chapters of this division of our work will be devoted to Hydraulics and Hydraulic Machines, and the third to a brief discussion of the various applications of Elastic Fluids. The transmission and storage of mechanical energy by elastic fluids is often considered as part of hydraulics, because the method of treatment is in many respects similar. In this treatise it will be called "Pneumatics." The relations between heat and mechanical energy form a distinct science called "Thermodynamics," the principles of which will only be referred to when absolutely necessary.

CHAPTER XIX.

ELEMENTARY PRINCIPLES OF HYDRAULICS.

SECTION I.—INTRODUCTORY.

240. *Velocity due to a Given Head.*—When the level of the surface of the water in a reservoir is above surrounding objects, a HEAD of water is said to exist, the magnitude of which is measured, relatively to any point, by the depth (h) of the point below the surface. If the water extend to this point a pressure is produced there which, so long as the water is at rest, is given in lbs. per sq. ft. by the formula

$$p = wh,$$

where w is the weight of a cubic foot of water, that is to say, about 62½ lbs. for fresh water, or 64 lbs. for salt. A ton of water occupies 36 cubic feet when fresh and 35 when salt. These values are of course only convenient round numbers; the exact value of w for pure water at 39° F. is 62·425, while at 100° F. it is only 62. At temperatures above 75° 62 is more accurate than 62½; but, on the other hand, water is seldom entirely free from solid matter, which increases its density.

Since the above formula may be written

$$h = \frac{p}{w},$$

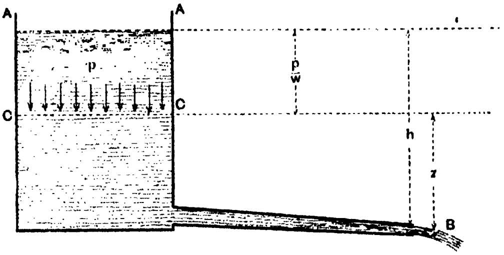
it appears that a pressure may be measured in terms of the head which would produce it. The fluid is usually water, for which h is reckoned in feet; and 1 lb. per sq. inch is equivalent to 2·3 feet of fresh, or 2·25 feet of salt water. For some purposes, however, mercury is employed, in which case the unit is generally 1 inch. One inch of mercury is equivalent to about ·49 lb. per sq. inch, that is, to a head of 1·1 feet of sea water, or 1·135 of fresh water. If the surface of the water be exposed to the atmosphere, the pressure p will be in excess of the atmospheric pressure, which must be added to obtain the absolute pressure. The mean value of the atmospheric pressure is 14·7 lbs. per sq. inch, which corresponds to a head of about 33 feet for salt, or 34 feet for fresh water.

When metric measures are employed, the unit of velocity is generally

1 metre per second (page 92), heads are reckoned in metres and pressures in kilogrammes per square metre. The value of w is then 1000, and that of g 9·81. The atmosphere was originally 76 centimetres of mercury, or 1·0333 kilogrammes per sq. centimetre, a value closely agreeing with that just given; but of late the "new atmosphere" of 1 kilogramme per sq. centimetre, or 14·233 lbs. per sq. inch, has been frequently used.

A head of water is a source of energy which may be employed in doing work of various kinds, or in simply transferring the water from one place to another. Let us take the second case, and imagine that, by means of a pipe, channel, or passage of any description, the water is delivered at B (Fig. 171), while at the same time, by a stream or otherwise, the surface of the water in the reservoir is kept constantly at the same level AA, so that the head h remains unchanged. The motion is then described as Steady, and consists simply in the transfer in each second of a certain weight of water from the stream to the reservoir, while an equal weight traverses the passage, and is delivered at B, the whole mass of water between AA and B remaining constantly in the same condition. The delivery at B may be supposed found by actual measurement; it is usually estimated in gallons per minute or cubic feet per second, as to which it need only be remarked that the gallon weighs 10 lbs., so that a cubic foot per second is about 375 gallons per minute. For large quantities, however, the cubic metre, which weighs about 1 ton, is also employed.

Fig. 171.



On delivery the water is moving with a certain velocity, but the definition and measurement of this quantity is not so simple. We must now suppose that the centre of gravity of the water delivered in some given time is observed and its velocity noted. This velocity will be the same whatever the time be, and will be a measure of the velocity of the mass of water considered as a whole. In some cases all particles of the water may be moving with this velocity, but in general this is not the case: it is then the mean velocity, and may be described as the "Velocity of Delivery." If the water be discharged by a channel which, near the exit, is of uniform transverse section A, a mean velocity may also be defined by the equation

$$v = \frac{Q}{A} = \frac{W}{wA},$$

where Q is the discharge in cubic feet per second, and W the weight of this quantity. The velocity thus defined is not identical with the velocity of the centre of gravity, a point considered further on (Art. 246).

The energy of motion of the water may now be separated into two parts, one external and the other internal (Art. 133, page 268), of which the first is

$$\text{Energy of Translation} = \frac{Wv^2}{2g},$$

while the second is due to the motions of the particles of water amongst themselves, and will be further considered as we proceed.

The whole energy of motion has been generated by the exertion of an amount of energy Wh due to the descent of the water from the level AA to the level B; and in cases where the internal energy may be neglected, we have, neglecting also friction,

$$\frac{v^2}{2g} = h,$$

where h the head is measured to the centre of gravity of the issuing water (page 181).

It has been here supposed that the surface of the water in the reservoir, and after delivery at B, is exposed to the atmosphere, but this is not always the case. Suppose in the figure the reservoir filled to the level CC only, but that the pressure on the surface has any value p instead of being simply that of the atmosphere. This pressure p may be produced by filling up the reservoir to the level AA where

$$h = z + \frac{p}{w};$$

and as the reservoir is supposed large, so that the water is sensibly at rest, except very near the exit, this can produce no change in the motion, which as before is given by

$$v^2 = 2gh = 2g\left(z + \frac{p}{w}\right).$$

In other words, in addition to the actual head z, we have a *virtual* head p/w, due to the difference of pressure p, thus giving a total head h.

The jet of water has been supposed to issue into the atmosphere, but the nature of the medium into which the discharge takes place has little influence, provided its pressure be duly taken into account. It has been proved by experiment that if the pressure of the atmosphere be artificially increased or diminished, the velocity is given by the same formula, modified as explained in the next article. This is also true if the efflux take place into a vessel of water.

241. *Hydraulic Resistances in General.*—The actual velocity v' with which the water is delivered is less than the value v just found, because a certain part of the available energy is always employed in overcoming certain resistances of the nature of friction, the origin of which we shall see gradually as we proceed. They are measured in two ways: (1) by comparing the actual velocity of delivery with that due to the head; (2) by considering how much energy is employed in overcoming them. In the first method we have only to introduce a co-efficient c given by

$$v' = cv,$$

which is called the Co-efficient of Velocity. It is of course always less than unity, and its value is found by experiment in each special case. In the second we write

$$h - h' = \frac{v'^2}{2g},$$

where h' is the "loss of head" due to the resistance. The value of h' is most conveniently expressed by connecting it with the *actual* velocity v, with which the water issues. For this purpose we replace h by $v^2/2g$ and v by v'/c, and thus obtain

$$h' = \left(\frac{1}{c^2} - 1\right)\frac{v'^2}{2g} = F\frac{v'^2}{2g},$$

where F is a new co-efficient called the Co-efficient of Resistance connected with the previous one by the equation

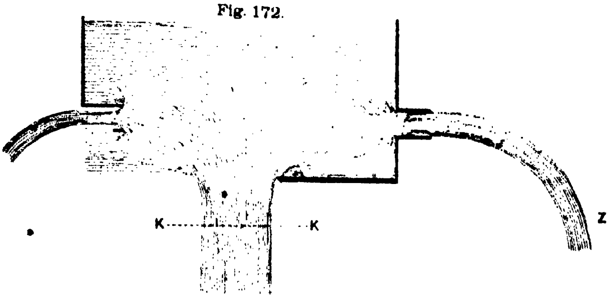
$$F = \frac{1}{c^2} - 1.$$

It is found by experience that the values of these co-efficients depend mainly on the form and nature of the bounding surfaces within which the water moves, and, subject to proper limitations, not on the pressure or velocity of the water—a fact which may be expressed by the following law of hydraulic resistance: *the energy lost by resistances is a fixed multiple of the energy of motion of the water.* This multiple is the co-efficient F which is sometimes fractional, but is often very large, as we shall see further on. The physical meaning of this law will be seen hereafter, and the apparent deviations from it which frequently occur will be accounted for.

242. *Discharge from Small Orifices.*—Fig. 172 shows a vessel of water discharging through a circular hole in the bottom which is flat. The whole is small, and its circumference is chamfered below to a sharp edge at the upper surface.

On observing the jet of water which issues, we see that it is nearly cylindrical but of diameter less than the diameter of the hole. The contraction is complete, so far as can be judged by the eye, at a distance of $d/2$ from the vessel; and by measurement is found to be in the ratio 4 : 5. that is, the sectional area of the jet is to the sectional area of the hole in the ratio 16 : 25.

Fig. 172.



If the hole be made in the vertical side of the vessel a contracted jet issues in the same way, but under the action of gravity it forms a curve which is very approximately parabolic in form, each particle moving nearly in the same way as a projectile *in vacuo*. This enables us to find the velocity of the efflux (v) by observing a point through which the jet passes, and we thus obtain experimentally the value of the co-efficient c which appears to be about ·97. The discharge is now given by the formula

$$Q = A_0 . v = ckA\sqrt{2gh},$$

where A_0, A are the contracted and actual areas of the orifice, and k is their ratio, which is a fraction called the Co efficient of Contraction; the discharge therefore depends on the product of the two co-efficients c and k which may be replaced by

$$C = ck,$$

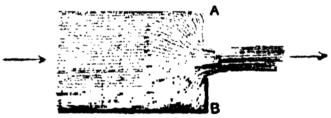
a quantity called the Co-efficient of Discharge.

The value of C can also be determined by direct measurement of the discharge, an observation which can be made with much greater accuracy than those of contraction and velocity on which it depends. In the present case it is usually about ·62, agreeing well with the product ·97 × ·64 of the values given above.

Some careful experiments have been made by Mr. Mair, from which it appears that very slight variations in sharpness of the edges of an orifice will produce a considerable effect on the co-efficient of discharge, the sharper the edge the lower the co-efficient, and a co-efficient as low as ·605 was thus obtained.* This is probably due to variations in the co-efficient of contraction. Until recently the only case in which a co-efficient of contraction had been found theoretically was that of a long narrow slit, for which Lord Rayleigh obtained the value of ·611; but in 1892 M. Boussinesq showed that the true value for a circular orifice was very approximately ·6. With orifices several inches diameter this is the actual value found by experiment: the larger values obtained with small orifices being probably due to friction against the flat side of the vessel before the orifice is reached.

With two forms of orifice the same co-efficients are used, but their numerical values are quite different. In the figure two cases are represented: on the right side of the vessel the water issues through a short pipe the entrance to which from the vessel is square-edged; on the left a similar pipe is employed, but it projects inwards instead of outwards. When the pipe projects outwards the water is found to issue in a jet the full diameter of the pipe, that is, k is unity; while, on the other hand, the velocity is much diminished, the value of c being only ·815. When it projects inwards the jet contracts greatly, the value of k being ·5 while the velocity is about the same as in a simple orifice. Thus C instead of being ·62 is ·815 and ·5 in the two cases. The causes of these remarkable differences will be seen hereafter, the results are only given here to illustrate the meaning of the co-efficients under consideration.

Fig. 173.



The contraction of the issuing jet depends on the average angle at which the moving particles converge towards the orifice before reaching it, and this is the reason why it is so great in the case of a short pipe projecting inwards. If the circumstances be such that the convergence is small the contraction diminishes. Fig. 173 shows a pipe of some size through an orifice in the flat end AB of which water is being forced, issuing into the atmosphere. The co-efficient k is found to depend on the proportion which the area of the original orifice A bears to that of the pipe S, because the smaller S is, the less is the angle of convergence. This has been expressed by an empirical formula due to Rankine which may be written

$$\frac{1}{k} = \sqrt{2\cdot618 - 1\cdot618\frac{A^2}{S^2}},$$

which will be found to give $k = \cdot618$ when S is infinite, as is nearly the case for a simple orifice as explained above, while for smaller values k

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*Proceedings of the Institution of Civil Engineers, vol. lxxxiv.

increases, becoming unity as it should when $S = A$. For a long narrow slit a rational formula has been obtained by Mr. (now Prof.) Michele.*

In a similar way if an orifice be near a corner of the vessel the contraction will be diminished. In these cases the contraction is usually described as "incomplete."

The passages through which the water is moving may be attached to a ship, locomotive, or other moving structure, in which case the velocity must be reckoned relatively to the structure, and the height due to the velocity must be reckoned as part of the head. If, for example, in the bow of a vessel moving through the water with velocity V an orifice be opened at the surface level, the water will enter through it, and if unacted on will move within the vessel with velocity V and will possess relatively to the vessel the energy $V^2/2g$ per unit of weight. If it be acted on during entrance by the head due to any difference of level or pressure, so that its velocity is changed from V to v, the corresponding change of energy will measure the work which is done, and therefore the equation $v^2 - V^2 = 2gh$ applies as before. The structure is here supposed to be moving uniformly in a straight line. A rotating casing will be considered in a later chapter.

243. *Steady Flow through Pipes. Conservation of Energy.*—Fig. 174 represents a vessel of water discharging through a large pipe, the section of which varies according to any law. If the pipe "runs full," that is, if it be always completely filled with water, the discharge is

$$Q = A_1u_1 = A_2u_2,$$

where u_1, u_2 are the velocities through two sections the areas of which are A_1, A_2. Hence the velocity is always inversely as the sectional area, and in an ordinary pipe in which the section is uniform

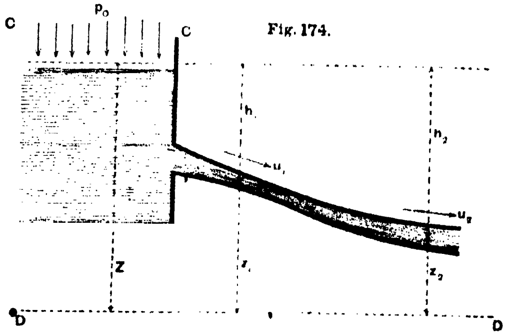

Fig. 174.

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* *Phil. Trans.*, 1891.

must be the same throughout. Let the pressures be p_1, p_2, and the actual head, that is to say, the depths below the water surface CC, h_1, h_2, then it appears from Art. 237 that

$$\frac{u_1^2}{2g} = h_1 + \frac{p_0 - p_1}{w}; \quad \frac{u_2^2}{2g} = h_2 + \frac{p_0 - p_2}{w},$$

where p_0 is the pressure on the surface CC.

Take now some convenient line DD at a depth Z below the water surface CC, and z_1, z_2, be the elevation of the section above this datum level so that

$$z_1 + h_1 = Z = z_2 + h_2,$$

then the above equations may be written

$$\frac{u_1^2}{2g} + \frac{p_1}{w} + z_1 = Z + \frac{p_0}{w} = \frac{u_2^2}{2g} + \frac{p_2}{w} + z_2.$$

This result shows that if u, p, z be the velocity, pressure, and elevation for any section of the pipe,

$$\frac{u^2}{2g} + \frac{p}{w} + z = \text{Constant}.$$

Each of the terms of this equation represents a particular kind of energy: the first is energy of motion, the third energy of position, the second is energy due to pressure, the origin of which will be further explained in the next chapter. The equation therefore shows that the total energy of the water remains constant as it traverses the pipe, and is accordingly the algebraical expression of the Principle of the Conservation of Energy. It supposes that no energy is lost by frictional resistances, and that any change in the internal motions of the particles amongst themselves may be disregarded. The word "head," the origin of which we have already seen, is frequently employed for the energy per unit of weight. (See Appendix.)

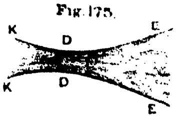
An important consequence of this principle is that where the sectional area of the pipe is least, and consequently the velocity greatest, there the pressure is least. Hence it follows that the velocity cannot exceed a certain limiting value u, found by putting $p = 0$. At an elevation z above datum level

$$u^2 = 2g\left(Z - z + \frac{p_0}{w}\right).$$

At a greater velocity a negative pressure would be required to preserve the continuity of the fluid mass, and under these circumstances the water breaks up with consequences to be hereafter considered.

It further appears that water can flow through a closed passage against a difference of pressure, provided the area of the passage vary so as to permit a corresponding reduction of velocity. An example of this occurs in the case of the discharge through a trumpet-shaped mouthpiece. In Fig. 175 water enters from a vessel at KK, an orifice provided with a mouthpiece, which first contracts to DD, and then expands to EE where the jet enters the atmosphere. The pressure at EE is that of the atmosphere, and therefore at DD is less than that of the atmosphere, that is, less than it would be if the trumpet were cut off at the neck. Hence the discharge is increased by the addition of the expanded portion. If the water issued into a vacuum the jet would not expand to fill the wide mouth of the trumpet, which would not in that case have any influence on the discharge. The increased discharge and partial vacuum at DD have been verified by experiment.*

Fig. 175.



SECTION II.—MOTION OF AN UNDISTURBED STREAM.

244. *Distribution of Energy in an Undisturbed Stream. Vortex Motion.*—If the reservoir in the last article be imagined to supply a stream running in a channel of any size either closed or open, that stream, if undisturbed by any of the causes mentioned hereafter, may be supposed made up of an indefinite number of elementary streams, each of which moves as it would do in a closed pipe, as just described, without in any way intermingling with the rest. The forms of these ideal pipes depend solely on the form of the channel in which the stream is confined. The equation

$$\frac{u^2}{2g} + \frac{p}{w} + z = Z + \frac{p_0}{w}$$

applies to the motion in every pipe, and from it we may draw two important conclusions. In the first place, it may be written in the form

$$\frac{p - p_0}{w} = Z - z - \frac{u^2}{2g};$$

and therefore *the pressure at any point is less than if the water were at rest by the height due to the velocity at that point.* Again, the equation interpreted as in the last article shows that the energy of all parts of the fluid is the same, or, as we may otherwise express it, *the energy of the fluid is uniformly distributed.*

From either way of stating the result it appears that the pressure is greatest where the velocity is least, and conversely. Now, if the water move in curved lines in a horizontal plane, each particle of water is at the instant moving in a circle, and to balance its centrifugal force (Art. 131) the pressure on its outer surface must be greater than

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* Readers to whom the subject is new are recommended to pass on at once to Art. 248, p. 483.

that on its inner. It follows therefore that, if a channel is curved so as to alter the direction of the stream, the pressure increases as we go from the inner side of the channel to the outer; while, on the other hand, the velocity is greatest at the inner side and least at the outer. The change is the greater the sharper the bend, for the centrifugal force is greater. In open channels the change at the surface where the pressure is constant is in elevation instead of in pressure.

The magnitude of the change can be calculated in certain cases (see Appendix), of which we can only here consider one which is of special importance. If the particles of water describe circles about a common vertical axis, the elementary streams will form uniform rings, the centrifugal force of which can be calculated as in Art. 145, page 288. The resultant force on the half ring is—employing the notation of the article cited—given by

$$P = w \cdot 2A \cdot \frac{V^2}{g}.$$

This is balanced by an excess pressure on the outer surface of the half ring, and if that excess be Δp the corresponding resultant force is $\Delta p \cdot 2r$, as shown on page 302. Equating this to P

$$\Delta p = \frac{w}{g} \cdot \frac{A}{r} \cdot V^2.$$

The ring is supposed of breadth unity, and for A we may write the thickness of the ring, which may be called Δr. Dividing by this, and proceeding to the limit

$$\frac{dp}{dr} = \frac{w}{g} \cdot \frac{V^2}{r},$$

an equation from which the pressure can be found if the law of velocity be given. If the fluid rotated about the axis like a solid mass, V would vary as r; but the case now to be examined is that in which V varies inversely as r, as expressed by the equation

$$Vr = \text{Constant} = k.$$

Substitute and integrate, then replacing k by Vr, it will be found that

$$\frac{p}{w} + \frac{V^2}{2g} = \frac{p_0}{w} + \frac{V_0^2}{2g},$$

where the suffix refers to a given point where the pressure is p_0 and the velocity V_0. This result shows that the energy is uniformly distributed, and we infer that if the direction of a moving current is changed so that the particles of water describe concentric circles, the velocity varies inversely as the distance from the centre.

A mass of rotating fluid is called a "vortex," and in the case just considered the vortex is described as "*free*," because the motion is that which is naturally produced (comp. Art. 273, p. 531). A free vortex is necessarily hollow, for to hold the water together a negative pressure would be required near the axis of rotation, but the hollow may be filled up by water moving according to a different law.

245. *Viscosity.*—When the motion of a mass of water is free from sudden changes of direction, loss of energy takes place only through the direct action of viscosity, a property of fluids which it will now be necessary briefly to consider. In Fig. 160, page 425, a block of plastic material is represented, and it was explained that to produce change of form a certain difference of pressure was necessary, depending on the hardness of the material. In a fluid a similar difference of pressure is necessary to produce a change of form at a *given rate*, and the magnitude of the difference is proportionate to the rate. If u be the rate at which the height of the block is diminishing and the breadth increasing, each reckoned per unit of dimension, the thickness remaining constant,

$$p = 2cu,$$

where c is a co-efficient called the "co-efficient of viscosity." Or to express the same thing differently, if ω be the *rate* at which a small rectangular portion of the fluid is distorting, as in Fig. 140, p. 352, q the corresponding distorting stress,

$$q = c \, . \, \omega.$$

Hence, when a fluid moves, any change of form requires an amount of work to be done which is proportionate to the speed at which the change takes place. In a free vortex the rate of distortion is twice the angular velocity of the particles round the axis, and varies inversely as the square of the distance; the changes of shape are therefore very rapid near the centre, and energy is consequently dissipated much more rapidly than in the stream from which the vortex is produced.

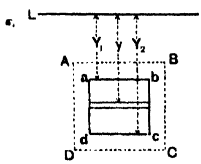
In the case of water the viscosity is so small that such changes of form as occur in an undisturbed stream are not rapid enough to absorb any large amount of energy. For example, in the discharge from orifices in a thin plate the loss of head is only 5 or 6 per cent. It is only when the water is disturbed by the neighbourhood of a rough surface over which it moves or in other ways described further on, that large quantities of energy are dissipated and frictional resistances of great magnitude produced.

246. *Discharge from Large Orifices in a Vertical Plane.*—When the orifices through which water is being discharged from a reservoir are not small compared with the head and the dimensions of the reservoir, the question becomes more complicated.

If the plane of the orifice be vertical the velocities of the several parts of the stream are not the same as in the case, so far as can be judged by the eye, when the orifice is small. On the contrary the velocity of that part of the stream which issues from the lower part of the orifice is visibly greater than that proceeding from the upper part. Hence it follows that the centre of gravity of the fluid issuing in a given time, to which the head is measured, is not on the same level as the centre of the contracted section, but lies below it. The corresponding point on the section may be described as the Centre of Energy. Also the velocity of the centre of gravity of the fluid is not the same as the velocity of mean flow Q/A, and the internal motions of the stream, even when undisturbed, are of sensible magnitude and cannot be neglected. To find the discharge therefore we must consider separately each of the elementary streams of which the whole stream may be imagined to be made up, and obtain the result by integration.

To illustrate these points let us consider the comparatively simple case of a rectangular orifice $ABCD$ (Fig. 176) from which water is being discharged from a reservoir, the level from which the head is measured being LL. The stream contracts on efflux, and the contracted section may be supposed rectangular. The position and dimensions of this section it will be necessary to suppose known by experiment; let its breadth be b, and let its upper and lower sides be at depths Y_1, Y_2 below LL. Divide the area into horizontal strips, and consider any one at depth y, then the velocity will be given by the formula

Fig. 176.



$$v^2 = 2gy.$$

The quantity discharged per second will be given by

$$Q = \int_{Y_1}^{Y_2} bv\,dy = b \,.\, \sqrt{2g}\int_{Y_1}^{Y_2} \sqrt{y} \,.\, dy,$$

which by integration gives

$$Q = \frac{2}{3} \,.\, b \,.\, \sqrt{2g}\,(Y_2^{\frac{3}{2}} - Y_1^{\frac{3}{2}}).$$

This determines the discharge, which is the same as with the mean velocity of flow,

$$V_0 = \frac{Q}{A} = \sqrt{2g} \,.\, \frac{2}{3} \,.\, \frac{Y_2^{\frac{3}{2}} - Y_1^{\frac{3}{2}}}{Y_2 - Y_1}.$$

The height due to this velocity is

$$h_0 = \frac{4}{9} \cdot \left(\frac{Y_2^{\frac{3}{2}} - Y_1^{\frac{3}{2}}}{Y_2 - Y_1} \right)^2,$$

which corresponds to a point called by Rankine the Centre of Flow, which lies somewhat above the centre of the section. When the head is measured to this point the discharge in the absence of hydraulic resistances is determined as if the orifice were small.

Again, the kinetic energy of the water discharged per second will be

$$U = w \int_{Y_1}^{Y_2} bv \cdot \frac{v^2}{2g} dy = w \cdot b\sqrt{2g} \int_{Y_1}^{Y_2} y^{\frac{3}{2}} dy,$$

which by integration gives

$$U = \frac{2}{5} wb\sqrt{2g}(Y_2^{\frac{5}{2}} - Y_1^{\frac{5}{2}}).$$

By dividing U by wQ we get the depth of the centre of gravity of the fluid discharged per second below LL, that is to say, the true head h is

$$h = \frac{3}{5} \cdot \frac{Y_2^{\frac{5}{2}} - Y_1^{\frac{5}{2}}}{Y_2^{\frac{3}{2}} - Y_1^{\frac{3}{2}}}.$$

The velocity of the centre of gravity which is the true velocity of delivery is

$$V = \frac{\int v^2 dy}{\int v dy} = \sqrt{2g} \cdot \frac{3}{4} \cdot \frac{Y_2^2 - Y_1^2}{Y_2^{\frac{3}{2}} - Y_1^{\frac{3}{2}}},$$

and the energy of translation on delivery is

$$U_0 = \frac{wQV^2}{2g},$$

a quantity less than the whole energy wQh by the energy due to internal motions.

In attempting to estimate the effect of the internal motions due to hydraulic resistance this method of analysis appears the most exact in principle. Practically, however, it is always necessary to obtain the discharge as above in terms of the dimensions of the orifice itself, and then allow for contraction and hydraulic resistance by a suitable co-efficient of discharge. Some additional examples will be found at the end of the present chapter.

Again, if the dimensions of the orifice be not small compared with the surface of the water in the vessel from which the discharge takes place, this surface will sink with a velocity V which is of sensible magnitude. If the area of the surface be S and that of the contracted section A_0, the discharge will be

$$Q = A_0 v = SV,$$

an equation which determines V. The water will now have a velocity V before descending through the height h, and the equation of energy is therefore

$$v^2 - V^2 = 2gh.$$

This may be written if we please

$$\frac{v^2}{2g} = h + \frac{V^2}{2g},$$

showing that in addition to the actual head h we must consider the *virtual* head $V^2/2g$ due to the initial velocity of the water. In many hydraulic questions it is inconvenient or impossible to measure the head from still water. It is then measured from some point where the water is approaching the orifice with a velocity determined by observation. The actual head h must then be increased by the height due to the velocity of approach.

247. *Similar Motions.*—When an incompressible fluid flows steadily through a pipe of small transverse section it was shown in Art. 243 that the total head is given by the equation

$$\frac{p}{w} + \frac{u^2}{2g} + z = \frac{p_0}{w} + \frac{u_0^2}{2g} + z_0,$$

while the discharge is

$$Q = Au = A_0 u_0,$$

where the suffix $_0$ refers to some given point.

Imagine now a precisely similar pipe constructed on an enlarged scale through which a fluid of different density is flowing, and let it be similarly placed relatively to the datum level; then, if large letters be used to denote the corresponding quantities in the large pipe,

$$\frac{P}{W} + \frac{U^2}{2g} + Z = \frac{P_0}{W} + \frac{U_0^2}{2g} + Z_0.$$

To each point in the small pipe will correspond a point in the large one; then at corresponding points if n be the ratio of enlargement, $Z = nz$, the sectional areas are in the ratio $n^2 : 1$, and the velocities must be in some fixed ratio depending on the relative discharge. Let us suppose the velocity-ratio to be $\sqrt{n} : 1$, the velocities are then said to correspond. Since in this case $U^2 = nu^2$ we find from the above equations

$$\frac{P - P_0}{W} = \frac{n(u_0^2 - u^2)}{2g} + n(z_0 - z) = n \cdot \frac{p - p_0}{w}.$$

Thus at corresponding velocities the difference of pressure-head at any two points of the large pipe is n times the difference at corresponding points of the small pipe. And if the pressure-ratio be $n : 1$ at any one pair of corresponding points it will be the same at any other pair.

In the motion of an undisturbed stream as already explained the complete stream may be analyzed into distinct elementary streams in each of which the flow goes on as it would in an isolated pipe: the forms of these elementary streams depending on the form of the surfaces within which the fluid is enclosed and by the uniformity of pressure on such parts of the surface as are free. Let us now suppose we have two streams, the boundary surfaces and elementary streams of which are similar and similarly placed, the ratio of enlargement being as before $n:1$. Further let the velocities be in the ratio $\sqrt{n}:1$, and the pressure-heads at some one pair of corresponding points in the ratio $n:1$; then the pressure-heads at every other pair of corresponding points will be in the same ratio, or in other words, the distribution of pressure will be the same. If then we have any actual motion on a small scale it must also be possible on a large scale when the velocities correspond. Such motions are said to be *similar*. In similar motions at corresponding speeds the distribution of pressure will be the same and conversely.

Let us take as an example the discharge of water from an orifice considered in Art. 246. Imagine two tanks, one large the other small, with similar orifices similarly placed with corresponding depths of water so that the heads are in the proportion $n:1$; then the principle of similar motions enables us to say that in the absence of hydraulic resistance, the co-efficients of contraction and discharge must be the same in the two cases, the velocities must be in the ratio $\sqrt{n}:1$, and the discharge in the proportion $n^2\sqrt{n}:1$. Strictly speaking however we must suppose the atmospheric pressures in the ratio $n:1$, a restriction which is probably *not actually necessary (p. 471)*.

Further if we consider any small area in the boundary surface of the small motion and a geometrically similar and similarly situated small area in the large motion, either the total pressures on those areas or their resolved parts in any given direction will when divided by the weight of a cubic foot of fluid be in the proportion $n^3:1$; and therefore will be in the proportion of the total weights of fluid in the two cases. The principal application *of this very important principle is in the* theory of the resistance of ships: it being equivalent to saying that in similar vessels at corresponding speeds the resistances (if any) when not influenced by causes of the nature of friction, must be in the proportion of their displacements. The needful qualifications of this principle and the mode of making use of it will be briefly noticed in the Appendix.

SECTION III.—HYDRAULIC RESISTANCES.

248.—*Surface Friction in General.*—We now proceed to study experimentally some of the more important causes of hydraulic resistance.

Fig.177.

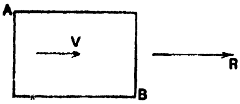


Fig. 177 shows a thin flat plate AB with sharp edges completely immersed in the water. The plate is moving edge-ways through the water with velocity V, then a certain resistance R is experienced which must be overcome by an external force. This resistance consists in a tangential action between the plate and the water, and so far is analogous to the friction between solid surfaces, but it follows quite different laws, which may be stated as follows:—

(1) The friction is independent of the pressure on the plate.

(2) It varies as the area of the surface in contact with the water.

(3) It varies as the square of the velocity.

These laws are expressed by the formula

$$R = fSV^2,$$

where f is a co-efficient which, as in the friction of solid surfaces, is described as the "co-efficient of friction." The value of this co-efficient depends on the degree of smoothness of the plate. Thus, for example, in some experiments, to be described presently, on thin boards moving through water it was found that the co-efficient was ·004 for a clean varnished surface, and ·009 for a surface resembling medium sand-paper, the units being pounds, feet, and seconds.

The first of these laws, so far as is known at present, is always strictly fulfilled, but to the second and third there are certain limitations, as in the ordinary laws governing the friction of solid surfaces. In the first place, if the velocity be below a certain limit the water adheres to the surface, and its velocity relatively to the surface is some continuous function of the distance from the surface so that the stream does not break up. This will be further referred to hereafter; for the present it is sufficient to say that the resistance then follows an entirely different law, varying nearly as the velocity instead of the (velocity)^2. The limiting velocity, however, at which this is sensibly the case is so low that in most practical applications the effect may be disregarded. In the second place, it is supposed that the water glides over all parts of the surface with the same velocity; but if the surface be any considerable length the friction of the front portion of the surface on the water furnishes a force which drags the water forward along with the surface and so diminishes the velocity with which it moves over the rear portion. The friction is thus diminished, and in large surfaces very considerably diminished. Thus Mr. Froude, experimenting on a surface 4 feet long, moving at 10 feet per second, found the value of f given above, but when the length was 20 feet and upwards, those values were diminished to ·0025 and ·005 respectively. Increasing the length beyond a certain amount produces very little change, and within a certain limiting length the effect is insensible. These limits must depend on the speed, but no exact observations have been made on this point. The power of the speed to which the friction is proportional has, however, been found to be diminished on long smooth surfaces, as shown below. The skin friction of vessels, on which the resistance chiefly depends at low speeds, is much diminished by the effect of length.

Experiments on surface friction were made by Colonel Beaufoy. They formed part of an elaborate series of experiments on the resistance of bodies moving through water, carried out during many years in the Greenland Dock, Deptford. Beaufoy employed the formula

$$R = f . SV^n$$

to represent his results, and for the index n obtained the values 1·66, 1·71, 1·9 in three series of experiments. The standard experiments on the subject are however due to the late Mr. Froude: they were made on boards $\frac{3}{16}$ inch thick, 19 inches deep, towed edgeways through the water. The boards were coated with various substances so as to form the surface to be experimented on.

The following table gives a general statement of Froude's results. In all the experiments in this table, the boards had a fine cutwater and a fine stern end or run, so that the resistance was entirely due to the surface. The table gives the resistances per square foot in pounds, at the standard speed of 600 feet per minute, and the power of the speed to which the friction is proportional, so that the resistance at other speeds is easily calculated.

Nature of Surface.	Length of Surface, or Distance from Cutwater, in Feet.											
	2 Feet.			8 Feet.			20 Feet.			50 Feet.		
	A	B	C	A	B	C	A	B	C	A	B	C
Varnish, -	2·00	·41	·390	1·85	·325	·264	1·85	·278	·240	1·83	·250	·226
Paraffin, - -	1·95	·38	·370	1·94	·314	·260	1·93	·271	·237	-	-	-
Tinfoil, - -	2·16	·30	·295	1·99	·278	·263	1·90	·262	·244	1·83	·246	·232
Calico, - -	1·93	·87	·725	1·92	·626	·504	1·89	·531	·447	1·87	·474	·423
Fine Sand, -	2·00	·81	·690	2·00	·583	·450	2·00	·480	·384	2·06	·405	·337
Medium Sand,	2·00	·90	·730	2·00	·625	·488	2·00	·534	·465	2·00	·488	·456
Coarse Sand, -	2·00	1·10	·880	2·00	·714	·520	2·00	·588	·490	—	-	-

Columns A give the power of the speed to which the resistance is approximately proportional.

Columns B give the mean resistance per square foot of the whole surface of a board of the lengths stated in the table.

Columns C give the resistance in pounds of a square foot of surface at the distance sternward from the cutwater stated in the heading.

To the three laws already mentioned may be added :

(4) In different fluids the friction varies as the density of the fluid. The grounds for this statement will be seen further on. It amounts to saying that surface friction is a kind of eddy resistance (p. 504). If we assume this, the laws of friction between a fluid and a surface are expressed by the equation

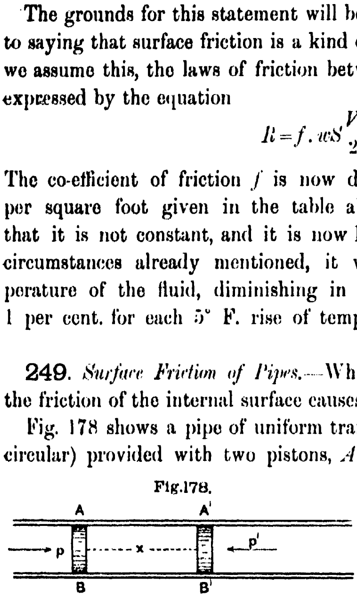
$$R = f \, . \, wS \frac{V^2}{2g} \, .$$

The co-efficient of friction f is now distinguished from the friction per square foot given in the table above. We have already seen that it is not constant, and it is now known that in addition to the circumstances already mentioned, it varies according to the temperature of the fluid, diminishing in water apparently as much as 1 per cent. for each 5° F. rise of temperature.

249. *Surface Friction of Pipes.*—When water moves through a pipe the friction of the internal surface causes a great resistance to the flow.

Fig. 178 shows a pipe of uniform transverse section (not necessarily circular) provided with two pistons, AB, $A'B'$, at a distance x enclosing between them a mass of water. The pistons and included water move forward together with velocity v under the action of a force R, required on account of the friction of the pistons and of the water on the pipe. Omitting piston friction the force R will be given by

Fig.178.



$$R = fwS \frac{v^2}{2g} = f \, . \, wsx \frac{v^2}{2g},$$

where S is the wetted surface and s the perimeter.

If we imagine the pipe full of water moving through it with velocity v, the force R is supplied by the difference of the pressures p, p' on the pistons, and therefore, if A be the sectional area

$$p - p' = fw \, . \, \frac{s}{A} \, . \, \frac{xv^2}{2g} .$$

The quantity A/s may be replaced by m and is described as the "hydraulic mean depth" of the pipe, a term derived from the case of an open channel to be considered hereafter. In the ordinary case of a cylindrical pipe $m = \frac{1}{4}d$. Further, we may reduce the pressures to feet of water by dividing by w and thus obtain for the difference of pressure h'

$$h' = f \, . \, \frac{x}{m} \, . \, \frac{v^2}{2g},$$

the value of the co-efficient f being determined by special experiment on pipes.

This formula for the head necessary to overcome surface friction is continually in use. The formula gives directly the head necessary for a length x of the pipe, when the water, by being enclosed between pistons, is constrained to move over the surface with a given velocity: when the pistons are removed and the water flows freely it represents the facts very imperfectly. The central parts of the stream move quicker than the parts in immediate contact with the pipe, and besides, though the circumstances are different, we cannot be sure that the velocity over the internal surface is not affected in the same way as in the case of a moving surface. The value of f has therefore to be obtained by special experiment, and the results of such experiments show that it varies very greatly according to the condition of the internal surface, and partly also on the diameter and velocity, the value being greater in small pipes than large ones, and at low velocities than high ones—a point considered further on. (See page 495.) For the present we assume ·0075 as roughly representing the facts when there is no special cause for increased resistance. For a pipe of circular section, length l, we have therefore

$$h' = 4f \cdot \frac{l}{d} \cdot \frac{v^2}{2g},$$

where for $4f$ we commonly assume the value ·03.

250. *Discharge of Pipes.*—The velocity v is the actual velocity with which the water moves, so that $v^2/2g$ is the energy of motion of each pound of the water. The loss of energy by friction is the same as that of raising the water through a height h', and is therefore equal to the energy of motion when

$$\frac{l}{d} = \frac{1}{4f} = 33 \text{ nearly},$$

that is, a length of pipe equal to 33 diameters absorbs an amount of energy equivalent to the whole energy of motion of the water. In pipes of any length, therefore, the effect of friction is very great, so much so that the size of a pipe is principally fixed by the loss of head which can be permitted. It is easily seen that to deliver water with a given velocity the loss varies inversely as the diameter, and that to deliver a given quantity it varies inversely as the fifth power of the diameter; thus, the smallest permissible diameter is fixed almost entirely by the value of h', which may be supposed already known.

The quantity discharged per second is given us by the formula

$$Q = Av = \frac{\pi}{4} d^2 v,$$

and on substitution this becomes

$$Q = \frac{\pi}{4} \sqrt{\frac{g}{2f}} \cdot \sqrt{\frac{h'}{l}} \cdot d^{\frac{5}{2}}.$$

All dimensions are here in feet, and Q is in cubic feet per second. If we require gallons per minute for a diameter of d inches, the formula will be

$$G = C \cdot \sqrt{\frac{h'}{l}} \cdot d^{\frac{5}{2}};$$

where C is a constant connected with $4f$ by the equation

$$C = \frac{4{\cdot}736}{\sqrt{4f}}.$$

For $4f = {\cdot}03$, this gives $C = 27{\cdot}3$, but for clean iron pipes not less than 9 inches in diameter the value 30 may be employed.

251. *Open Channels.*—Returning to Fig. 178, suppose the pipe, instead of being horizontal, is laid at an angle θ (see Fig. 179, next page), so that the difference of level of the two ends in $y = l . \sin\theta$, then the difference of pressure-head is

$$\frac{p - p'}{w} = f \cdot \frac{l}{m} \cdot \frac{v^2}{2g} - y,$$

and therefore may be made zero if the slope of the pipe be

$$\sin\theta = f \cdot \frac{1}{m} \cdot \frac{v^2}{2g} = \frac{h'}{l}.$$

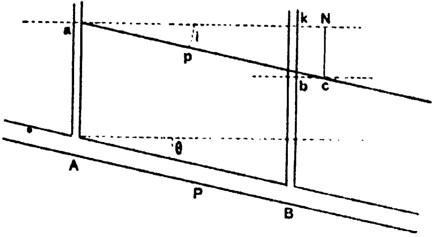
But if the pressure be constant we may remove the upper surface of the pipe and thus obtain the case of an open channel. The quantity m is now the sectional area of the channel divided by the wetted perimeter, and is therefore the actual depth in a very board shallow channel, but in other cases less in a ratio dependent on the form of section. As before stated it is described as the "hydraulic mean depth" of the channel.

We can now find the velocity and discharge of a stream of given dimensions and fall, provided that we know the value of f, or conversely the size of channel for a given discharge and fall. The value of f, however, varies for the same reasons as in pipes which indeed apply with still greater force, so that the limits of variation are wider. The average value does not differ very widely from ·0075, already adopted for pipes; but to obtain results of even moderate accuracy a special study of the experiments on the subject is necessary, which will not be attempted in this treatise.

252. *Virtual Slope of a Pipe.*—If the pipe be laid at any other angle the pressure will not be constant, and the mode in which it varies is best seen by a graphical construction.

Suppose small vertical pipes Aa, Bb to be placed at points A, B of the pipe we are considering (Fig. 179), then (if they enter the water square, without being bent towards the direction of motion) the water will rise in them to a level representing the pressure in feet of water at these points. If there were no friction the level would be the same in both and the difference (bk in the figure) therefore represents the loss by friction. Now draw a horizontal line through b, and take c on it, so that $ac = AB = l$, then the angle caN is given by the equation

Fig. 179.



$$\sin i = \frac{h}{l},$$

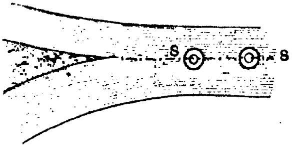
and is therefore the slope of a channel of the same length and hydraulic mean depth which would give the same discharge. This angle is therefore called the VIRTUAL SLOPE of the pipe. At any point P in the pipe, the water would rise to the level of the corresponding point p in the virtual channel, found by taking $ap = AP$. The construction would of course fail if h were equal to, or greater than l, but this case does not occur in practice; on the contrary, in pipes as in channels the angle i is nearly always small. The virtual slope is frequently one of the data of the question. The line ac is variously described as the "pressure line," "line of virtual slope," or "hydraulic gradient."

The pipe need not be straight; it may be curved or be laid in sections at different slopes, there will still be a continuous hydraulic gradient, provided the diameter be the same throughout; but if the sections be of different diameters each section will have its own slope. In practice care must be taken that the pipe does not rise above its hydraulic gradient, for otherwise there will be a partial vacuum: the pipe then acts as a syphon which is liable to fail on account of leakage and the presence of air in the water.

253. *Loss of Energy by Eddies and by Broken Water.*—We now proceed to consider other causes of frictional resistance.

In Fig. 180 two streams of water, moving with different velocities, converge towards each other and unite into one. Each stream, so far as can be judged by the eye, moves originally without disturbance in the manner described in Art. 244. On union, however, near the junction indicated by the dotted line SS in the figure, small depressions are observed, which move for some distance along with the stream, and then disappear. On examination these depressions are found to consist of small portions of the fluid in a state of rotation, the speed of rotation being greatest at the centre and gradually dying away towards the circumference. A motion of this kind was called a "vortex" in Art. 244, and in the present case is also described as an "eddy"; it is independent of the general motion of the stream, and its energy is therefore of the internal kind. The disappearance of the eddies thus formed is due to viscosity, the effect of which is much greater in the eddy than in the stream as already explained. After the eddies have disappeared the two streams are found to have become a single one, moving with a velocity intermediate between those of the streams which form it, but possessing less energy. Theoretically there is nothing to prevent two streams of a perfect fluid from moving side by side with different velocities, but such a motion is always unstable, and will not long continue without the formation of eddies by a sudden change of direction (Art. 244) in small portions of the fluid which separate from the rest. The instability is greater the more nearly perfect the fluid is. Whenever the water in motion intermingles with water at rest, or moving with a different velocity, internal motions of a complex kind are produced, representing a considerable amount of energy of the internal kind which is virtually lost even before its final dissipation by fluid friction.

Fig.180.



Again, in order that a mass of water may form a continuous whole, sufficient pressure must exist on the bounding surface to prevent the pressure at any point within the mass from becoming zero, as explained in Art. 243. If this condition is not satisfied the water breaks up more or less completely, and the result is a confused mass with complex internal motions rapidly disappearing as before by fluid friction. When waves break on a beach, or when paddles strike the water and drive it upwards in a mass of foam, the process takes place on a large scale before our eyes; but the same thing occurs in most cases where the velocity of a mass of water is suddenly changed, and of this we will now consider some examples.

Fig. 181*a* shows a jet of water filling a tank. Here the water pouring in possesses the kinetic energy $Wv^2/2g$ due to the original velocity of the water, and the height from which it falls into the tank. If it be of some size as compared with the tank the water will be completely broken up; if it be small it will penetrate the water in the tank without much apparent disturbance at the surface: in either case the result is a mass of water at rest as a whole, so that its energy is all of the internal kind. If the jet be shut off the water rapidly settles down to rest, the whole energy is then dissipated by fluid friction.

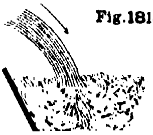
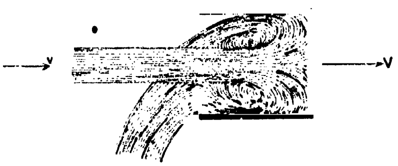


Fig. 181a.

Fig. 181*b* shows a bucket moving horizontally, bottom foremost, with velocity V, while a horizontal jet moving with greater velocity strikes it centrally: the bucket is then filled with broken water which pours out under the action of gravity. In water-wheels a series of buckets are filled in succession, and the broken water carried on with the wheel. Here if the bucket were at rest the loss of energy would be, as before, $Wv^2/2g$: but as it is moving with velocity V, the striking velocity on which the breaking depends will be $v-V$, and the loss of energy is

Fig. 181b.



$$U = W\frac{(v-V)^2}{2g},$$

where W is the weight of water acted on in the time considered. Both these cases may be treated as examples of the collision of two bodies considered on page 270, one of the bodies being indefinitely great. The energy of collision is employed in breaking up the water. It is represented in the first instance by internal motions, and subsequently dissipated by fluid friction.

Fig. 182.

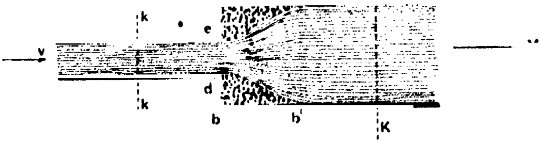


Fig. 182 represents a pipe which is suddenly enlarged from the diameter *ed* to the diameter *ab*. The water is moving through the small part of the pipe with velocity v, and, on passing through *ed* spreads out so as to fill the larger part. At some distance from the enlargement it moves in a continuous mass with velocity V, but in its immediate neighbourhood we have broken water, as in the case of the bucket, from which it only differs in the enclosure of the water in a casing. The loss or energy per unit of weight may be expected to be the same as before, and is therefore

$$h' = \frac{(v - V)^2}{2g},$$

a formula which gives us the "loss of head." If the sectional areas of the two parts of a pipe be A, a the discharge is

$$Q = AV = av,$$

so that if m be the ratio of areas,

$$h' = (m - 1)^2 \frac{V^2}{2g} = \left(1 - \frac{1}{m}\right)^2 \frac{v^2}{2g}.$$

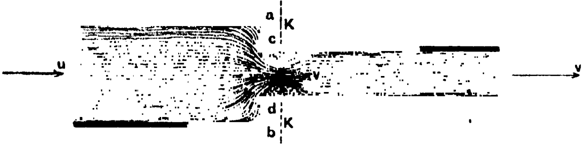
The coefficient of resistance is therefore

$$(m - 1)^2 \quad \text{or} \quad (1 - 1/m)^2,$$

according as the velocity to which it is referred is that in the large pipe or that in the small one.

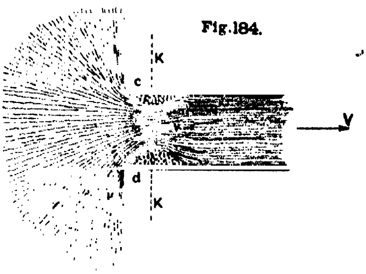
Instead of the water moving from a small pipe into a large one, we may have the converse case of a suddenly contracted pipe as in Fig. 183. The loss here is due to precisely the same cause, namely a sudden enlargement, which is produced as follows. In the figure the stream of water moving with velocity u contracts on passing through *cd* nearly as it would if the small part of the pipe were removed, as in Fig. 173, p. 474, until it reaches a contracted section KK, and is then moving with a velocity v which is greater than u in the ratio of the area of the large pipe to the *contracted* area KK. The loss of head in this part of the process is not large. After passing KK, however, an expansion takes place to the area of the small pipe, and this is accompanied by breaking up, the space between the contracted jet and the pipe being filled up with broken water.

Fig.183.



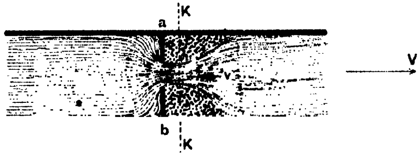
In Fig. 184 we have the extreme case, in which the large pipe is a vessel of any size. We thus obtain the case of a pipe with square-edged entrance which has already been referred to in Art. 242. Another modification is that of a diaphragm in a pipe, as in Fig. 185. The small pipe is here larger than the orifice through which the water enters, and in the figure we have simply a single pipe divided into parts by a diaphragm with an orifice in the centre. The stream of water, after passing the contracted section *KK*, expands to fill the pipe. In cocks when partially closed, a loss of head of the same kind occurs, which may be increased to any extent by closing the cock further.

Fig. 184.



In all these cases the loss of head may be calculated approximately by means of the formula for a sudden enlargement, but the ratio of enlargement is not known exactly, on account of the uncertainty of the value of the co-efficient of contraction to be assumed. Losses of head of this kind are indeed always subject to variation within certain limits from accidental causes; in general and on the average the quantity of water broken up will bear a certain proportion to the whole quantity passing, and in consequence we have the general law of hydraulic resistance stated on page 472, but the ratio may vary from time to time, and cannot be stated with precise accuracy. The causes of this uncertainty will be clearly understood on considering somewhat more closely the manner in which the loss takes place.

Fig. 185.



In Figs. 183, 185 two plane surfaces at right angles meet at *a*, forming an internal angle through which water is flowing. The particles of water there describe curves which are all convex towards *a*, and in conformity with the general principle explained in Art. 245, the pressure must increase and the velocity diminish on going towards *a*. The water then moves slowly and quietly round the angle without disturbance. But when compelled by the general movement of the stream to move round an external angle such as *kea* in Fig. 182, the case is very different; the particles then describe curves which are concave round *e*; and consequently the pressure diminishes in going towards *e*, while the velocity increases. To hold the particles of water in contact with the surface, an infinite pressure would be required in the other parts of the fluid. The particles of water therefore leave the surface at *e*, and describe a path *ea'*, regaining the surface farther on; *ea'* is then described as a "surface of separation," as it separates the moving mass of water from a portion enclosed within it which is in a state of violent disturbance. Such are the surfaces shown in Figs. 182-186. It is not, however, to be supposed that these surfaces are sharply defined, and that they permanently separate different masses of water. On the contrary, no such equilibrium is possible; the surfaces are continually fluctuating, and a constant interchange takes place between the so-called "dead" water and the stream. In this intermingling eddies are produced nearly as in the comparatively simple case of two streams given on page 490. The process is always essentially the same, and consists in sudden changes of direction being communicated to parts of the stream which become detached from the rest.

254. *Bends in a Pipe. Surface Friction.*—In some other cases the process of breaking up by which energy is lost is less obvious, and the ratio is subject to greater variations.

When a pipe has a bend in it, if the internal surface of the pipe were perfectly smooth and free from discontinuity of curvature, there would be no disturbance of the current of water, which would flow as described in Art. 248. These conditions, however, are not satisfied by actual bends in pipes, and there is always a loss of head due to them in addition to the loss by surface friction. This loss can only be determined by experiment, but it is easy to conjecture that the loss will be proportional to the angle through which the pipe is bent, and that it will be greater the quicker the bend, that is, the smaller the radius of the bend is as compared with the diameter of the pipe. The extreme case of a bend is a knee, but the loss is not in this case proportional to the angle of the knee, but follows a complex law. For details respecting bends and knees the reader is referred to the treatises cited at the end of this chapter, but some common examples are given in the table on page 496.

In the case of surface friction the loss of energy is represented in the first instance by eddies formed at the surface and thrown off. In almost all practical cases of the motion of water in pipes and channels, even when to all outward appearance quite undisturbed, the fluid is in fact in a state of eddy motion throughout, and dissipation of energy at every point is going on much more rapidly than would be the case if the motion were of the simple kind described in Art. 248. The quantity of water broken up, however, is not generally in a fixed proportion to the quantity passing, for reasons already partly indicated in Art. 249. In the first place, as in the case of a board moving edgewise through water, the friction per sq. ft. is proportional to the n^{th} power of the velocity, where n is an index which, in smooth surfaces, is somewhat less than 2. Secondly, the disturbance caused by the friction at a given velocity is less at some distance from the surface than in its immediate neighbourhood, and hence the central portion of the water in the pipe is less disturbed than the boundaries, and that the less the greater the size of the pipe. The loss of head therefore at a given velocity is less in large pipes than small ones. The various experiments on the discharge of pipes have been very thoroughly examined by Professor Unwin,* who has shown that they are represented very closely by a formula originally given in a slightly different form by Hägen,

$$h' = \mu \cdot \frac{l}{d^{1+x}} \cdot \frac{v^{2-y}}{2g},$$

where x and y are small fractions measuring the deviation from the simple formula already used, and μ is a co-efficient. The values of μ, x, y stated below are selected from a number of cases given by Professor Unwin in the paper already cited: the values of μ being for diameters in feet.

KIND OF PIPES.	μ	x	y
Wrought Iron (Gas),	·0226	·21	·25
New Cast Iron,	·0215	·168	·05
Cleaned Cast Iron,	·0243	·168	·0
Incrusted Cast Iron,	·044	·16	·0

The result for cleaned cast iron is equivalent to taking in the simple formula

$$4f = \frac{\cdot 036}{\sqrt[6]{d}}, \qquad (d \text{ in inches})$$

a formula which may be used for any case of a clean surface not of the smoothest description. In the smoothest surfaces the value of μ is slightly smaller, and the velocity must also be considered; but as in the case of open channels a special study of the experiments on the subject is necessary to obtain fairly accurate results. In very rough surfaces the value of $4f$ may be doubled. The value ·03 employed in preceding articles, allows a certain margin for incrustation, except in pipes less than 2 or 3 inches in diameter. For Darcy's formula see Appendix.

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* *Formulae for the Flow of Water in Pipes.* Reprinted from *Industries*, 1886.

255. *Summation of Losses of Head.*—The total loss of energy due to a number of hydraulic resistances of various kinds is found by adding together the losses of head due to each cause taken separately. The velocity of the water past each obstacle will not generally be the same for all, and it is then necessary to select some one velocity from which all the rest can be found by multiplication by a suitable factor for each obstacle. If n be this multiplier the loss of head will be

$$h' = \Sigma F n^2 \frac{V^2}{2g},$$

where V is the velocity selected for reference. The value of V is then found for motion under a given head H by the formula

$$(1 + \Sigma F n^2) \frac{V^2}{2g} = H.$$

The various values of F already given are collected with some additions in the annexed table:—

CO-EFFICIENTS OF HYDRAULIC RESISTANCE.

NATURE OF OBSTACLE.	VALUE OF F.	REMARKS.
Orifice in a Thin Plate.	·06	
Square-edged Entrance of a Pipe.	·5	
Sudden Enlargement of a Pipe in the ratio $m : 1$.	$(m-1)^2$	Referred to Velocity through large part of Pipe.
Bend at right angles in a Pipe.	·14	Radius of Bend = 3 × Diameter of Pipe.
Quick Bend at right angles.	·3	Radius of Bend = Diameter of Pipe.
Common Cock partially closed.	·75, 5·5, 31	Handle turned through 15°, 30°, 45° from position when fully open.
Surface Friction of a Pipe the length of which is n times the diameter.	$4f . n$	For a clean Cast-iron Pipe d inches diameter. $4f = \frac{\cdot 036}{\sqrt[6]{d}}$.
Knee in a Pipe at right angles.	Unity.	In Bends the coefficient is proportional to the angle of the Bend, but in Knees the law is much more complex.

SECTION IV.—PRINCIPLE OF MOMENTUM.

256. *Direct Impulse and Reaction.*—The generalized form of the second and third laws of motion, described as the Principle of Momentum in Chapter XI. of this work, may be employed with great advantage when the motion of water in large masses is under consideration, because the total momentum of a fluid mass depends solely on the motion of the centre of gravity (p. 267), and not on the very intricate motions of the parts of the fluid amongst themselves. Further, the energy dissipated by frictional resistances is accounted for by these internal motions, or by the mutual actions of the fluid particles, and the total momentum is therefore independent of these resistances. Hence it follows that results may be obtained which are true notwithstanding any frictional resistances, and in some cases the loss of energy by them may be determined *a priori*. Also the pressures on fixed surfaces may be found which do no work, and to which therefore the principle of work does not directly apply.

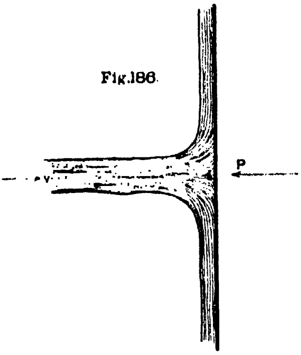


Fig. 186.

Fig. 186 shows a jet of water striking perpendicularly a fixed plane of infinite extent, and exerting on it a pressure P. The magnitude of this pressure is found by considering that the plane exerts an equal and opposite pressure on the water, which changes its velocity. The water originally moving with velocity v, spreads out laterally, and any motion which it possesses is parallel to the plane. In time t the impulse is Pt, and the change of momentum is Mvt, where M is the mass of water delivered per second. Equating these we have

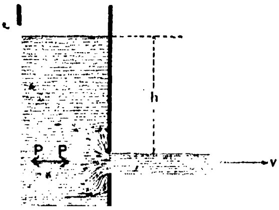
$$P = Mv = \frac{W}{g} \cdot v,$$

where W is the weight of water delivered per second.

If the plane be smooth, and gravity be neglected, the motion of the water will be continuous; but if it be rough to any extent, so that breaking-up occurs, the result will still be correct, provided only the roughness be symmetrical about the axis of the jet. And the action of gravity parallel to the plane does not affect the question.

In Fig. 187 we have the converse case of water issuing from a vessel with a lateral orifice. Here the water, which originally was at rest, issues with velocity v, and the momentum generated in time t is Mvt. To produce this momentum a corresponding impulse is required, which is derived from the resultant horizontal pressure P of the sides of the vessel upon the water. We have as before

Fig. 187.



$$P = Mv = \frac{Wv}{g}.$$

A pressure equal and opposite to P is exerted by the water on the vessel this is described as the "reaction" of the water; and if the vessel is to remain at rest, must be balanced by an external force supplied by the supports on which it rests.

A remarkable connection exists between the change of pressure on the sides of the vessel consequent on the motion and the co-efficients of contraction and resistance.

First, suppose the water at rest, the orifice being closed, then the value of P is zero, and the pressure on the area of the orifice is $w \,.\, A \,.\, h$, the notation being as in Art. 240. When the orifice is opened the pressure on that side is diminished, first, by the quantity $w \,.\, A \,.\, h$; secondly, by an unknown diminution S due to the motion of the water (p. 477) over the surface near the orifice. Now

$$P = S + w \,.\, A \,.\, h = \frac{wA_0v'^2}{g} - 2wA_0(h - h'),$$

the notation still being as in the article cited. Replacing A_0 by kA we obtain

$$S = wA\left\{2k(h - h') - h\right\} = wAh\left(\frac{2k}{1 + F} - 1\right).$$

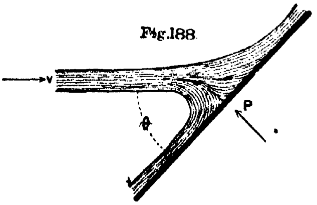
Since S is always positive the least value of k is

$$k = \frac{1 + F}{2}.$$

If there be no frictional resistance $k = \cdot 5$, and this is the smallest value k can have under any circumstances. For a small pipe projecting inwards as in Fig. 172, p. 473, these conditions are approximately realized, the water being at rest over the whole internal surface of the vessel.

257. *Oblique Action. Curved Surfaces.*—When a jet impinges obliquely on an indefinite place (Fig. 188), the water spreads out laterally as before, but the quantity varies according to the direction. In the absence of friction the velocity of individual particles is the same as that of the jet in whatever direction the water passes. At the same time the velocity of the whole mass of water parallel to the plane cannot

Fig. 188.



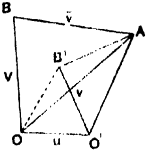
be altered by the action of the plane, and is therefore $v . \cos \theta$, where θ is the angle the jet makes with the plane. It immediately follows that any small portion of water diverging from the centre of the jet at an angle ϕ with the jet must be balanced by another portion diverging in the direction immediately opposite, and the quantities so diverging must be in the ratio $1 - \cos \phi : 1 + \cos \phi$, being inversely as the changes of velocity parallel to the plane. But if the circumstances be such that breaking-up takes place, the motion of the water parallel to the plane will be undetermined, and in general there will be a tangential action on the plane of the nature of friction.

The normal pressure on the plane is in all cases the same, being given by the formula

$$P = Mv . \sin \theta = \frac{W}{g} . v . \sin \theta.$$

If the surface on which the water impinges be curved it is necessary to know the average direction and magnitude of the velocity with which the water leaves the surface. In the absence of friction, as already noticed, the velocity of the individual particles is unaltered unless the water be enclosed in a pipe so that the pressure can be varied—a case for subsequent consideration; the direction, however, will depend on the way in which the water is guided. In cases which occur in practice it will generally be found either that the whole of the water is guided in some one direction, or that it leaves the surface in all directions symmetrically.

Fig. 189.



Taking the first case, suppose the original velocity (v) of the water to be represented by OA (Fig. 189), and the final velocity to be diminished to V by friction, and altered in direction so as to be represented by OB. Then the change of velocity in the most general sense of the word (p. 265) is represented by AB. If this be denoted by r the change of momentum per second is

$$P = \frac{Wr}{g}.$$

The resultant pressure on the surface is parallel to AB and numerically equal to P.

In applications to machines the curved surface is frequently a vane which is not fixed, but moves with a given velocity; the pressure can then be found by a simple addition to the diagram. Through O draw OO', representing the velocity (u) of the moving surface in direction and magnitude, then $O'A$ represents the velocity with which the water strikes the surface. Considering the vane as fixed, the velocity is now estimated with which the water would leave it, and $O'B'$ drawn to represent it: the change is now AB' instead of AB. If the absolute velocity is required with which the water leaves the surface, it may be found simply by joining OB', which will completely represent it; the change of velocity being AB', whether the velocities are absolute or relative to the moving surface.

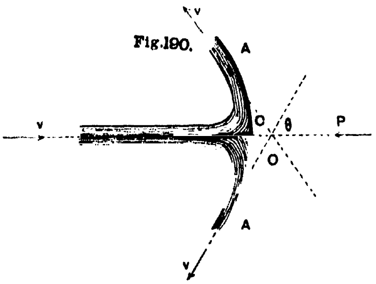


Fig. 190.

The cup vane ACA (Fig. 190), against which a small jet of water impinges centrally, may be taken as an example where the water spreads in all directions symmetrically. If OA be tangent to the vane at A, making an angle θ with the centre line of the jet, the water leaves the vane in the direction OA with unaltered velocity (neglecting friction). The resultant pressure P is in the direction of the jet, and the velocity in that direction is altered from v to $v\cos\theta$ in the opposite direction, so that the change of velocity is $v(1+\cos\theta)$. Thus we have

$$P = \frac{Wv}{g}(1+\cos\theta).$$

258. *Impulse and Reaction of Water in a Closed Passage.*—When the water is moving in a closed passage the resultant pressure to be considered in applying the principle is not merely that on the sides of the passage, but also that on the ideal surfaces which separate the mass of water we are considering from the complete current. In the previous cases the pressure of the atmosphere on the free surface bounding the fluid was the same throughout, and was balanced by an equal pressure of the surface against which it impinges, which is not included in the preceding results. This is now no longer the case.

An important example is that of the sudden enlargement in a pipe already referred to in Art. 253. In Fig. 182, page 491, take ideal sections, KK, kk of the large and small portions of the pipe, and consider the whole mass of water between them. This mass is acted on (1) by the pressure (p) on the transverse section kk, (2) by the pressure (P) on the transverse section KK, and (3) by the pressure of the sides of the pipe. If we resolve in the direction of the length of the pipe, the only part of (3) which we need consider is the pressure (p') on the annular surface, *ae*, *bd*, the area of which is $A - a$, and the whole resultant pressure is therefore $PA - pa - p'(A - a)$ in the direction opposite to the motion of the water. Now let W be the weight of water delivered in one second, then in that space of time W passes from the small pipe, where its velocity is v, to the large pipe, where it has a velocity V, so that if we equate the resultant pressure to diminution of momentum,

$$PA - pa - p'(A - a) = \frac{W}{g}(v - V) = \frac{wAV(v - V)}{g},$$

a formula which may be written

$$\frac{P}{w} - \frac{p}{w} = \frac{V(v - V)}{g} + \frac{p' - p}{w}\left(1 - \frac{1}{m}\right),$$

m being as in Art. 253 the ratio of enlargement. Let now H be the total head in the large pipe and h in the small one, then subtracting $(v^2 - V^2)/2g$ from both sides and re-arranging the terms

$$h - H = \frac{(v - V)^2}{2g} + \frac{p - p'}{w}\left(1 - \frac{1}{m}\right).$$

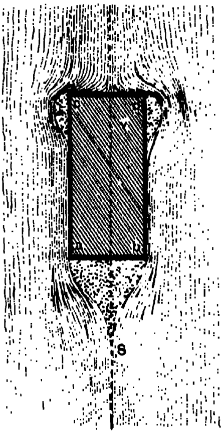
Comparing this result with that obtained in the article cited, it appears that the value of the loss of head there given is a necessary consequence of supposing $p = p'$, but cannot otherwise be correct. That the pressure in the broken water at *ae*, *bd* is nearly equal to the pressure in the small pipe may be considered probable *a priori*, independently of the experimental verification which the formula has received.

SECTION V.—RESISTANCE OF DEEPLY IMMERSED BODIES.

259. *Eddy Resistance.*—The subject of the resistance of ships is outside the limits of this treatise, for the ship moves on the surface of water, exposed to the atmosphere, on which waves are produced; whereas in the branch of mechanics now under consideration, the water is supposed to move within fixed boundaries. A certain part of the subject, however, may properly be considered as belonging to Hydraulics. If a body be deeply immersed in a fluid, that part of the fluid alone which is in its immediate neighbourhood will be affected by its motion, and the question is not essentially different from the cases already considered of the movement of water in pipes and channels.

Fig. 191 shows a parallelopiped *abcd* moving through water in the direction of its length, the face *ad* being foremost. To an observer whose eye travels along with the body the water will appear to move past the solid in a stream of indefinite extent. At some distance away the action of the solid is insensible, but it becomes increasingly great as the solid is approached, and is greatest for that part of the water which moves in immediate contact with it. At *c* and *d* eddies are formed in passing round the corners exactly as in the case at the same points in Figs. 183, 184—the stream in fact is suddenly contracted in the same way as in passing from a large pipe to a small one, the diminution of area in this case being the transverse section of the solid. After this the water moves in actual contact with the solid until it reaches the corners *ab*, when it describes the curves *aS*, *bS*, meeting in *S* (see p. 494), after which it forms a continuous stream as before. The two curves enclose between them a mass of eddying water exactly similar to the eddies at *a* and *b* in Fig. 182—the stream, in fact, suddenly expands, just as in passing from a small pipe to a large one, the increase of area being in this case the sectional area of the solid. The eddies thus formed during the passage of the solid through the water absorb energy, which must be supplied by means of an external force, which drags the body through the water. This kind of resistance to the movement of a body through water is called Eddy Resistance, and may be almost entirely avoided by employing "fair" forms, that is by avoiding all discontinuity of curvature in the solid itself, and in the junction of its surface with the direction of motion. The way in which it is created by the action of the eddies will be discussed further on.

Fig. 191.



A general formula for eddy resistance is derived thus. As already stated the water suffers no sensible disturbance at a certain distance from the solid. If then we imagine a certain plane area A attached transversely to the solid, and moving with it, all the water affected by the solid will pass through this plane, and its quantity will be

$$Q = AV,$$

where V is the velocity. In similar solids this area must be proportioned to the sectional area S of the solid, so that we write $A = cS$, where c is a constant depending on the form. Of this water a certain fraction will be disturbed by eddies, and the velocity of each particle of water will be some fraction of the velocity of the solid. Hence it follows that the energy U generated per second in the production of eddies must be

$$U = c'wQ \cdot \frac{V^2}{2g} = cc'wS \cdot \frac{V^3}{2g},$$

where c' is a co-efficient. Now this amount of energy is generated by means of a force which drags the solid through the water, at the rate of V feet per second, notwithstanding an equal and opposite resistance R. We have then

$$RV = cc'wS \cdot \frac{V^3}{2g},$$

or dividing by V, and replacing cc' by a single constant k,

$$R = kwS\frac{V^2}{2g}.$$

The co-efficient k is to be determined by experiment for each form of solid. In the case of the parallelopiped shown in the figure, the value of k depends little on the length, unless it be so short that the eddies at the corners *cd* coalesce with those in the rear of the solid, and it then becomes the same as that of a plate moved flatwise. Further it is nearly the same, if the transverse section be circular instead of square, and does not greatly differ from unity. For the flat plate it is greater and may be taken as 1·25. It must be remarked, however, that resistance of this kind is very irregular, and may vary considerably even in the course of the same experiment. To reach a permanent regime it is necessary that the velocity should be perfectly uniform through a run of considerable length, a condition most nearly attained in the experiments made by Beaufoy (p. 485), and recently by Mr. R. E. Froude at the Admiralty works. Their results are 1·13 and 1·1 respectively, but by some authorities much larger values are given. The same remarks apply to the case of a sphere for which the value may be taken as about ·4. For a cylinder moving perpendicular to its axis it is probably about ·5.

In all cases the value of k is independent of the units employed. It is also to a great extent independent of the kind of fluid, being roughly approximately the same for example in air as in water; but this would not hold good for fluids of very different viscosity; nor is it even approximately true for high speeds in air, because the compressibility of the air affects the question. The same remarks apply to the co-efficient (F) of hydraulic resistance employed above. It has been found that co-efficients of surface friction are greater in salt water than in fresh in the ratio of the densities of these fluids, as we might anticipate, since surface friction is a kind of eddy resistance.

Let us now consider more particularly the way in which the resistance is produced.

When a solid rests in any given position in a fluid the resultant horizontal pressure over the whole surface is zero, or in other words, if the solid be divided by any vertical plane the resultant pressure on the rear half is equal and opposite to that on the front. When the solid is set in motion in a given direction, the current of fluid passing it is separated by it into parts, which may be regarded as distinct streams having a single point or a line of points on the front of the solid at which the division takes place. At these dividing points the fluid is reduced to rest relatively to the solid, and (p. 477) the pressure there exceeds the hydrostatic pressure which would exist were the solid at rest by the quantity $wV^2/2g$. As each stream gliding over the surface moves away from the points of division its velocity increases, and consequently the excess pressure diminishes, till at length at a certain distance it vanishes. Over a certain area, then, in front of the solid, the resultant horizontal pressure is in excess of that which would exist were the solid at rest.

Now, in the absence of eddies, the streams on uniting again behind the solid would be brought to rest at one or more points of union lying in corresponding positions on the hinder surface, and in consequence there would be a corresponding excess pressure behind which would be found exactly to balance the excess in front, so that there would be no resistance to movement. Take, for example, a solid, the front and rear of which are exactly alike; if there were no dissipation of energy of any kind, the motion of the fluid in front and rear would necessarily be the same, for no alteration is conceivable merely by reversing the direction of movement. The difference between front and rear consists in the instability of the motion in the rear, in consequence of which the streams do not fully unite on the surface of the solid, but leave a space between filled with eddies which lower the pressure there, reducing it in general below the hydrostatic pressure which would exist were the solid at rest. Any eddies which are produced at sharp corners like *c*, *d* (Fig. 191) lower the pressure in the streams, and the reduction is ultimately transmitted to the rear of the solid, and takes effect in the same way. There is a strictly analogous difference between the motion in a pipe through a sudden contraction (Fig. 183) and a sudden enlargement (Fig. 182).

The co-efficient k is frequently regarded as the sum of two parts m and n, of which the first represents the *plus* pressure in front, and the second the *minus* pressure or suction in the rear; the terms plus and minus being used with reference to the hydrostatic pressure which would exist were the solid at rest. The eddies have little influence on the co-efficient m, which, when the motion is perfectly steady and uniform, is necessarily less than unity, and can in many cases be approximately calculated; they chiefly affect the co-efficient n, which (on the same supposition) would otherwise be equal and opposite to m, but actually has a certain value only capable of being determined by experiment, or inferred from its value in some similar case (Ex. 8, p. 510). It has a maximum possible value depending on the depth of immersion, for the minus pressure evidently can never be greater than the hydrostatic pressure due to the depth.

260. *Oblique Moving Plate.*—The case of a flat plate moving obliquely through a fluid may now be briefly mentioned, being of great technical importance. The plate, in the first instance, is supposed rectangular, of indefinite breadth, and immersed in an infinite fluid, through which it moves in a line perpendicular to its longer side.

Turning to Fig. 188, p. 498, suppose the jet represented to be of indefinite breadth, perpendicular to the plane of the paper, then the difference between this and the present case consists in the isolation of the jet and the infinite extent of the plane. These circumstances, however, make no difference in the character of the motion in front of the plane; the current of fluid passing is still divided into two, as indicated in the figure, the points of division lying on a line perpendicular to the plane of the paper, which is parallel to the longer axis of the rectangle. The streams are of different magnitudes, that which makes an acute angle (θ) with the current being the smaller, for reasons given in the article cited, which apply also to the present case. Hence the line of division moves away from the centre when θ is diminished, and when θ becomes very small approaches nearly to the edge of the rectangle. The line of division, however, always exists, and along it the excess pressure is $wV^2/2g$ as already described.

The total excess pressure upon the front of the plane is, as before,

$$P = \frac{W}{g} \cdot V \cdot \sin\theta,$$

only in the present case we do not know directly the quantity of water which is acted on. If we write

$$W = w \cdot SV$$

S will be the unknown area of an ideal isolated jet, which would produce the same effect and

$$P = w \cdot S \cdot \frac{V^2}{g} \cdot \sin\theta,$$

ormula which may be written

$$P = \mu \, . \, A \, . \, \frac{V^2}{2g} \, . \sin \theta,$$

where A is the area of the plate and μ a co-efficient depending on the quantity of water acted upon. The value of the *plus* portion of the co-efficient of resistance is now $\mu \, . \sin \theta$. Behind the plane, eddies are formed, the effect of which is represented by the *minus* portion n of the co-efficient. The total co-efficient k is now $\mu \, . \sin \theta + n$.

To determine k two methods may be adopted:—(1) By direct experiment on planes set at various angles in a stream, various formulæ have been obtained, of which, perhaps, the best is that devised by Duchemin, and adopted by Poncelet in the second edition (1839) of the *Mécanique Industrielle*, namely,

$$k = \frac{2 \, . \sin \theta}{1 + \sin^2 \theta} \, . \, k_0,$$

where k_0 is the value of k when the plate is at right angles to the stream.

(2) By methods of calculation which cannot be explained here, Lord Rayleigh has shown that the plus portion of the co-efficient is

$$m = \frac{2\pi \, . \sin \theta}{4 + \pi \, . \sin \theta},$$

it being pre-supposed that behind the plane there is an indefinite mass of fluid at rest relatively to the plane, and separated from the moving current by fixed surfaces of separation. The actual value of m may probably be nearly the same as in the actual case where eddies are formed, but the minus part of the total co-efficient, which does not exist in the ideal case, must still be found by experiment. If $\theta = 90°$, m becomes ·88, and adopting 1·25 as the value of k, n is found to be ·37. When θ is very small it will be seen that μ becomes constant, being equal to $\pi/2$ or 1·57, a conclusion which might have been foreseen, for at small angles there appears no reason why the effective breadth of the current of water acted on by the plate should vary. The suction at the back of the plate has the same general effect as the excess pressure in front, namely, of deflecting a current of water, the breadth of which is approximately constant for small values of θ. Thus, when θ is small (not exceeding 10° or 15°), the value of k is $a \sin \theta$ where a is constant. The value of a was taken by Froude as 1·7 for thin flat plates, but there can be little doubt that it is much greater when the back of the plate is convex, so that the eddies extend over the whole area, instead of being localized at the back of the leading edge. According to Duchemin's formula, it will be seen that $a = 2k_0$, or about 2·5.

It must be remembered that the resistance considered in the present article is the force normal to the plate. The resistance in the direction of motion is obtained by multiplication by $\sin\theta$, and to it must be added the component in the direction of motion of the tangential force on the plate. If the plate is very thin and perfectly flat on both sides, the tangential force is due to surface friction only being at small angles nearly the same as if it moved edgewise; but otherwise it will be much greater, and must be ascertained by experiment. The ratio which it bears to the normal force is much less variable, and may be taken as ·005. The value given by Froude is ·0047.

From what has been said it is clear that the line of action of the normal pressure on the plate does not pass through the centre; if therefore it be mounted on an axis parallel to the longer side the plate cannot be in equilibrium if the axis passes through the centre, but will always tend to place itself perpendicular to the direction of motion. This is also true for a square or circular plate, and so far as is known the value of k in this case is not very different.

261. *Pressure of a Current against an Obstacle.*—When an obstacle is placed below the surface of a stream a pressure is experienced by the obstacle which is due to the same causes as when a solid moves through still water, and, since the relative motion is the same in the two cases, should be given by the same formula

$$P = kwS\frac{V^2}{2g}.$$

In fact, however, the cases are often very different, because a uniform steady current is seldom to be met with in nature. The motion of the water is often unsteady and almost always disturbed by eddies due to the neighbourhood of the boundaries or other solid bodies. Experience shows that the value of k is generally considerably greater than in the case of motion through still water. For a flat plate fixed at right angles to a stream Dubuat found k to be 1·86, and this estimate being confirmed by other experimentalists, has been very generally accepted.

The irregularity and uncertainty characteristic of experiments on fluid resistance, when the solids exposed to its action are of unfair form, is especially marked in the case of wind pressure for sufficiently obvious reasons. This question, together with that of the resistance of the atmosphere to moving bodies, is outside the range of this work, but a short statement of results will be found in the Appendix, in which a brief account is also given of the theory of the resistance and propulsion of ships.

EXAMPLES.

FIRST SERIES (SECTIONS I. AND III.).

1. The injection orifices of the jet condenser of a marine engine are 5 feet below the surface of the sea, and the vacuum is 27 inches of mercury: with what velocity will the water enter the condenser, supposing three-fourths the head lost by frictional resistances? Also find the co-efficients of velocity and resistance and the effective area of the orifices to deliver 100,000 gallons per hour. *Ans.* Velocity = 23·6′ per second; Area = 27 sq. inches.

2. Water is discharged under a head of 25′ through a short pipe 1″ diameter with square-edged entrance; find the discharge in gallons per minute. *Ans.* 66⅓.

3. Water issues from an orifice the area of which is ·01 sq. feet in a horizontal direction and strikes a point distant 4′ horizontally and 3′ vertically from the orifices. The head is 2′ and the discharge 25 gallons per min.; find the co-efficients of velocity, resistance, contraction, and discharge. *Ans.* $c = \cdot 816$, $F = \cdot 5$, $k = \cdot 72$, $C = \cdot 59$.

4. The wetted surface of a vessel is 7,500 sq. feet, find her skin resistance at 8 knots and the H.P. required to propel her, taking the resistance to vary as V^2 with a co-efficient of ·004. *Ans.* Resistance = 5,500 lbs., H.P. = 135.

5. The diameter of a screw propeller is 18′, the pitch 18′, and the revolutions 91 per min. Neglecting slip find the H.P. lost by friction per square foot of blade at the tips, taking a co-efficient ·008 to include both faces of the blade. *Ans.* Friction = 65 lbs. per square foot. H.P. = 10·6.

6. Two pipes of the same length are 3″ and 4″ diameter respectively: compare the losses of head by skin friction (1) when they deliver the same quantity of water, (2) when the velocity is the same. *Ans.* Ratio = 4·21 and 1·33.

7. Water is to be raised to a height of 20′ by a pipe 30′ long 6″ diameter: what is the greatest admissible velocity of the water if not more than 10 per cent. additional power is to be required in consequence of the friction of the pipe? *Ans.* 8½′ per sec.

8. Two reservoirs are connected by a pipe 6″ diameter and three-fourths of a mile long. For the first quarter mile the pipe slopes at 1 in 50, for the second at 1 in 100, while in the third it is level. The head of water over the inlet is 20 feet and that over the outlet 9 feet. Neglecting all loss except that due to surface friction, find the discharge in gallons per min., assuming $f = \cdot 0087$. *Ans.* $v_{\cdot} = 3\cdot 43$ f.s. Discharge = 253 gallons per min.

9. A river is 1000′ wide at the surface of the water, the sides slope at 45°, and the depth is 20′; find the discharge in cubic feet per sec. with a fall of 2′ to the mile, assuming $f = \cdot 0075$. *Ans.* 154,000.

10. A tank of 250 gallons capacity is 50′ above the street. It is connected with the street main, the head in which is 52′ by a service pipe 100′ long: find the diameter of the pipe that the tank may be filled in 20 min. What must the head in the main be to fill the tank in five min. with this service pipe? *Ans.* $d = 1\cdot 6''$. Head in main = 82′.

11. Water is discharged from a vessel by a long pipe: show that the discharge is the same for all pipes of the same length and diameter with the discharging extremity in the same horizontal line. Draw the hydraulic gradient and examine the case of a syphon.

12. In question 2 suppose the pipe instead of being short to be 25″ long, find the discharge, assuming for surface friction $f = \cdot 01$. *Ans.* 52.

13. A horizontal pipe is reduced in diameter from 3″ to ½″ in the middle, the reduction being very gradual. The pressure head in the pipe is 40′, what would be the greatest velocity with which the water could flow through it, all losses of head being neglected? *Ans.* 1·4′ per sec.

14. A pipe 2″ diameter is suddenly enlarged to 3″. If it discharge 100 gallons per min., the water flowing from the small pipe into the large one, find the loss of total head and the gain of pressure head at the sudden enlargement. State the two values of the co-efficient of resistance.

Ans. Loss of head = 8½″. *F* = 1·56 or ·31.
Gain of pressure = 1′ 2″.

15. In the last question suppose the water to move in the reverse direction. Find the loss of head and the change of pressure consequent on the sudden contraction, assuming the co-efficient of contraction to be ·66.

Ans. Loss of head = 7½″.
Diminution of pressure = 2′ 5¼″.

16. A horizontal pipe 30′ long is suddenly enlarged from 2″ to 3″ and then suddenly returns to its original diameter. Length of each section = 10′. Draw the hydraulic gradient when the pipe is discharging 100 gallons per min. into the atmosphere, assuming as co-efficient of surface friction $4f = \cdot 03$. Find the total loss of head. *Ans.* Total loss of head = 10′ 2½″.

17. A pipe contains a diaphragm with an orifice in it the area of which is one fifth the sectional area of the pipe. Find the co-efficient of resistance of the diaphragm, assuming the contraction on passing through the orifice the same as that on efflux from a vessel through a small orifice in a thin plate. *Ans.* *F* = 46.

18. Find the loss of head in inches due to a bend through 45° of radius 6″ in a pipe 2″ diameter, the velocity of the water being 12′ per sec. *Ans.* 2″.

19. In question 1 suppose the ship moving at 10 knots and the orifice of entry so arranged as to cause no additional resistance: find the velocity of delivery. *Ans.* Additional head = 4·42′: velocity = 25′ per sec.

20. Water is supplied by a scoop to a locomotive tender at a height of 7′ above the trough. Assuming half the head lost by frictional resistance, what will be the velocity of delivery when the train is running at 40 miles per hour, and what will be the lowest speed of train at which the operation is possible? *Ans.* 36′ per sec.; 14½ miles per hour.

21. If *m* be the hydraulic mean depth of a channel of rectangular section, sides in the ratio *n* : 1 : show that the h.m.d. of a circular section of the same area is

$$m^1 = \frac{m}{\sqrt{\pi}}\left(\sqrt{n} + \frac{1}{\sqrt{n}}\right).$$

22. A pipe is suddenly enlarged to double its diameter (1) all at once, (2) by two stages; compare the losses of head, the stages in (2) being arranged so that the loss may be the least possible. *Ans.* Ratio = ½.

EXAMPLES.

SECOND SERIES (SECTIONS II., IV., AND V.).

1. A stream of water delivering 500 gallons per min. at a velocity of 15 feet per sec. strikes an indefinite plane (1) direct, (2) at an angle of 30°: find the pressure on the plane. *Ans.* (1) 39 lbs.; (2) 19½ lbs.

2. Employ the principle of momentum to prove the formula on page 478 for the resultant centrifugal force of one-half a rotating ring of fluid.

3. A plane area moves perpendicularly through water in which it is deeply immersed: find the resistance per sq. foot at a speed of 10 miles per hour. Deduce the pressure of a wind of 20 miles per hour using the same co-efficient. *Ans.* Resistance = 269 lbs. Wind pressure = 1·312 lbs.

4. Compare the resistance of an area moving flatwise through the water with its resistance moving edgewise so far as due to surface friction, the co-efficient for which is ·004. *Ans.* Ratio = 312.

5. Water is being discharged from a tank with vertical sides, by a sharp-edged rectangular notch 8 inches wide, the lower edge of which is 4 inches below the level of still water. Co-efficient of discharge, ·6. Find the discharge in gallons per minute. *Ans.* 154.

NOTE.—A notch is treated as an orifice the upper edge of which is at the still water level. Hence in the formula of page 481, b is to be taken as 8 inches, Y_1 zero, and Y_2 4 inches. Contraction and hydraulic resistance are then allowed for by multiplication by the co-efficient which varies to some extent according to the proportions which the head and the breadth of the tank bear to the width of the notch.

6. Obtain a formula for the discharge from a triangular notch with sides inclined at an angle θ to the vertical, the apex being downwards and at a depth h below still water.

$$Ans.\ Q = \frac{8}{15}c \,.\, \sqrt{2g} \,.\, \tan\theta \,.\, h^{\frac{5}{2}}.$$

NOTE.—The co-efficient of discharge c varies somewhat with the angle θ being about ·6 when the angle is 45°: but by the principle of similar motions (p. 482) will be nearly independent of the head in a notch of moderate size, a considerable practical advantage.

7. When a sphere moves in a straight line through a fluid the velocity with which the fluid glides over the surface, at a point the angular distance of which from the central line is θ, is $\frac{3}{2} \,.\, \sin\theta$. Assuming this, find the plus portion of the co-efficient of resistance. *Ans.* $\frac{2}{9}$.

8. In the last question assuming the motion in front the same as before notwithstanding the formation of eddies at the rear: and further, assuming the suction to extend over the same area as the excess pressure with a co-efficient the same as for a flat plate, find k. *Ans.* $k = ·386$.

REFERENCES.

For further information on subjects connected with the present chapter, the reader is referred to a treatise on Hydraulics by Professor W. C. Unwin, M.I.C.E., forming part of the article Hydro-Mechanics in the "Encyclopaedia Britannica."

CHAPTER XX.

HYDRAULIC MACHINES.

262. *Preliminary Remarks.*—Hitherto the energy exerted by means of a head of water has been supposed to be wholly employed in overcoming frictional resistances, and in generating the velocity with which the water is delivered at some given point. We now proceed to consider the cases in which only a fraction of the head is required for these purposes; the remainder then becomes a source of energy at the point of delivery by means of which useful work may be done. A machine for utilizing such a source is called an Hydraulic Motor.

Hydraulic energy may exist in three forms, according as it is due to motion, elevation, or pressure. In the first two cases it is inherent in the water itself, being a consequence of its motion or its position as in the case of any other heavy body. In the third it is due to the action of gravity or some other reversible force, sometimes on the water itself, but oftener on other bodies, as, for example, the load on an accumulator ram. The water is then only a transmitter of energy and not directly the source of it. As, however, the energy transmitted is proportional to the weight of water delivered, just as in the two other cases, the water is, as before, described as possessing energy. The energy per unit of weight is called "head," as sufficiently explained in the preceding chapter, and the "total head" is the sum of the "velocity head," the "actual head," and the "pressure head."

Hydraulic motors are classed according to the mode in which the water operates upon them, which may be either by weight, or by pressure, or by impulse, including in the last term also "reaction."

Most hydraulic motors are capable of being reversed, and then become machines for raising water, commonly described as Pumps.

SECTION I.—WEIGHT AND PRESSURE MACHINES.

263. *Weight Machines.*—To utilize a head of water, consisting of an actual elevation (h) above a datum level at which the water can be delivered and disposed of, a machine may be employed in which the direct action of the weight of the water, while falling through the height h, is the principal motive force.

The common overshot water-wheel (Fig. 2, Plate III., p. 141) may be taken as a type. Here the driving pair is a simple turning pair, and the driving link is the force of gravity upon the falling water, which acts directly on buckets open to the atmosphere. If G be the delivery in gallons per minute, the energy exerted in foot-pounds per minute is

$$E = 10Gh.$$

The head h is here measured from the level of still water in a reservoir which supplies the wheel. If v be the velocity of delivery to the wheel, the portion $v^2/2g$ is converted into energy of motion before reaching the buckets and operates by impulse. In a wheel of this class, therefore, the water does not operate wholly by weight. The speed of the wheel is limited to about 5 feet per second by the centrifugal force on the water, which, if too great, causes it to spill from the buckets. It will be seen hereafter that the velocity of the water should be about double this, so that v is about 10 feet per second, and the part of the fall operating by impulse is therefore about 1·5 feet. The remainder operates by gravitation, but a certain fraction is wasted by spilling from the buckets, and emptying them before reaching the bottom of the fall. More than one half the head operating by impulse is always wasted (Art. 270), and this class of wheels is therefore only suitable for falls exceeding 10 feet. The great diameter of wheel required for very high falls is inconvenient, but examples may be found of wheels 60 feet diameter and more. The efficiency of these wheels under favourable circumstances is ·75, and is generally about ·65.

In "breast wheels" the buckets are replaced by vanes which move in a channel of masonry partially surrounding the wheel. The water is admitted by a moveable sluice through a grating of fixed blades in the upper part of the channel. The channel is thus filled with water, the weight of which rests on the vanes and furnishes the motive force on the wheel. There is a certain amount of leakage between the vanes and the sides of the channel, but this loss is not so great as that by spilling from the buckets of the overshot wheel. The efficiency is found by experience to be as much as ·75. As the diameter of the wheel is greater than the fall a breast wheel can only be employed for moderate falls.

In both these machines the water virtually forms part of the piece on which it acts. This link of the kinematic chain forms one element of the driving pair, while that attached to the earth forms the other.

In the overshot wheel the water is contained in open buckets, in the breast wheel it is contained in a closed chamber or channel. A third class of weight machines is referred to farther on under the head of pumps.

264. *Hydraulic Pressure Machines in Steady Motion.*—A water wheel of great diameter is a slow-moving cumbrous machine, and for heads of 100 feet and upwards it is therefore necessary to employ a pressure or an impulse machine. Such machines are also often more convenient for low falls.

In pressure machines the driving link is compressed water, which is forced between the elements of the driving pair by some source ot the energy which supplies the necessary head. The head is sometimes an actual elevation either natural or artificial: in the docks at Great Grimsby the hydraulic machinery is operated from a tank placed on a tower 200 feet high. It is however difficult to get a considerable pressure in this way, and an apparatus called an Hydraulic Accumulator is therefore generally resorted to. Two forms occur, of which one is shown in Plate IX. In the first a plunger or ram is forced into a cylinder by heavy weights placed in a plate-iron cage suspended from it and stayed by iron rods. The accumulator is supplied by pumps generally worked by steam, which is the ultimate source of the energy, the accumulator merely serving the purpose of a store of energy which can be drawn on at pleasure. For ordinary hydraulic machinery the pressure is limited to 750 lbs. per square inch from the difficulty of obtaining pipes of sufficient strength and of working slide valves under heavy pressures. In machines for riveting and other special purposes, however, pressures of 1,500 lbs. per square inch and upwards are employed. The accumulator then consists of a cylinder *B* (Fig. 1, Plate IX., p. 525), loaded with ring weights *EE*, sliding on a fixed spindle *F*, divided into two lengths of which the upper portion is of smaller diameter than the lower.

In either form the accumulator provides a store of compressed water which can be supplied by suitable pipes to any number of machines, placed often at considerable distances. A head of 1,700 feet is thus readily obtained, and for special purposes much more: differences of level may therefore be disregarded as of small importance, and the water considered as operating wholly by pressure.

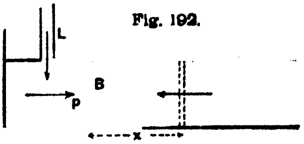


Fig. 192.

The driving pair of the machine forms a chamber of variable size which is alternately enlarged by the pressure of the water, and contracted to expel it. In most cases it is a simple cylinder C and piston B (Fig. 192): the water is admitted by a port from a pipe L, transmitting it from the accumulator at pressure p. Let the piston move through a space x, let A be its area, then

$$\text{Energy exerted} = pAx = p\,.\,X,$$

where X is the volume swept through by the piston. If w be as usual the weight of a cubic foot, $w\,.\,X$ is the weight of water which enters the cylinder as the piston moves through the distance x, and therefore

$$\text{Energy exerted per lb. of water} = \frac{p}{w} = \text{pressure-head in cylinder.}$$

This might have been anticipated from what was said in the last chapter as to the meaning of the term "head," and in fact it is equally true if the driving pair be not a simple piston and cylinder, but of any other kind.

The head in the cylinder is less than that in the accumulator, on account of the friction in the supply pipe and other frictional resistances, and it is on the action of these resistances that the working of the machine depends. Let V be the velocity of the piston in its cylinder, p_0 pressure in accumulator, F the co-efficient of hydraulic resistance *referred to the velocity of the piston* (Art. 255), then, neglecting differences of level, also the heights due to velocities of working and accumulator pistons,

$$\frac{p_0 - p}{w} = F\,.\,\frac{V^2}{2g}.$$

If the machine be moving steadily the pressure p will be equal to the useful resistance which the piston is overcoming, increased by the friction of the piston in its cylinder. Thus p and p_0 will be known quantities, a certain definite velocity V_0 will then be determined, which may be described as the "speed of steady motion": it is given by the equation

$$V_0^2 = \frac{2g}{wF}(p_0 - p).$$

Since the hydraulic resistances may be increased to any extent at pleasure by the turning of a cock, it follows that the speed of an hydraulic pressure machine can be regulated at pleasure. Further, if the resistance to the movement of the piston be diminished, the speed will increase only by a limited amount, and can, under no circumstances, be greater than is given by

$$V_0^2 = p_0 \frac{2g}{wF'},$$

which can be regulated as before. The surplus energy is here absorbed by the frictional resistances, and an hydraulic pressure machine therefore possesses the very important, and for many purposes, valuable characteristic that *it contains within it brakes which work automatically.*

265. *Hydraulic Pressure Machines in Unsteady Motion.*—Although the speed of a pressure engine cannot exceed a certain limit, which is easily found, yet it does not follow that the limit will ever be reached. When the engine starts, the piston and the water in the pipes have to be set in motion, the force required to do this is so much subtracted from that available to overcome resistances. A considerable time therefore elapses before a condition approaching steady motion can be obtained.

In Figs. 178, p. 486, water is supposed flowing through a pipe with a velocity u. Two pistons at a distance x enclose water between them, as in Art. 249, then the difference of pressure $p_1 - p_2$ in the case of steady motion is simply balanced by the surface friction, but in unsteady motion is partially employed in accelerating the flow of the water. Neglecting friction the acceleration g' will be given by the formula

$$(p_1 - p_2)A = W \cdot \frac{g'}{g},$$

where A is the sectional area of the pipe and W is the weight of the water between the pistons. Replacing W by $Ax \cdot w$, as in the preceding article

$$\frac{p_1 - p_2}{w} = x \cdot \frac{g'}{g},$$

which gives a simple formula for the change of pressure-head due to inertia. Now if nA be the area of the working piston, the velocity of the water in the pipe is n times the velocity of the piston, and the accelerations are necessarily in the same ratio; and hence it follows that the difference of pressure-head between cylinder and accumulator due to an acceleration g' of the piston is for a length of pipe l

$$\frac{p_1 - p_2}{w} = nl \cdot \frac{g'}{g}.$$

In addition to this the piston itself requires a certain pressure to accelerate it. Let q_0 be the "pressure equivalent to that weight," being the actual weight divided by the area, as in Art. 109, p. 224, then the pressure due to inertia is

$$q = q_0 \cdot \frac{g'}{g};$$

hence, adding the length (s) of cylinder containing water

$$\frac{p_1 - p_2}{w} + \frac{q}{w} = \left(nl + s + \frac{q_0}{w}\right)\frac{g'}{g} = \lambda \frac{g'}{g},$$

where λ is a certain length. This may be described as the "length of working cylinder equivalent to the inertia of the moving parts," and may always be approximately calculated for any given engine. In cranes and other hoisting machines the weight raised multiplied by the square of the velocity ratio between it and the ram must be included in the weight of the ram. The pressure in feet of water necessary to overcome inertia will then always be given by the simple formula

$$\text{Pressure-head due to inertia} = \lambda \frac{g'}{g}.$$

It will now be seen that the weight of water in the pipes and cylinders is so much added to the weight of the piston, that in the pipes being multiplied by the square of the ratio of areas of cylinder and pipe. A water-pressure engine is therefore a machine with very heavy moving parts, a circumstance which greatly limits its speed irrespectively of frictional resistances. The smaller the pipes the heavier the parts virtually are, and this must be considered as well as friction (p. 487) in fixing their diameter.

It will be advisable to consider a particular case more in detail. Suppose, as is sometimes the case in practice, that a water-pressure engine is employed to turn a crank, and let us suppose that the crank shaft rotates nearly uniformly as in Ch. IX., then the difference between the pressure in the accumulator and that transmitted to the crank pin may be represented graphically thus:

Let V be the velocity of the crank pin and let the stroke be $2a$ or AA in the diagram (Fig. 193). Set up

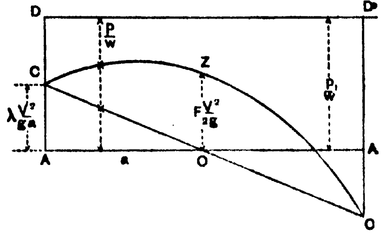
$$CA = \lambda \,.\, \frac{V^2}{ga},$$

and draw the sloping line COC. Then, as in Art. 109, already cited, the ordinate of that line represents the pressure-head necessary to overcome the inertia of the piston and the water connected with it. Again, supposing, as usual, F to be the co-efficient of hydraulic resistance, set up

$$OZ = F\frac{V^2}{2g} = CA \,.\, \frac{Fa}{2\lambda},$$

and on the oblique base COC draw the parabola CZC, then (comp. Arts. 20, 109) the ordinate of this parabola will represent the pressure necessary to overcome the hydraulic resistances at every point. If then the horizontal line DD be drawn at a height representing the pressure in the accumulator, the intercept between that line and the parabola will represent the pressure transmitted to the crank pin at each point of the stroke. The slope of CC and the height of the parabola increase rapidly with the speed, which must never be great enough to cause the parabola to touch DD, otherwise a violent shock will occur. The same effect will be produced by any falling off in the useful resistance: the angular acceleration of the crank shaft then raises the central part of the line *CC* and with it the line of frictional resistances. It should be observed that the curve of frictional resistances may also be taken to represent the kinetic energy of the piston, both these quantities being proportional to the square of the velocity of the piston. It is therefore the graphical integral of the curve of acceleration (Ch. IX.).

Fig. 193.



The simple example here given will serve as an illustration of the great variations of pressure which occur in water-pressure engines and their consequent liability to shocks. For which reason escape valves or air chambers must be provided to relieve the pressure when it becomes excessive. Unless the resistance be very uniform an additional accumulator is required as near as possible to the machine.

266. *Examples of Hydraulic Pressure Machines.*—Water-pressure engines form a large and interesting class of hydraulic motors, of which a few examples will now be given.

(1) In direct-acting lifts a weight is raised by the direct action of fluid pressure on a ram the stroke of which is equal to the height lifted. The weight here rests on a cage or platform fixed to the upper end of the ram and sliding in guides. The water is frequently supplied from a tank at a moderate elevation, so that the pressure-head diminishes as the lift rises. This is a very convenient arrangement for the purpose, as it supplies an additional pressure at the bottom of the stroke where it is required to overcome inertia at starting, and a diminished pressure at the top where the lift requires to be stopped. The useful resistance is here constant and the pressure head would be represented by the ordinates of a sloping line. A diagram of speed and acceleration may be constructed by a process similar to that given in the last article.

(2) A direct-acting lift necessarily occupies a great space, and the stroke of the working cylinder is therefore often multiplied by the use of blocks and tackle as shown in Fig. 2, Plate IX. The cylinder may be placed in any convenient position, and the chain passes from the blocks over fixed pulleys to the cage which is suspended from

it. The friction of the pulleys is here considerable, and there is a liability to breakage; but for convenience the arrangement is one which is frequently employed.

(3) In hydraulic cranes the working cylinder is sometimes placed below and sometimes occupies the crane post which is tubular. The stroke is multiplied by tackle as in the previous case, the chain passing through the crane post and over fixed pulleys to the extremity of the jib. An example is shown in Fig. 2, Plate IX., p. 525.

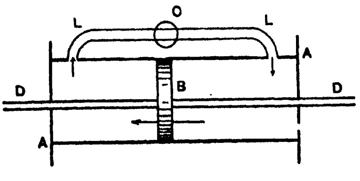
(4) A water-pressure engine may be employed to turn a crank. Three working cylinders inclined at 120° are frequently used as shown in Fig. 1, Plate X., p. 525. They are single acting and drive the same crank as in the small steam engines of the same type employed where great speed is required. The water is admitted to the outer ends of the cylinders, so that the piston rods are always in compression.

(5) The hydraulic mechanism applied to work heavy guns on board ship consists of a cylinder in which works a piston attached to a rod, the sectional area of which is one-half that of the cylinder. If water be admitted at both ends of the cylinder the piston moves outwards, but if to the inner end only, it moves inwards. The motive force in either case is the same, being due to the difference of areas. This apparatus serves also as a brake of the kind described in the next article. For details and illustrations the reader is referred to the *Gunnery Manual*.

267. *Hydraulic Brakes.*—It has been sufficiently explained that hydraulic resistances absorb an amount of energy which varies as the square of the speed. An hydraulic machine therefore may be employed as a brake, and it is in this way that large amounts of surplus energy are most easily disposed of. Moreover, by its use the speed of any machine to which it is applied is readily controlled.

An hydraulic brake is constructed by interposing a mass of fluid between the elements of a pair so that any motion of the pair causes a breaking-up of the fluid with a corresponding resistance.

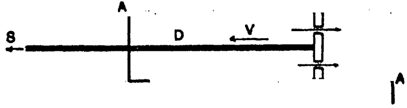
Fig. 194a.



A common case is that of a sliding pair consisting of a piston and cylinder filled with water or oil, which passes from one side of the piston to the other whenever the piston moves. Two examples of this apparatus are shown in skeleton in Figs. 194*a*, 194*b*. In the first (Fig. 194*a*), the piston rod *DD* projects through both cylinder covers, and communication is made between the two ends of the cylinder by a pipe LL provided with a cock C, which can be closed at pleasure. At D the rod is attached to the piston rod of a steam cylinder employed to obtain the very considerable force necessary to work the starting and reversing gear of large marine engines. The resistance of this brake is zero when the piston begins to move, but increases as the square of the speed, and thus effectually prevents it from moving too rapidly. The maximum speed is controlled by turning the cock. For a detailed description of this gear the reader is referred to a treatise on the *Marine Engine*, by Mr. Sennett.

In the second (Fig. 194*b*) the water passes from one end of the cylinder through orifices in the piston itself. This is the common "compressor" or Service Buffer.* The piston rod in this case passes out at one end only of the working cylinder, and is attached to the gun, the recoil of which is to be checked. The theory of this apparatus is of some interest, and will now be briefly considered.

Fig. 194b.



Let n be the ratio of the area of the piston to the *effective* area of the orifices, then the loss of head consequent on the sudden enlargement will be

$$\frac{p_1 - p_2}{w} = (n-1)^2 \frac{V^2}{2g},$$

where V is the speed of piston and p_1, p_2 are the pressures on the two sides of the piston. Hence the pull

$$S = wA(n-1)^2 \frac{V^2}{2g}$$

on the piston rod is necessary to overcome the hydraulic resistance at this speed. The gun is gradually brought to rest by this resistance, aided by the friction of the slide.

At the instant of firing, a certain amount of kinetic energy is generated in the gun given by the formula

$$\text{Energy of Recoil} = \frac{WV_0^2}{2g} \text{ (Art. 133, p. 268)},$$

where V_0 is the maximum velocity of recoil. As the gun recoils its velocity diminishes, and if P_0 be the friction of the slide the retarding force will be

$$S + P_0 = wA(n-1)^2 \frac{V^2}{2g} + P_0.$$

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* *Manual of Gunnery for Her Majesty's Fleet*, p. 68.

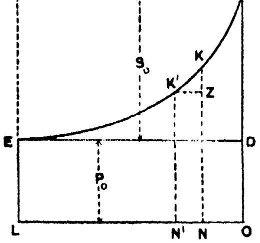
The maximum value of S will be found by writing V_0 for V, and may be denoted by S_0.

To represent this graphically, in Fig. 195 draw a curve in which the ordinate KN at any point N represents the retarding force after the gun has recoiled through the space ON from the point O, at which the action of the powder pressure ceases, and the gun has its maximum velocity V_0. This curve will start from a point A such that

$$OA = S_0 + P_0;$$

and will reach the horizontal DE at a height P_0 above the base line at a point E, such that OL is the complete recoil. The area $OAEL$ of this curve represents the energy of recoil which has all been absorbed by the frictional resistance of the slide and the hydraulic resistance of the compressor. Further, the area $KNN'K'$ between two ordinates will represent the diminution of energy as the gun recoils through the space NN' between them, a circumstance which enables us to construct the curve, for if VV' be the velocities of the recoiling gun at NN' respectively,

Fig. 195.



$$\text{Area } KNN'K' = \frac{W(V^2 - V'^2)}{2g}.$$

But if S, S' be the corresponding values of S,

$$KZ = S - S' = wA(n-1)^2 \frac{V^2 - V'^2}{2g};$$

and if the ordinates be taken near together the area in question will be nearly $KN.NN'$. We have therefore, by division,

$$\frac{KZ}{KN} = NN'.\frac{wA(n-1)^2}{W}.$$

That is, if a number of equidistant ordinates be drawn near together the ratio of consecutive ordinates is constant. The curve may be roughly traced from this property; it is identical with the curve already drawn in Art. 123, p. 252, except that it is a linear instead of a polar curve.

The mean resistance to recoil is given by the equation

$$(\bar{S} + P_0)l = \text{Energy of Recoil},$$

where l is the distance traversed. It would, of course, be advantageous to have a uniform resistance to recoil, because the maximum pressure in the compressor would be diminished and less strain thrown on the gear. This is the object of the various modified forms of the compressor, in which the orifices are not of constant area, but become smaller as the recoil proceeds. In order that the resistance may be constant we must have

$$\bar{S} = wA(n-1)^2 \frac{V^2}{2g},$$

so that $(n-1)V$ is constant. Further, since the retardation is uniform

$$V^2 = 2g \cdot \frac{\bar{S}+P_0}{W} x,$$

where x is the distance from the end of the recoil. It appears therefore that the orifices should vary in such a way that $(n-1)^2 x$ should be constant. Descriptions of two forms of compressor, with varying orifices, will be found in the *Gunnery Manual.*

Instead of a sliding pair we may employ a turning pair. This is the common "fan" or "fly" brake used to control the speed and absorb the surplus energy of the striking movement of a clock, or in other similar cases. A friction dynamometer (p. 279) was designed by the late Mr. Froude for the purpose of measuring the power of large marine engines, in which the ordinary block or strap surrounding a shaft or drum is replaced by a casing in which a wheel works. Vanes attached to the wheel and the fixed casing thoroughly break up a stream of water passing through the casing. Any amount of energy may thus be absorbed without occasioning any considerable rise of temperature. Siemens' combined brake and regulator has been mentioned already on page 278.

268. *Transmission of Energy by Hydraulic Pressure.*—Energy may be distributed from a central source, and transmitted to considerable distances with economy by hydraulic pressure. The delivery in gallons per minute of a pipe d'' diameter is

$$G = 27\sqrt{\frac{h'}{l}} \cdot d^{\frac{5}{2}} \text{ (Art. 250).}$$

Assume now that the pipe supplies an hydraulic machine at a distance of l feet from an accumulator in which h is the head. Further, suppose that n per cent. is lost by friction of the pipe, then the power transmitted in foot-lbs. per minute is

$$10Gh = 270h\sqrt{\frac{nh}{100l}} \cdot d^{\frac{5}{2}},$$

and the distance to which N horse-power can be transmitted with a loss of n per cent. is in feet

$$l = \frac{h^3 d^5 n}{1{,}500{,}000 N^2} \text{ (nearly).}$$

With the usual pressure in accumulators of 750 lbs. per square inch, or 1700 feet of water, this gives the simple approximate formula

$$l = 3300\frac{d^5 n}{N^2}.$$

Thus, for example, 100 horse-power may be transmitted by a 5″ pipe to a distance of 4 miles, or 10 horse-power by a 1″ pipe to a distance of 220 yards, with a loss by friction not exceeding 20 per cent. The diameter of pipe is limited by considerations of strength and cost.

The power of a motor supplied by a given pipe does not increase indefinitely as its speed increases, but is greatest when one-third of the head is lost by friction.* The maximum possible power is therefore given by the formula

$$H.P. = 220\sqrt{\frac{d^5}{l}} \text{ (approximately).}$$

This is of course two-thirds the value of N in the preceding formula.

269. *Pumps.*—If the direction of motion of an hydraulic motor be reversed by the action of sufficient external force applied to drive it, while, at the same time, the direction of the issuing water is reversed so as to supply the machine at the point from which it originally proceeded, we obtain a machine which raises water instead of utilizing a head of water. Every hydraulic machine therefore may be employed to raise water as well as to do work, and most of them actually occur in this form; they are then called PUMPS, though in some cases this name would not be used in practice. Much of what has been said about motors applies equally well to pumps: the principal difference lies in the fact that the useful resistance which the pump overcomes is always reversible, whereas in the motor this is not necessarily the case. The principles of action and the classification of hydraulic machines are, in the main, the same in both cases. Some points omitted while considering motors as being of most importance in pumps, and certain differences of action between the two will now be briefly noticed. Certain machines occurring principally as pumps will be mentioned.

(1) If the direction of motion of an overshot wheel be reversed a machine is obtained which is known as a "Chinese Wheel." It picks up water in its buckets and raises it to a height somewhat less than the diameter of the wheel. This machine is little used, but a reversed breast wheel is frequently employed in drainage operations, under the name of a "scoop," or "flash" wheel. The working pair is here

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* This result was pointed out to the writer by Mr. (now Prof.) Hearson. It appears to be little known.

a turning pair, but in the chain pump we find an example in which one of its elements is a chain passing over pulleys. The chain is endless and is provided with flat plates fitting into a vertical pipe, the lower end of which is below the surface of the water, and through which the water is raised. In the common dredging machine the closed channel (p. 512) is replaced by buckets. In a third class of weight machines the water occupies the moveable chamber and forms with it a kinematic pair with only one solid element, while it forms with the link attached to the earth, a working pair which has also but one solid element. The Archimedian screw, and certain varieties of "scoop" wheel, in which the water enters the scoop at the circumference of the wheel and is delivered at the centre, are examples of this kind.

(2) The most common forms of pumps are the "lift" or "force" pumps, which consist of a chamber which expands to admit the water to be lifted and contracts in the act of lifting; they are therefore pressure machines like those considered in Arts. 264-5, but reversed. The name "pump" originally applied to these machines alone.

Fig. 196

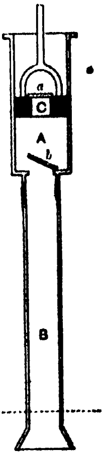

Fig. 196 shows a common lift pump. *A* is a cylinder at a certain height h_1 above the water to be raised, *C* is a piston working in the cylinder by the action of which the water is lifted. The piston has orifices in it which permit the water to pass through. The orifices are closed by a valve, as is also the opening at the bottom of the cylinder. These valves are simple "flaps" which open on hinges to permit the water to pass upwards, but close the passage to motion in the opposite direction, thus acting as a ratchet (p. 156). Assuming the piston at the bottom of its stroke, at rest close to the bottom of the cylinder, let it be supposed to rise; the valve *b* will rise and allow air to pass if any. After several strokes the air will be nearly exhausted, and if h_1 be not too great the empty space will be filled with water raised from the tank by atmospheric pressure. Thus the water will pass into the cylinder closely following the piston. At the top of the stroke the piston commences to descend, *b* closes and *a* opens, allowing the water to pass above the piston. This water is now raised by the piston to any required height. In force pumps the process is the same, but the water passes out through an orifice in the bottom of the cylinder instead of through the piston; the raising of the water above the level of the cylinder is done in the down stroke instead of the up

PLATE IX.

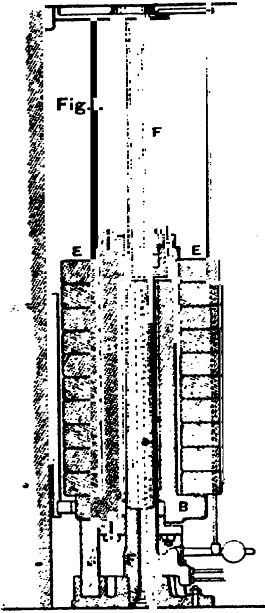

Fig. 1.

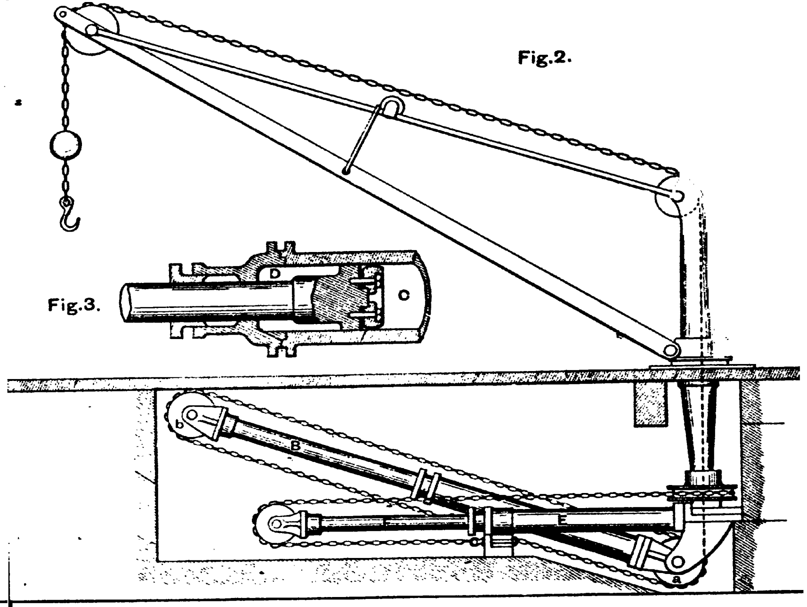

Fig. 2.

Fig. 3.

To face page 525.

PLATE X.

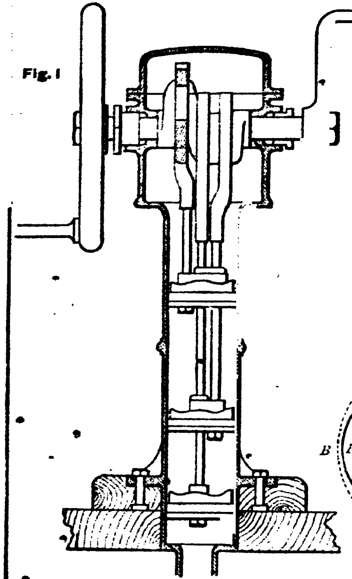

Fig. 1

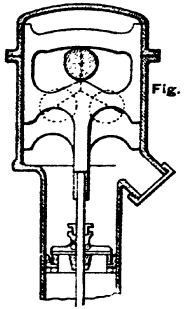

Fig. 2

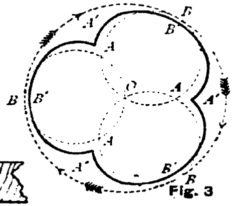

Fig. 3

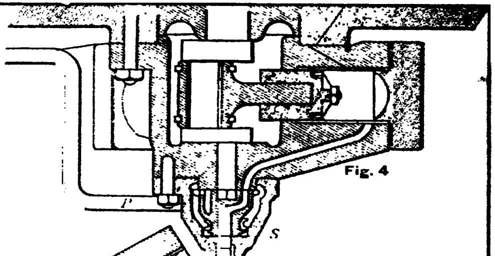

Fig. 4

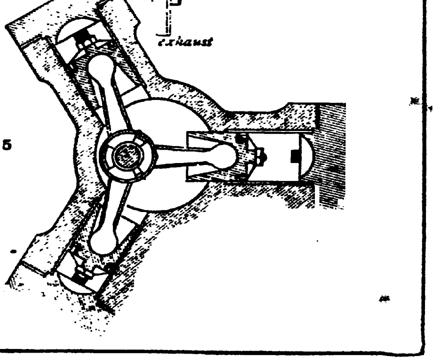

Fig. 5

To face page 525.

strokes: in this example the slip was 12 per cent. The efficiency of the pumps and mechanism of the engine was found to be 66 per cent. by careful experiments.*

In raising water from great depths in mines, force pumps at the bottom of the mine are used, worked by heavy "spear" rods from a beam engine at the surface. The weight of the rod supplies the motive force during the downward stroke of the pump; while the engine, which is single-acting, raises the rods again during the downward stroke of the steam piston.

DESCRIPTION OF PLATES IX. AND X.

In order further to illustrate the action of water-pressure machines Plates IX. and X. have been drawn.

Fig. 1, Plate IX., shows the differential accumulator described on page 513.

In Fig. 2 is represented an hydraulic crane, designed by Sir W. Armstrong, for lifting weights of 2 to 3 tons. In it the hydraulic power is applied to rotate the crane as well as to lift the weight.

In order to effect the lift the high-pressure water from the accumulator is admitted to the cylinder *A*, and forces out the plunger *B*. There are two pulleys at *a* and two at *b*. One end of the chain is secured to the cylinder *A*, it is led round *b*, then round *a*, again round *b*, then under the second pulley at *a* up through the hollow crane post on to the weight as shown. The effect of this arrangement is that any movement of the plunger *B* is at the hook multiplied four times.

If *B* is simply a plunger working in a stuffing box, then the expenditure of energy is always the same whatever weight is being lifted, and the amount must be equal to that which corresponds to lifting the maximum possible weight.

This is an objection which is common to all such machines. The surplus energy is expended in overcoming frictional resistances (p. 514). To mitigate this evil, in cranes of high power the plunger has a piston end, which fits a bored cylinder, and is provided with a cup leather, as shown in Fig 3. The sectional area of the plunger is about one-half that of the cylinder. If a light weight is to be lifted, water is admitted to both sides of the piston, and the difference of the pressures, equal to what would be exerted on a simple plunger, is available for effecting the lift. When it is required to lift a heavy weight water is admitted to the side *C* only of the piston, the annular space *D* being put in communication with the atmosphere. Thus the full pressure due to the area of the piston is exerted with the corresponding expenditure of water.

For the purpose of rotating the crane a pair of cylinders, *E*, are provided, of which one only is shown in the figure. The thrusting out of the plunger *F* of one of them by the pressure of the water causes the other to be drawn in by means of a chain which passes around a recessed pulley secured to the crane post.

In Plate X., Figs. 1 and 2 show the construction of Downton's Pump, so much used on board ship. In the barrel work three buckets with flap valves, as shown in Fig. 2. The rods to which the upper and second buckets are attached are necessarily out of centre. The rods to the lower buckets pass through deep stuffing boxes in the buckets above, and thus the buckets are maintained from canting seriously. The movement of the buckets is effected by a three-throw crank, the crank pins, which are not round, being set at 120° apart. These pins fit and work in a curved slot in the bucket rod heads. Assuming the admission of no air but water only from below, the discharge of the pump will at each instant equal the displacement of the fastest upward moving bucket. Accordingly the rate of discharge may be represented by a curve, as in Fig. 3. If the slot in the rod head were straight and the pin round, then, the crank moving uniformly, in direction shown, the velocity of discharge would be represented by the radii from *O* to the dotted curve *BABABA*, which is made up of parts of three circles, the position of the radius being that of either of the three cranks. The effect of the curved slot is to diminish the maximum and increase the minimum discharge, as shown by the full curve *B'A'B'A'B'A'*.

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* *Minutes of Proceedings of the Institution of Civil Engineers*, vol. lxvi.

Figs. 4 and 5 of this Plate are sections of the hydraulic engine referred to on page 518, employed to drive a capstan. It need only be further added that a single rotating valve *V* suffices for admission and exhaust of all three cylinders. The high-pressure water is supplied by the pipe *P* to the passage *S* surrounding the valve and exhausted from the cylinders through the central passage.

EXAMPLES.

1. In estimating the power of a fall of water it is sometimes assumed that 12 cubic feet per second will give 1 H.P. for each foot of fall: what efficiency does this suppose in the motor? *Ans.* ·72.

2. An accumulator ram is 9 inches diameter, and 21 feet stroke; find the store of energy in foot-lbs. when the ram is at the top of its stroke, and is loaded till the pressure is 750 lbs. per square inch. *Ans.* 1,000,000 foot-lbs.

3. In a differential accumulator the diameters of the spindle are 7 inches and 5 inches; the stroke is 10 feet: find the store of energy when full, and loaded to 2,000 lbs. per square inch. *Ans.* 377,000 foot-lbs.

4. A direct-acting lift has a ram 9 inches diameter, and works under a *constant* head of 73 feet, of which 13 per cent. is required by ram friction and friction of mechanism. The supply pipe is 100 feet long and 4 inches diameter. Find the speed of steady motion when raising a load of 1,350 lbs., and also the load it would raise at double that speed.

Ans. Speed = 2 feet per second.
Load = 150 lbs.

5. In the last question, if a valve in the supply pipe is partially closed so as to increase the equivalent of resistance by $5\frac{1}{2}$, what would the speed be? *Ans.* 1·6 f.s.

6. Eight cwt. of ore is to be raised from a mine at the rate of 900 feet per minute by a water-pressure engine, which has four single-acting cylinders, 6 inches diameter, 18 inches stroke, making 60 revolutions per minute. Find the diameter of a supply pipe 230 feet long, for a head of 230 feet, not including friction of mechanism. *Ans.* Diameter = 4 inches.

7. Water is flowing through a pipe 20 feet long with a velocity of 10 feet per second. If the flow be stopped in one-tenth of a second, find the intensity of the pressure produced, assuming the retardation during stoppage uniform. *Ans.* 62 feet of water.

8. If λ be the length equivalent to the inertia of a water-pressure engine, *F* the co-efficient of hydraulic resistance, both reduced to the ram, v_0 the speed of steady motion; find the velocity of ram, after moving from rest through a space x against a constant useful resistance. Also find the time occupied.

$$Ans.\ v^2 = v_0^2\left(1 - \epsilon^{-\frac{Fx}{\lambda}}\right);\ t = \frac{\lambda}{Fv_0}\log_\epsilon\frac{v_0+v}{v_0-v}.$$

9. An hydraulic motor is driven from an accumulator, the pressure in which is 750 lbs. per square inch, by means of a supply pipe 300 feet long 4 inches diameter; what would be the maximum power theoretically obtainable, and what would be the velocity in the pipe at that power? Find approximately the efficiency of transmission at half power. *Ans.* H.P. = 240; $v = 22$; efficiency = ·96 nearly.

10. A gun recoils with a maximum velocity of 10 feet per second. The area of the orifices in the compressor, after allowing for contraction, may be taken as one-twentieth the area of the piston: find the maximum pressure in the compressor in feet of liquid.

Ans. 560 to 594.

11. In the last question assume weight of gun 12 tons; friction of slide 3 tons; diameter of compressor 6 inches; fluid in compressor water: find the recoil.

Ans. 4 feet $2\frac{1}{4}$ inches.

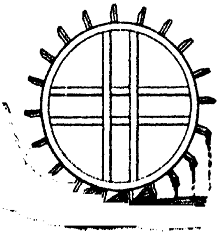
12. In the last question find the mean resistance to recoil. Compare the maximum and mean resistances each exclusive of friction of slide.

Ans. Total mean resistance = 4·4 tons. Ratio = 2·2.

SECTION II.—IMPULSE AND REACTION MACHINES.

270. *Impulse and Reaction Machines in General.*—The source of energy may be a current of water or the head may be too small to obtain any considerable pressure, and it is then necessary to have some means of utilizing the energy of water in its kinetic form. A machine for this purpose operates by changing the motion of the water and utilizing the force to which the change gives rise. If the water strikes a moving piece and is reduced to rest relatively to it, the machine works by "impulse," and if it be discharged from a moving piece, by "reaction." There is no difference in principle between these modes of working, and both may occur in the same machine. In either case, the motive force arises from the mutual action between the water and the piece which changes their relative motion. Machines of this class are also employed for high falls when the low speed of pressure machines renders their use inconvenient or impossible. The water is then allowed to attain a velocity equivalent to a considerable portion of the head immediately before entering the machine, so that its energy is, in the first instance, wholly or partially converted into the kinetic form.

Fig. 197.



The simplest machine of this kind is the common undershot wheel, consisting of a wheel (Fig. 197) provided with vanes against which the water impinges directly. Let the velocity of periphery of the wheel be V, then the water after striking the vanes is carried along with them at this velocity. If, then, the original velocity of the water be v, the diminution of velocity due to the action of the vanes will be $v-V$. Let W be the weight of water acted on per second, then the impulse on the wheel must be

$$P=\frac{W(v-V)}{g},$$

but if A be the sectional area of the stream,

$$W = Avw,$$

this being the weight of water per second which comes in contact with all the vanes taken together,

$$\therefore\ P = \frac{w}{g} Av(v - V).$$

The power of the wheel is PV foot-lbs. per second, and the energy of the stream is $Wv^2/2g$, therefore

$$\text{Efficiency} = \frac{2V(v - V)}{v^2}.$$

This is greatest when $V = \frac{1}{2}v$ and its value is then ·5, showing that the wheel works to best advantage when the speed of periphery is one-half that of the stream, but that the efficiency is low, never exceeding ·5.

Such wheels may be seen working a mill floating in a large river, or in other similar circumstances, but they are cumbrous and, allowing for various losses not included in the preceding investigation, their efficiency is not more than 30 per cent. In the early days of hydraulic machines, they were often used for the sake of simplicity or, as in the example shown in the figure, from a want of comprehension of their principle.* In mountain countries, where unlimited power is available, they are still found. The water is then conducted by an artificial channel to the wheel, which sometimes revolves in a horizontal plane. When of small diameter their efficiency is still further diminished.

In overshot wheels and other machines operating chiefly by weight the head corresponding to the velocity of delivery is partly utilized by impulse, and the speed of the wheel is determined by this consideration. In all cases of direct impulse, if h is that part of the head operating by impulse, the speed of maximum efficiency is

$$V = \frac{1}{2}\sqrt{2gh} = 4\sqrt{h},$$

or in practice somewhat less, and at that speed at least half that head is wasted. The great waste of energy in this process is due partly to the velocity V with which the water moves onward with the wheel, and partly to breaking-up during impulse. It is in fact easy to see that one-fourth the head is wasted by each of these causes. To avoid it, the water must be received by the moving piece against which it impinges without any sudden change of direction, and must be discharged at the lowest possible velocity, effects which may be produced by a suitably-shaped vane curved so as to deflect the water gradually and guide it in a proper direction. The principle on which such a vane is designed may be explained by the annexed diagram. In Fig. 198 *AB* is a vane moving with velocity *V* in a given direction, against which a jet strikes. Drawing a diagram of velocities, let *Oa* represent *v*, the velocity of the jet, and let *OO'* represent *V*. Then as before (p. 499) *O'a* represents the velocity of the jet relatively to the vane, and, in order that the water may impinge without shock, the tangent to the vane at *A* must be parallel to *O'a*. The vane is now curved so as gradually to deflect the water, in doing which there is a mutual action between the jet and the vane furnishing the motive force which drives the wheel. If the water leave the vane at *B*, its velocity relatively to the vane is represented by *O'b* drawn parallel to the vane at *B*, and somewhat less than *O'a* in magnitude, to allow for friction, unless the water be enclosed in a passage, when it will bear some given proportion to *O'a*. The absolute velocity with which the water moves at *B* is now represented by *Ob*, and this may be arranged to deliver the water in a convenient direction with a velocity just sufficient to clear the wheel and no more.

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* See Fairbairn's *Millwork and Machinery*, from which this figure is taken, vol. i., p. 149.

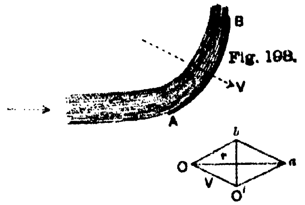

Fig. 198.

Two examples of the use of such vanes may now be mentioned.

(1) In the Pelton wheel recently introduced in America, the buckets of an ordinary vertical water wheel, receiving a jet of water tangentially under a considerable head, are divided in the middle, and each half curved so as to form a cup or pocket facing the jet. The inner edges of the two halves are now united so as to form a dividing edge, upon which the jet impinges centrally and by which it is separated into two parts, each diverging laterally and then turning through an angle of nearly 180°. The double pocket with its dividing edge is not essential, a simple cup vane (Fig. 190, p. 500) would suffice; but it probably renders the jet less liable to breaking up from unsteadiness or in consequence of the angular motion of the bucket. A wheel of this kind at the Comstock mines, Nevada, U.S., works under a head of 2,100 feet with a velocity of periphery of 180 f.s.* Their efficiency is very considerable, in many cases exceeding 80 per cent.

(2) Of much older date are the vanes applied to vertical water wheels by Poncelet in order to utilize as far as possible a head of moderate amount. The water in this case impinging below the wheel at *A*, ascends to *B*, and then while the vane is moving onwards

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* *The Practical Engineer*, June 17, 1892.

descends again to A under the action of gravity. The velocity of the water relatively to the wheel is thus reversed: $O'b$ being approximately equal and opposite to $O'a$.

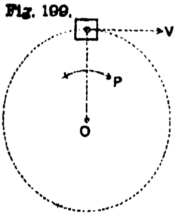
In all impulse and reaction machines there is a speed of maximum efficiency which, as in the simple case first considered, is given by the formula

$$V = k\sqrt{2gh},$$

where k is a fraction depending on the type of machine.

271. *Angular Impulse and Momentum.*—The most important of these machines are those in which the change of motion produced in the water is a motion of rotation, and it is needful to consider that form of the principle of momentum which is applicable to such cases.

Fig. 199.



In Fig. 199, W is a weight describing a circle round O with velocity V; then the product of its momentum by the radius r is called the "moment of momentum" of the weight about O. If O represent an axis to which W is attached rigidly, we may imagine it turning under the action of a force P at a radius R. The moment of P multiplied by the time during which it acts is called the "moment of impulse."

During the action of P the weight will move quicker and quicker and the motion is governed by the principle expressed by the equation

$$\text{Moment of Impulse} = \text{Change of Moment of Momentum.}$$

If L be the moment of P, then taking the time as one second,

$$L = \text{Change of Moment of Momentum per second.}$$

This equation is true, not only for a single weight and a single force, but also for any number of weights and any number of forces. As in other forms of the principle of momentum it is also true, notwithstanding any mutual actions or any relative movements of the weights or particles considered. Further, any radial motions which the particles possess may be left out of account, for they do not influence the moment of momentum. A particular case is when $L = 0$, then the moment of momentum remains constant, a principle known as the Conservation of Moment of Momentum. The terms "moment of momentum" and "moment of impulse" are often replaced by "angular momentum," "angular impulse."

A weight rotating about an axis is capable of exerting energy in two ways. First, it may move away from the axis of rotation, overcoming by its centrifugal force a radial resistance which it just overbalances.

Secondly, it may overcome a resistance to rotation in the shaft to which it is attached.. In either case the work done will be represented by a diminution in the kinetic energy of the weight.

If the shaft be free, the diminution of kinetic energy must be equal to the work done by the centrifugal force, and it may be proved in this way, that if V be the velocity of rotation of the weight, r the radius,

$$Vr = \text{Constant},$$

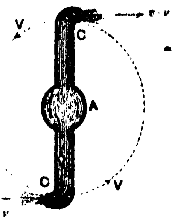
an equation equivalent to the conservation of the moment of momentum.

Conversely, energy may be applied to a rotating weight either by moving it inwards against its centrifugal force, or by a couple applied to the axis of rotation.

In turbines both modes of action occur together as we shall see presently; and the employment of the principle of momentum, though not necessary, is on the whole the most convenient way of dealing with the question.

272. *Reaction Wheels.*—Fig. 200 shows a reaction wheel in its simplest form. CAC is a horizontal tube communicating with a vertical tubular axis to which it is fixed, and with which it rotates. Water descends through the vertical tube, and issues through orifices at the extremities of the horizontal tube so placed that the direction of motion of the water is tangential to the circle described by the orifices. The efflux is in opposite directions from the two orifices, and a reaction is produced in each arm which furnishes a motive force. There are two methods of investigating the action of this machine which are both instructive. Frictional resistances are, in the first instance, neglected.

Fig. 200.



(1) Let the orifices be closed, and let the machine revolve so that the speed of the orifices in their circular path of radius r is V. Centrifugal action produces a pressure in excess of the head h existing when the arms are at rest, the magnitude of the excess in feet of water being $V^2/2g$. This is so much addition to the head, which now becomes

$$H = h + \frac{V^2}{2g}.$$

This quantity H may also be considered as the head "relative to the moving orifices" estimated as on p. 4[illegible].

When the orifices are opened, the water issues with velocit given by

$$v^2 = 2gH = V^2 + 2gh;$$

thus the water issues with a velocity greater than V, and after leaving the machine has the velocity $v - V$ relatively to the earth. The energy exerted per lb. of water is h, and this is partly employed in generating the kinetic energy corresponding to this velocity. The remainder does useful work by turning the wheel against some useful resistance, so that we have per lb. of water

$$\text{Useful Work} = h - \frac{(v-V)^2}{2g} = \frac{V(v-V)}{g},$$

and, dividing by h,

$$\text{Efficiency} = \frac{V(v-V)}{gh} = \frac{2V}{v+V}.$$

(2) A second method is to employ the principle of the equality of angular impulse and angular momentum already given in Art. 271. Originally the water descends the vertical tube without possessing any rotatory motion, but after leaving the machine it has the velocity $v - V$ its angular momentum is therefore for each lb. of water,

$$\text{Angular Momentum} = \frac{(v-V)}{g}\,.\,r.$$

Now according to the principle the angular momentum generated per second is also the angular reaction on the wheel which, when multiplied by V/r, the angular velocity of the wheel, gives us the useful work done per second. Performing this operation, and dividing by the weight of water used per second, we get per lb. of water

$$\text{Useful Work} = \frac{V(v-V)}{g}.$$

This is the result already obtained, and the solution may now be completed by adding the kinetic energy on exit.

From the result it appears that the proportion which the waste work bears to the useful work is $v - V : 2V$, which diminishes indefinitely as v approaches V; but in this case the velocities become very great, since $v^2 - V^2$ is always equal to $2gh$. The frictional resistances then become very great, so that in the actual machine there is always a speed of maximum efficiency which may be investigated as follows:

Let F be the coefficient of hydraulic resistances referred to the orifices, then

$$(1+F)\frac{v^2}{2g} = H = h + \frac{V^2}{2g}.$$

The useful work remains as before, and therefore

$$\text{Efficiency} = \frac{2V(v-V)}{v^2 - V^2 + F\,.\,v^2},$$

a fraction which can readily be shown to be a maximum when

$$v - V = V\sqrt{\frac{F}{1+F}}, \quad \text{or} \quad v = V\left\{1 + \sqrt{\frac{F}{1+F}}\right\},$$

which value of v, when substituted in the preceding equation, will give the value of V in terms of h for maximum efficiency. The existence of a speed of maximum efficiency is well known by experience with these machines. In general it is found to be about that due to the head, so that

$$V^2 = 2gh,$$

a value which corresponds to $F = \cdot125$, and gives an efficiency of ·67. This is about the actual efficiency of these machines under favourable circumstances; of the whole waste of energy two-thirds, that is two-ninths of the whole head, is spent in overcoming frictional resistances, and the remaining one-third, or one-ninth the whole head, in the kinetic energy of delivery.

The reaction wheel in its crudest form is a very old machine known as "Barker's Mill." It has been employed to some extent in practice as an hydraulic motor, the water being admitted below and the arms curved in the form of a spiral. These modifications do not in any way affect the principle of the machine, but the frictional resistances may probably be diminished.

273. *Turbine Motors.*—A reaction wheel is defective in principle, because the water after delivery has a rotatory velocity in consequence of which we have seen a large part of the head is wasted. To avoid this, it is necessary to employ a machine in which some rotatory velocity is given to the water before entrance in order that it may be possible to discharge it with no velocity except that which is absolutely required to pass it through the machine. Such a machine is called in general a TURBINE, and it is described as "outward flow," "inward flow," or "parallel flow," according as the water during its passage through the machine diverges from, converges to, or moves parallel to, the axis of rotation.

Fig. 201*a* shows in plan and section part of an annular casing forming a wheel revolving about an axis XX through which water is flowing, entering at the centre and spreading outwards. The water leaves the wheel at the outer circumference. Fig. 201*b* is similar, but the flow is inward instead of outward.

If we consider a section *aa* made by a concentric cylinder of length y and radius r, the flow will be

$$Q = u \,.\, 2\pi ry,$$

where u is the radial velocity or, as we may call it, the "velocity of flow." The area of the section ($2\pi ry$) may conveniently be called the "area of flow." The value of Q is everywhere the same, and therefore ury must be constant. It is generally desirable to make u constant or nearly so, and then the form of the casing is such that *ry* is constant. Whether this be so or not, the value of *u* can always be calculated at any radius for a given wheel with a given delivery.

Fig. 201a.

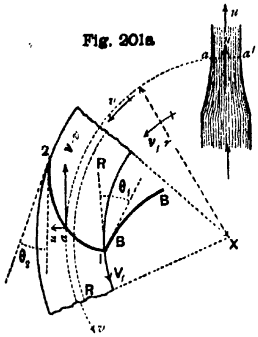
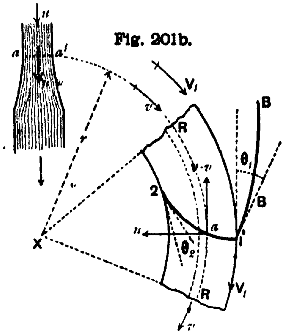


Fig. 201b.



The water which at any given instant is at a given distance *r* from the axis may be considered as forming a ring *RR*, which rotates while at the same time it expands or contracts according as the flow is outward or inward. The velocity of the periphery of this ring may be described as the "velocity of whirl," and if it be called *v*, the moment of momentum of a ring, the weight of which is *W*, is

$$M = \frac{W}{g} . vr.$$

If the wheel has no action on the water, this quantity cannot be altered, and we must then have

$$vr = \text{Constant.}$$

The water then forms what we have already called a "free vortex" (Art. 244), with the addition of a certain radial velocity *u*, in consequence of which the rings change their diameter. The paths of the particles of water are then spirals, the inclination of which depends on the proportion between *u* and *v*.

The case now to be considered is that in which the moment of momentum of the rotating rings is gradually reduced during their passage through the wheel by the action of suitable vanes attached to it. An impulse is thus exerted on the wheel which furnishes the motive force. The moment of this impulse is given by the equation,

$$L = \frac{wQ}{g}(v_1r_1 - v_2r_2),$$

where *wQ* is the weight of all the rings passing through the machine in a second, and the suffixes 1, 2 refer to entrance and exit respectively as indicated in the figures for the two cases of outward flow and inward flow. In this article the turbine is supposed to work to best advantage when the water is discharged without any whirl, that is when $v_2=0$, and putting aside friction the only loss then is that due to the velocity of flow u, which may be made small by making the wheel of sufficient breadth at the circumference where the water is discharged.

In practice there are of course always frictional resistances, but, for given velocities, the impulse on the wheel is not altered by them, so that the moment of impulse is always given by the above equation. Suppose, now, h the *effective* head found from the actual head by deducting (1) the height due to the velocity of delivery, (2) the friction of the supply pipe and passages in the wheel, (3) the loss (if any) by shock on entering the wheel; then

$$\text{Work done per second} = wQh.$$

But, if V_1 be the speed of periphery of the wheel at the radius r_1 where the water enters, V_1/r_1 is the angular velocity of the wheel, and $L.V_1/r_1$ is the work done per second. We have then for the case where there is no whirl at exit

$$V_1v_1=gh.$$

The effective head h in this formula includes (1) a part equivalent to the useful work, and (2) a part equivalent to the frictional resistances to the rotation of the wheel, such as friction of bearings and friction of the water surrounding the wheel (if any) on its external surface. This last item is often described as "disc friction." If H be the actual head, the efficiency, apart from external friction, is

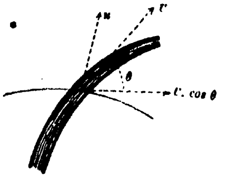
$$\text{Efficiency} = \frac{h}{H} = \frac{V_1v_1}{gH}.$$

The whirl before entrance is communicated by fixed blades BB, curved, as shown in the figures, so as to guide the water in a proper direction on entrance to the wheel. It is the use of these guide blades which characterizes the turbine as distinguished from the reaction wheel.

The whirl at different points, either in the wheel or outside it, depends on the angle of inclination of the vanes or guide blades to the periphery. These blades are so numerous that the water moves between them nearly as it would do in a pipe of the same form. If θ be the angle such a pipe (Fig. 202) makes with the periphery at any point at which the water is flowing through it with velocity U, the radial and tangential components of that velocity will be $U.\sin\theta$ and

$U.\cos\theta$. The first of these is always the velocity of flow u, whether the pipe be fixed or whether it be attached to the revolving wheel.

Fig. 202.



In the fixed pipe the second is the velocity of whirl which we may call v', and for motion along a fixed guide blade before entering the wheel,

$$v' = u\,.\cot\theta.$$

In the moving pipe, however, it is the velocity of whirl relatively to the revolving wheel, and this is $V - v$, therefore

$$V - v = u\,.\cot\theta.$$

Case I.—Suppose the vanes of the wheel are radial at the circumference where the water enters. In order that the water may have no velocity of whirl relatively to the wheel on entrance, and that the water may enter without shock, we must then have $v' = V_1$, that is, the value of θ for the fixed guide blade at entrance should be given by

$$\tan\theta_1 = \frac{u}{V_1}.$$

Further, the water should be discharged without whirl, that is, v should be zero at the circumference where the water leaves the wheel, hence

$$\tan\theta_2 = \frac{u}{V_2}.$$

The inclination of the fixed blades at entrance, and of the vanes at entrance and exit is thus determined. At intermediate points it would be desirable that it should so vary that vr should diminish uniformly from entrance to exit in order that the action of all parts of the vane upon the water may be the same. This condition would completely determine the form of the vane, but, in practice, any "fair" form would be a sufficient approximation.

Supposing the vanes thus designed $v_1 = V_1$, and the speed of periphery of the wheel, at the circumference where the water enters, is then given by the simple formula

$$V_1 = \sqrt{gh},$$

a value which applies to the outer periphery of an inward-flow and the inner periphery of an outward-flow turbine.

The flow through the wheel instead of being radial may be parallel to the axis, and in this case the formula is still applicable if V_1 be taken as the mean of the speeds of the outer and inner peripheries.

Case II.—In drawing Figs. 201*a*, 201*b*, it has been supposed that the vanes are radial at entrance, but this restriction is not necessary; they may be supposed inclined at a given angle to the periphery. The speed of periphery of the wheel may then be reduced by a proper choice of the angle (Ex. 11, p. 549).

Many forms of outward-flow turbines exist, of which the best known was invented by Fourneyron, and is commonly known by his name. The inward-flow or vortex turbine was invented by Prof. James Thomson. Parallel flow, often described as Jonval turbines, from the name of the original inventor, are also a common type (compare p. 544). For descriptions and illustrations of these machines the reader is referred to the treatises cited at the end of this chapter.

The efficiency of turbines when working under the best conditions is as much as 80 per cent. Their action will be further investigated in the Appendix.

274. *Turbine Pumps.*—Impulse and reaction machines are always reversible, and every motor may therefore be converted into a pump by reversing the direction of motion of the machine and of the water passing through it. If, for example, in the reaction wheel of Fig. 200 we imagine the wheel to turn in the opposite direction with velocity V, while by suitable means the water is caused to move in the opposite direction with velocity $v - V$, so as to enter the orifices with velocity v, it will flow through the arms to the centre and be delivered up the central pipe. The only difference will be that the lift of the pump will not be so great as the fall in the motor on account of frictional resistances. So, any turbine motor is at once converted into a turbine pump by reversing the direction of its motion and supplying it with water moving with a proper velocity. An inward-flow motor is thus converted into an outward-flow pump, and conversely.

No inward-flow pump appears as yet to have been constructed, though it has occasionally been proposed. The "centrifugal" pump so common in practice is, of course, always an outward-flow machine.

The earliest idea for a centrifugal pump was to employ an inverted Barker's Mill, consisting of a central pipe dipping into water connected with rotating arms placed at the level at which water is to be delivered. This machine, which must be carefully distinguished from the true reversed Barker's Mill mentioned above, operates by suction. Its efficiency, which may be investigated as in Art. 272, is very considerable (Ex. 4, p. 548), but there are obvious practical inconveniences which prevent its use in ordinary cases. The actual centrifugal pump is a reversed inward-flow turbine.

All that was said about motors in the last article applies equally well to pumps, and the same formula

$$Vv = gh$$

applies, V being the speed of rotation of the wheel, now usually called the "fan" and v that of the water, both reckoned at the outer periphery where the water issues. The quantity h is now the *gross* lift found by adding to the actual lift, the head corresponding to the velocity of delivery, the friction of the ascending main, the friction of the suction pipe and passages through the wheel into the main, and the losses by shock at entrance and exit.

A pump, however, works under different conditions from a motor, and corresponding differences are necessary in its design. The energy of a fall can, by proper arrangements, be readily converted, wholly or partially, into the kinetic form without any serious loss by frictional resistances, and the water can, therefore, be delivered to the wheel with a great velocity of whirl to be afterwards reduced by the action of the wheel to zero. When such a motor is reversed, the water enters without any velocity of whirl, and leaves with a velocity, the moment of momentum corresponding to which represents the couple by which the wheel is driven. To carry out the reversal exactly, this velocity ought to be reduced to as small an amount as possible in the act of lifting. Now the reduction of a velocity without loss of head is by no means easy to accomplish, and (see Appendix) always requires some special arrangement.

In Thomson's inward-flow turbine, when reversed, the water is discharged with a velocity of whirl which is equal to the speed of periphery V, and given by the formula

$$V = \sqrt{gh}.$$

The corresponding kinetic energy represents at least half the power required to drive the pump, and if it be wasted, as was the case in some of the earlier centrifugal pumps constructed with radial vanes, the efficiency is necessarily less than ·5, and in practice will be at most ·3. One method of avoiding this loss is to cause the wheel to revolve in a large "vortex chamber," at least double the diameter of the wheel from the outer circumference of which the ascending main proceeds. The water before entering the main forms a free vortex, and its velocity is reduced one-half as it spreads radially from the wheel; three-fourths the kinetic energy is thus converted into the pressure form. The speed of periphery in pumps of this class is that due to *half* the *gross* lift. Assuming their efficiency as ·65, the gross lift is found by an addition of 50 per cent. to the actual lift.

Many examples of vortex-chamber pumps exist, but they are comparatively rare, probably because the machine is more cumbrous; in practice a different method of reducing the velocity of discharge is generally employed. Instead of the vanes being radial at the outer periphery, they are curved back so as to cut it at an angle θ, given by the formula (p. 536)

$$V - v = u \cdot \cot \theta,$$

the velocity of whirl is thus reduced from V to kV, where k is a fraction, and the speed is then

$$V = \sqrt{g\frac{h}{k}}.$$

If the efficiency be supposed ·65, and the velocity be reduced in this way to one-half its original value, this gives about $10\sqrt{H}$ for the speed where H is the actual lift. The greater speed is a cause of increased friction as compared with the vortex-chamber arrangement, but on the other hand the friction of the vortex is by no means inconsiderable, and this is so much subtracted from the useful work done.

The centrifugal pump in this form was introduced by Mr. Appold in 1851, and is commonly known by his name.

Another important point in which the pump differs from the motor is in the guidance of the water outside the wheel. In the motor there are four or more fixed blades which guide the water to the wheel; but in the pump the outer surface of the chamber surrounding the wheel forms a single spiral guide blade. The whole of the water discharged from the wheel rotates in the same direction, and in order that the discharge may be uniform at all points of the circumference the sectional area of this chamber should increase uniformly from zero at one side of the ascending main to a maximum value at the other side. In some of the earlier designs of centrifugal pumps it was supposed that some of the water would rotate one way, and some the other, but in fact all the discharged water rotates with the wheel, and the passage should be so designed as to permit this, the area corresponding to the proposed velocity of whirl. There are, however, examples in which the water is discharged in all directions into an annular casing, and guided by spiral blades parallel to the axis of rotation. (See a paper by Mr. Thomson, *Min. Proc. Inst. C.E.*, vol. 32.)

Centrifugal pumps work to best advantage only at the particular lift for which they are designed. When employed for variable lifts, as is constantly the case in practice, their efficiency is much reduced and does not exceed ·5. It is often much less.

275. *Approximate investigation of the Efficiency of a Centrifugal Pump.*

(1) Few centrifugal pumps utilize more than a small fraction of the energy of motion possessed by the water at exit from the wheel, and an investigation of their efficiency on the supposition that this energy is wholly wasted is therefore of considerable interest.

Let h_0 be the actual lift, and let all frictional losses except that specified be neglected; then, if u be the velocity of flow, and v the velocity of whirl at exit, the loss of head is $(u^2+v^2)/2g$, and the gross lift is

$$h = h_0 + \frac{u^2+v^2}{2g}.$$

Substituting this value of h in the formula for V, and replacing u by its value $(V-v)\tan\theta$, we obtain

$$Vv = gh = gh_0 + \frac{v^2 + (V-v)^2\tan^2\theta}{2}.$$

Adding $\frac{1}{2}V^2$ to each side, and re-arranging the terms,

$$\tfrac{1}{2}V^2 = gh_0 + \tfrac{1}{2}(V-v)^2 . \sec^2\theta,$$

a formula from which we find

$$\text{Efficiency} = \frac{h_0}{h} = \frac{V^2 - (V-v)^2\sec^2\theta}{2Vv},$$

$$= \sec^2\theta\left(1 - \tfrac{1}{2}\left(\frac{v}{V} + \frac{V}{v}\sin^2\theta\right)\right).$$

This result shows that the efficiency is greatest when

$$v = V . \sin\theta;$$

and on substitution we find

$$\text{Maximum efficiency} = \sec^2\theta(1-\sin\theta) = \frac{1}{1+\sin\theta}.$$

The speed of maximum efficiency is found from the equation

$$\tfrac{1}{2}V_0^2 = gh_0 + \tfrac{1}{2}V_0^2\sec^2\theta . (1-\sin\theta),$$

which gives

$$V_0^2 = (1+\operatorname{cosec}\theta)gh_0.$$

The proper velocity of flow is

$$u_0 = V_0 . \tan\theta(1+\sin\theta),$$

and the area of flow through the periphery of the wheel should be made to give this velocity with the intended delivery.

At any other speed V the velocity of flow will be given by

$$u^2 = (V^2 - 2gh_0)\sin^2\theta,$$

and the efficiency may be found by the preceding formula.

(2) In the preceding investigation it is supposed that the whole of the energy of motion on exit from the wheel is wasted, and it follows as a necessary consequence that the efficiency is much greater when the

vanes are curved backward than when they are radial. This conclusion has been verified experimentally, and till recently has been very generally accepted, yet there can be no doubt that so great a waste is not a necessity in a pump with radial vanes but is in great measure a consequence of improper design of the chamber surrounding the wheel and its connection with the delivery pipe. Let A be the sectional area of the chamber at a point the angular distance of which from the point of junction with the delivery pipe is ϕ: then if there be no whirlpool chamber

$$A = A_0 \cdot \frac{\phi}{2\pi},$$

the chamber consisting of a simple spiral passage the section of which increases uniformly from zero to its maximum value A_0. The fan will now discharge uniformly at all points of its periphery with a radial velocity u connected with v the velocity of entrance to the delivery pipe by the equation

$$A_0 v = Su,$$

where S is the area of flow. The junction with the delivery pipe must be knife-edged next the wheel and form a continuation of the spiral passage gradually expanding till the full size of the pipe is reached.

The area A_0 will generally be such that v is less than the speed of periphery V at any ordinary speed of working, and the water issuing from the radial vanes with velocity of flow u and of whirl V will intermingle with water which has simply the smaller velocity v with which the water moves through the spiral passage. The consequent loss of head may be taken as $\{u^2 + (V - v)^2\}/2g$. The other resistances for the purposes of this calculation are taken as due, (1) to surface friction of pipes and passages, (2) to losses at entrance to the wheel, and (3) to the gradual enlargement after entering the delivery pipe. By suitably curving the vanes at the inner periphery (2) may be reduced and made to depend only on the velocity of flow u, which is proportional to v. We have therefore

$$V^2 = gh = gh_0 + \frac{\beta v^2 + (V - v)^2}{2},$$

where β is a co-efficient, whence we find

$$\text{Efficiency} = \frac{h_0}{h} = 1 - \frac{\beta v^2 + (V - v)^2}{2V^2}.$$

This is greatest when $v = \frac{V}{1+\beta}$, $\therefore V = \sqrt{\frac{1+\beta}{1+\frac{1}{2}\beta} \cdot gh_0}$, and the corresponding

$$\text{Maximum efficiency} = 1 - \frac{1}{2} \cdot \frac{\beta}{1+\beta},$$

By increasing the size of the suction and delivery pipes and the area of outflow S, the resistance of these pipes and of the passage through the wheel can be reduced to a small amount, while the part of the coefficient β which measures the friction of the spiral passage outside the wheel can hardly exceed ·35, or at most ·4. Little is known as to the loss in a gradual enlargement, but in many cases, as for example in a trumpet-shaped orifice, it is small: in any case it can only be a fraction of that due to a sudden enlargement. If then the precautions mentioned above are taken in designing the chamber, the value of β will not exceed 2 and may probably be capable of being reduced to unity, giving an efficiency ranging from ·66 to ·75. From one-third to one-half the energy of motion on exit from the wheel is now utilized. Of the waste-work from two-thirds to one-half is due to the sudden change of velocity on entrance to the spiral passage and the rest to surface friction.

Curving back the vanes has the effect of reducing the velocity of whirl *only* when the area S is small enough to increase the velocity of flow u to an amount which causes a considerable loss of head on passing through the wheel. The speed of periphery is also increased, and for these reasons it is probable that, especially at high lifts, a properly designed pump with radial vanes is more efficient. If a whirlpool chamber be added the spiral passage now forms part of the chamber, and care must still be taken that at the junction with the delivery pipe no obstruction is offered to the rotation of the water. The influence of the form of the vanes is further discussed in the Appendix.

When a centrifugal pump is started the fan is filled with water which, in the first instance, rotates as a solid mass with the fan. If the radius of the inner periphery be m times that of the outer where m is a fraction, it will not commence to deliver water till the speed reaches the value

$$V^2(1-m^2)=2gh_0.$$

But when once started, the speed may be reduced below this value without stopping the delivery, provided that some of the energy of motion on exit from the wheel is utilized. This has been observed to occur in practice, and it will serve as a test of efficiency.

276. *Limitation of Diameter of Wheel.*—For a given fall in a motor or lift in a pump the diameter of wheel in a turbine is in many cases limited, because some of the frictional resistances increase rapidly with the diameter.

Let u as usual be the velocity of flow, d the diameter, b the inside effective breadth of the wheel at exit after allowing for the thickness of the vanes; then the delivery in cubic feet per second is

$$Q = Su = ub\pi d.$$

Now, if the breadth b be too small as compared with the diameter, the surface friction of the passages through the wheel will be too great, as in the case of a pipe the diameter of which is too small for the intended delivery. Thus b is proportional to d: also, we have seen that u in most of these machines is proportional to V_1, that is to $\sqrt{h}$, and it follows therefore, by substitution for b and u that

$$Q = Cd^2\sqrt{h},$$

where C is a co-efficient.

If the wheel be wholly immersed in the water the surface friction (Ex. 8, p. 548) is relatively increased by increasing the diameter. On investigating how great the diameter may be without too great a loss we arrive at the same formula.

Where it is of importance to have as large a diameter as possible to reduce the number of revolutions per minute, the diameter of wheel in a pump or a turbine is therefore found by the formula

$$d = \sqrt{\frac{G}{c\sqrt{h}}}.$$

If G be the delivery in gallons per minute, h the actual fall in feet, d the external diameter also in feet, the value of c for an outward-flow turbine is about 200.

This formula is frequently used in the case of a centrifugal pump with a value of the constant not differing greatly from that just given: but it must be understood that it is only suitable for a pump in which the velocity of whirl at exit from the wheel is reduced by curving back the vanes and increasing the velocity of flow as already described. When the vanes are radial the velocity of flow may be reduced at pleasure. If now D be the diameter of the suction pipe determined for a given delivery Q in the usual way (pp. 488, 495), u_0 the velocity in this pipe,

$$Q = u \, . \, \pi bd = u_0 \, . \, \frac{\pi}{4} D^2.$$

If u be proportional to u_0, and, as before, b proportional to d, this shows that d should be proportional to D. Assuming $u = u_0$ and $b = \frac{1}{8}d$ the ratio d/D is 2: but a somewhat larger value is probably desirable, at least for high lifts. It should be observed that in this case the diameter of fan does not depend on the lift, but only on the delivery.

Centrifugal pumps cannot generally be employed for very high lifts, partly because it becomes increasingly difficult to utilize the energy of motion on exit from the wheel, and partly on account of disc friction. The fan rotates much faster than the wheel of a turbine, and the disc friction is consequently much greater.

277. *Impulse Wheels.*—The formula

$$V_1^2 = gh,$$

which gives the speed of a turbine wheel in terms of the effective head, when the vanes are radial at entry, also gives the velocity of whirl at entrance, and therefore shows that, of the whole head employed in driving the wheel and producing the velocity of flow, one-half operates by impulse. When the vanes are not radial (p. 537), a certain fraction, depending on the inclination and sometimes less than one-half, operates by impulse. The remainder operates by pressure, and turbines of this class are consequently not simple impulse, but impulse-pressure machines. It is necessary therefore that the wheel should revolve in a casing, and that the passages should be always completely filled with water. The diameter of wheel is then limited as explained in the last article, and for a small supply of water and a high fall the number of revolutions per minute becomes abnormally great. This consideration and the necessity o adaptation to a variable supply of water render it often advisable to resort to a machine in which the passages are actually or virtually open to the atmosphere. The whole of the energy of the fall is then converted into the kinetic form before reaching the wheel, and consequently operates wholly by impulse.

A wheel of this kind approaches closely in principle to the Poncelet water wheel mentioned in Art. 270, but is often still described as a "turbine," because the water is guided by fixed blades before reaching the wheel. A common example is a Girard turbine with axial flow. The flow of the water is here parallel to the axis of the wheel, spiral guide blades being ranged round the circumference of a cylinder like the threads of a screw in order to give the necessary whirl to the water before entrance. The wheel is provided with a similar set of spiral vanes curved in the opposite direction, which reduce it to rest as it passes through. In the French *roue à poire* the wheel is conical, the water enters at the circumference, and, guided by spiral vanes, descends to the apex where it is discharged.

Impulse wheels, which are sometimes described as "Girard" turbines even when the flow is radial, appear not to be so efficient as a pressure-turbine working at its best speed. The Pelton wheel (p. 529) may be taken as an exception. On the other hand their efficiency is very little diminished by a considerable falling off in the supply of water, and this advantage is so great that they are much employed in cases where the supply of water is subject to variation.

The propulsion of ships is effected by machines, which are virtually impulse wheels reversed. The subject is outside the limits of this work, but some information respecting it will be found in the Appendix.

278. *Stress due to Rotation.*—Machines of the class considered in this section often work under a very considerable head, and the speed of periphery determined by the formula

$$V = k\sqrt{2gh}$$

becomes very great. Thus in the example of the Pelton wheel given on p. 529, the head is 2100 feet, and the speed of periphery 180 f.s. If λ be the stress on the rim of the wheel due to centrifugal action, reckoned in feet of material, as on p. 81 and elsewhere in this treatise, the formula for the centrifugal stress produced given on p. 288, may be put in the form

$$V^2 = g\lambda,$$

and thus λ is simply proportional to the head. In the example just mentioned, this gives a value of λ of about 1000 feet. In fans and other machines working with elastic fluids, the head is often much greater. The greatest permissible value of λ is given, subject to the qualifications there stated, in the Table on p. 455.

SECTION III.—MACHINES IN GENERAL.

279. *Equation of Steady Flow in a Rotating Casing.*—When water moves in a pipe or passage of any kind rigidly attached to a wheel or drum rotating about a fixed axis: a general equation can be found for steady flow, as in the case of a fixed pipe considered in the last chapter.

Referring to Fig. 202, p. 536, let the pipe there represented be fixed in any position to a rotating wheel, and consider a point in the pipe, the velocity of which is

$$V = 2\pi n r,$$

where n is the number of revolutions per second, and r is the distance from the axis. Let the velocity of flow through the pipe at this point be U, and resolve this velocity into $U.\cos\theta$ along the periphery of the circle described by the point, and $U.\sin\theta$ perpendicular to this periphery. In the question considered on the page cited this second component was radial, the pipe lying in a plane perpendicular to the axis of rotation. We now take the general case in which the pipe is inclined to this plane, and $U . \sin\theta$ is consequently the resultant of a radial velocity and an axial velocity, each independent of the velocity of rotation. *As before, the component $U . \cos\theta$ is the velocity of whirl relatively to the rotating pipe and*

$$v = V - U . \cos\theta$$

will be the absolute velocity of whirl.

Let Q be the flow per second, and consider two points in the pipe specified by the suffixes 2 and 1. Then if L be the couple applied to the part of the pipe between these points

$$L = \frac{wQ}{g}(r_2 v_2 - r_1 v_1),$$

and since V/r is the same at all points, being the angular velocity $2\pi n$,

$$\text{Energy exerted per second} = \frac{wQ}{g}(v_2 V_2 - v_1 V_1).$$

This is the amount of energy exerted per second on the water as it passes from the point 2 to the point 1, and is employed in increasing the head.

Now the absolute velocity (U') is given by the equation

$$\begin{aligned} U'^2 &= v^2 + U^2 . \sin^2\theta \\ &= V^2 - 2VU . \cos\theta + U^2 \\ &= U^2 - V^2 + 2vV. \end{aligned}$$

Hence if as usual p/w be the pressure-head and z the elevation, the change of head will be

$$h_2 - h_1 = \frac{p_2 - p_1}{w} + z_2 - z_1 + \left[\frac{U^2 - V^2 + 2vV}{2g}\right]_1^2.$$

Multiplying this by wQ and equating it to the energy exerted the terms containing vV disappear, and omitting the suffixes

$$\frac{p}{w} + z + \frac{U^2 - V^2}{2g} = \text{Constant},$$

which is the general equation of steady flow. The equation may also be written

$$\frac{p}{w} + z + \frac{U^2}{2g} = \text{Constant} + \frac{V^2}{2g},$$

showing that the total head in the pipe is increased in consequence of the rotation by the quantity $V^2/2g$, which is the so-called "head due to centrifugal force."

If H_1 be the head outside the casing before the water enters, then from the value of the absolute velocity given above it appears that

$$H_1 = \frac{p_1}{w} + z_1 + \frac{U_1^2 - V_1^2}{2g} + \frac{v_1 V_1}{g},$$

where the suffix 1 refers to the point of entrance. Hence by substitution

$$\frac{p}{w}+z+\frac{U^2-V^2}{2g}=H_1-\frac{v_1V_1}{g}.$$

Similarly if H_2 be the head after leaving the casing,

$$\frac{p}{w}+z+\frac{U^2-V^2}{2g}=H_2-\frac{v_2V_2}{g},$$

where the suffix 2 refers to the point of exit. When there is no loss of head by hydraulic resistances within the casing

$$H_1-H_2=\frac{v_1V_1-v_2V_2}{g},$$

which, as before, gives the head employed in driving the wheel in a motor; or, when negative, the increased head created by the external forces driving the wheel in a pump.

The losses of head by hydraulic resistance are determined directly from the velocity U just as if the casing were at rest.

In questions relating to turbines and centrifugal pumps the general equation here given is often very useful.

279A. *Similar Hydraulic Machines.* If two machines, whether motors or pumps, are imagined differing only in scale, the heads of water or lifts as the case may be being in the same proportion, the velocities for a given efficiency in the absence of friction by the general principle of similar motions (p. 182) will be as the square roots of their linear dimensions. The same will be true approximately when hydraulic resistances are taken into account. Taking for example the formula

$$h'=4f\cdot\frac{l}{d}\cdot\frac{v^2}{g},$$

which gives the loss of head in a pipe, we see at once that the losses of head by pipe friction in the two cases compared will be the same fraction of the actual head or lift, and therefore the efficiencies will be the same; and the same argument applies to all the hydraulic resistances. The efficiency on the small scale, however, will be relatively diminished because the value of $4f$ is greater in the small scale motion just as the skin friction of a model is relatively greater than that of a vessel.

The delivery in similar machines at corresponding speeds varies as $h^{\frac{5}{2}}$ and the power as $h^{\frac{7}{2}}$, where h is the linear dimension or head.

In comparing ventilating or blowing fans with centrifugal pumps this principle must be borne in mind. Unless the fan be of great size its action is only comparable with that of a centrifugal pump of great lift and small delivery.

EXAMPLES.

1. In a reaction wheel the speed of maximum efficiency is that due to the head. In what ratio must the resistance be diminished to work at four-thirds this speed, and what will then be the efficiency? Obtain similar results when the speed is diminished to three-fourths its original amount.

Ans. Efficiency = ·63 or ·64.
Ratio = ·84 or 1·14.

2. Water is delivered to an outward-flow turbine, at a radius of 2 feet, with a velocity of whirl of 20 feet per second, and issues from it in the reverse direction at a radius of 4 feet, with a velocity of 10 feet per second. The speed of periphery at entrance is 20 feet per second, find the head equivalent to the work done in driving the wheel. *Ans.* 24·22 feet.

3. In a Fourneyron turbine the internal diameter of the wheel is $9\frac{1}{2}$ inches, and the outside diameter 14 inches. The effective head (p. 535) is estimated at 270 feet; find the number of revolutions per minute. *Ans.* 2,200.

NOTE.—These data are about the same as those of a turbine erected at St. Blasien in the Black Forest.

4. An inverted Barker's Mill (p. 537) is used as a centrifugal pump. If the coefficient of hydraulic resistances referred to the orifices be ·125, show that the speed of maximum efficiency is that due to twice the lift, and find the maximum efficiency. *Ans.* Maximum efficiency = ·75.

5. A centrifugal pump delivers 1,500 gallons per minute. Fan 16 inches diameter. Lift 25 feet. Inclination of vanes at outer periphery to the tangent 30°. Find the breadth at the outer periphery that the velocity of whirl may be reduced one-half, and also the revolutions per minute, assuming the *gross* lift $1\frac{1}{2}$ times the actual lift. *Ans.* Breadth = $\frac{3}{4}$ inch. Revolutions = 700.

6. In the last question find the proper sectional area of the chamber surrounding the fan (p. 541) for the proposed delivery and lift. Also examine the working of the pump at a lift of 15 feet. *Ans.* 24 sq. inches.

7. A jet of water moving with a given velocity, strikes a plane perpendicularly. Find how much of the energy of the jet is utilized in driving the plane with given speed. Determine the speed of the plane for maximum efficiency, and the value of the maximum efficiency. *Ans.* Speed of maximum efficiency = one-third that of jet. Maximum efficiency = $\frac{8}{27}$.

8. Assuming the ordinary laws of friction between a fluid and a surface, and supposing that any motion of the fluid due to friction does not affect the question: find the moment of friction (L), and the loss of work per second (U), when a disc of radius a rotates with speed of periphery V.

$$Ans.\ L = f.\frac{2\pi}{5}.a^3V^2;\ U = f.\frac{2\pi}{5}.a^2.V^3.$$

9. If the rotating disc in question 8 be surrounded by a free vortex of double its diameter, show that the loss by friction of the vortex on the flat sides of the vortex chamber is $2\frac{1}{2}$ times the loss by friction of the disc.

10. Show that the loss of head by surface friction in the spiral passage (p. 541) of a centrifugal pump is the same as in a passage of uniform transverse section of the same area of length about $2\frac{1}{2}$ times the diameter of the fan.

11. The vanes of a turbine wheel are inclined at an angle i to the radius at entrance, the angle being measured in the direction of motion of the wheel. Find the speed of

periphery for no shock at entrance and no whirl at exit. Also find the necessary connection between the angles of the vanes at entrance and exit and the angle of the guide blades. *Ans.* With the notation of Case I., p. 536, and further supposing $r_2/r_1 = m$.

$$V_1^2 = gh(1 - \tan i \tan \theta_1)$$
$$\cot \theta_2 = m(\cot \theta_1 - \tan i).$$

REFERENCES.

The subject of hydraulic machines is very extensive, and it is impossible within the limits of a single chapter to do more than give a general idea of their working. For descriptive details and illustrations the reader is referred, amongst other works, to

GLYNN. *Power of Water.* Weales' Series.

FAIRBAIRN. *Millwork and Machinery.* Longman.

COLYER. *Water-Pressure Machinery.* Spon.

BARROW. *Hydraulic Manual.* Printed by authority of the Lords Commissioners of the Admiralty.

As a text-book on turbines and pressure engines may be mentioned

BODMER. *Hydraulic Motors.* Whittaker. 1889.

CHAPTER XXI.

ELASTIC FLUIDS.

280. *Preliminary Remarks.*—An elastic fluid under pressure is a source of energy which, like a head of water in hydraulics (p. 470), may be employed in doing work of various kinds by a machine, or simply in transferring the fluid from one place to another. In hydraulics we commence with the case of simple transfer, but the density of gases is so low that, unless the differences of pressure considered are very small, the inertia and frictional resistances of the fluid employed in a pneumatic machine have little influence: it is the elastic force which is the principal thing to be considered. In studying pneumatics, therefore, we commence with machines working under considerable differences of pressure and then pass on to consider the flow of gases through pipes and orifices together with those machines in which the inertia and frictional resistances of the fluid cannot be neglected.

SECTION I.—MACHINES IN GENERAL.

281. *Expansive Energy.*—The special characteristic of an elastic fluid is its power of indefinite expansion as the external pressure is diminished. While expanding, it exerts energy of which the fluid itself is, in the first instance, the source, whereas the energy exerted by an incompressible fluid is transmitted from some other source. Expansive energy is utilized by enclosing the fluid in a chamber which alternately expands and contracts; the common case being that of a cylinder and piston.

Fig. 203 represents in skeleton a cylinder and piston enclosing a mass of expanding fluid. Taking a base line aa' to represent the stroke, set up ordinates to represent the total pressure S on the piston in each position; a curve $1Q2$ drawn through the extremities of these ordinates is the Expansion Curve. Reasoning as in Art. 90, p. 182, the area of this curve represents the energy exerted as the piston moves from the position 1, where the expansion commences, to the position 2, where it terminates. One common case was considered in the article cited, namely, that in which the expansion curve is a common hyperbola. This is included in the more general supposition,

$$Sy^n = S_1y_1^n = S_2y_2^n,$$

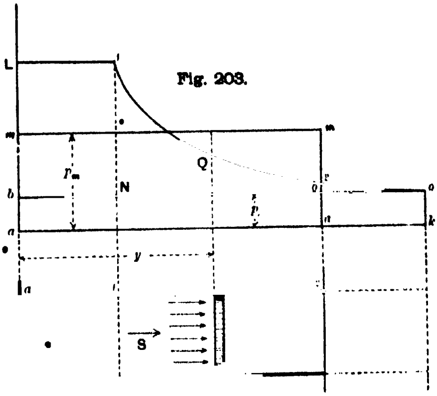


Fig. 203.

where y is the distance of the piston from the end a of its stroke, and n is an index which, for the particular case of the hyperbola, is unity. Most cases common in practice may be dealt with by ascribing a proper value to n; for air it ranges between 1 and 1·4, and for steam it is roughly approximately unity. The suffixes indicate the points at which the expansion commences and terminates.

If, now, E be the energy exerted during expansion,

$$E = \int_{y_1}^{y_2} S\,dy = S_1y_1^n\int_{y_1}^{y_2} y^{-n}\,dy = S_1y_1^n \cdot \frac{y_2^{1-n} - y_1^{1-n}}{1-n}.$$

This formula may be written in the simpler form

$$E = \frac{S_1y_1 - S_2y_2}{n-1},$$

in applying which, the terminal pressure S_2 is supposed to have been previously found from the equation

$$S_2 = S_1 \cdot \left(\frac{y_1}{y_2}\right)^n.$$

It is, for brevity, convenient to write

$$y_2 = ry_1; \quad S_2 = \mu \cdot S_1,$$

where r is a number known as the "ratio of expansion," and μ is a fraction which may be described as the "pressure fraction" connected with r by the formula[*]

$$\mu=\left(\frac{1}{r}\right)^n.$$

The formula for E then takes the simpler form

$$E=S_1y_1\cdot\frac{1-\mu r}{n-1}=P_1V_1\cdot\frac{1-\mu r}{n-1}.$$

The product μr employed for simplicity in this and other formulæ which follow is given by the equation

$$\mu r=\left(\frac{1}{r}\right)^{n-1}=\mu^{\frac{n-1}{n}}.$$

If $n=1$ the formula fails and is replaced by

$$E=S_1y_1\log_e r=P_1V_1\log_e r.$$

The value of E is here, in the first instance, expressed in terms of the total pressure on the piston, but, reasoning as in Art. 264, p. 513, we may, if P be the pressure, replace S by PA, and Ay by V, so that S_1y_1 is replaced by P_1V_1. In "rotatory" engines and pumps the expanding chamber is not a simple cylinder and piston, but is formed from a turning pair. Or, more generally, the chamber pair may be formed from any two links of a kinematic chain which it may be convenient to select for the purpose. In its last form the formula is applicable in every case. If the expansion curve be not given in the form supposed, the value of E is determined graphically by measuring the area of the curve, in doing which, when the chamber is not a simple cylinder, the base of the diagram must represent the volume swept out by the chamber pair, and the ordinates the pressures per unit of area.

282. *Transmitted Energy.* The energy exerted by an elastic fluid consists not merely of that derived from the expansive power of the fluid pressing against the piston, but also of that which is transmitted in the same way as would be the case if it were incompressible. The fluid is supplied from a reservoir, which may either be an accumulator in which it is stored by the action of pumps, or a vessel in which, by the action of heat, it is generated or its elasticity increased. In any case, so long as the cylinder remains in communication with the reservoir the fluid enters at nearly constant pressure, and energy is exerted on the piston just as in the water-pressure engine. During this period of admission the energy exerted is

$$L=S_1y_1=P_1V_1=144p_1V_1,$$

---

[*] The symbol x was used for the pressure-fraction in former editions; the change to μ has been made to avoid confusion, x being used for the "dryness-fraction" of steam later on in this chapter.

the notation being as in the last article. It is usually convenient to express volumes in cubic feet and pressures (p) in lbs. per square inch. We must then replace P by $144p$.

The whole energy exerted on the piston is now

$$U = L + E = L \cdot \frac{n - \mu r}{n - 1},$$

which for the case of the hyperbola becomes

$$U = L(1 + \log_e r).$$

The mean pressure on the piston is conveniently denoted by p_m, and is represented in the figure by the ordinate of the line mm so drawn that the area of the rectangle ma is equal to the area of the diagram. Its value is given by the formulæ

$$p_m = \frac{p_1}{r} \cdot \frac{n - \mu r}{n - 1}; \quad p_m = p_1 \cdot \frac{1 + \log_e r}{r}.$$

A reservoir filled with an elastic fluid at high pressure is an accumulator, the absolute amount of energy stored in which is the expansive energy or the total energy according as the pressure is not, or is, maintained by the addition of fresh fluid in place of that discharged, the expansion being supposed indefinite in either case. With the law of expansion already supposed, when n is greater than unity, μr vanishes when the expansion curve is prolonged indefinitely. The total absolute energy is then

$$U_1 = P_1 V_1 \cdot \frac{n}{n - 1},$$

where V_1 is the whole volume of fluid considered. When n is not greater than unity, U_1 is infinite.

283. *Available Energy.*—Of the whole amount of energy thus calculated only a part is *available* for useful purposes, because in practice there is always a "back" pressure P_0 on the working piston, or, more generally, on the sides of the chamber in which the fluid is enclosed. In overcoming this, the work $P_0 r V_1$ is done, and nothing is gained by prolonging the expansion beyond the point 0 at which the terminal pressure P_2 has fallen to P_0. The corresponding ratio r_0 is given by the formula

$$\log r_0 = \frac{1}{n} \log\left(\frac{1}{\mu_0}\right) = \frac{\log P_1 - \log P_0}{n}.$$

The available energy is found by writing $r = r_0$, $\mu = \mu_0$ in the value of U, and subtracting $P_0 r_0 V_1 = P_1 V_1 \mu_0 r_0$. This result is

$$\frac{n}{n - 1} \cdot P_1 V_1 (1 - \mu_0 r_0) = U_1 - U_0,$$

being the difference of the values of U when the expansion commences at 1 and at 0. It is always finite, and is graphically represented by the area $L10b$, shaded in the annexed figure.

In the transmission and storage of energy by elastic fluids this quantity plays the same part as the "pressure-head" in hydraulics, o which indeed it reduces if n be supposed very great, r_0 unity, and

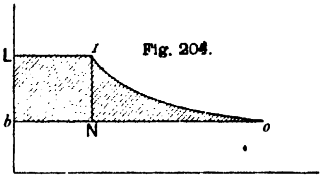


Fig. 204.

V_1 the volume of a lb. of water. It is the energy of a given quantity of fluid due to a given difference of pressure, for which, as before, the term "head" may be used when the quantity considered is 1 lb.

Two cases may now be mentioned which are of special importance.

(1) Let the reservoir contain air at pressure P reckoned in *atmospheres* of 14·7 lbs. per sq. inch, or 2116 lbs. per sq. ft., and let $n = 1·4$, then

$$U_1 - U_0 = 3·5 \times 2116(P_1V_1 - P_0V_0),$$

from which we find, writing $P_1 = P$, $P_0 = 1$, and substituting for V_0

$$\text{Available energy} = 7400(P - P^{\frac{5}{7}})V,$$

where V is the volume of the weight of air considered.

(2) Let $n = 1$ instead of 1·4, then

$$U_1 - U_0 = 2116P_1V_1 \log_e r_0,$$

which gives

$$\text{Available Energy} = 2116V . P \log_e P.$$

In either case by putting $V = 1$ we get the available energy per cubic foot of compressed air, which, it should be observed, depends solely on the pressure.

The available energy is here calculated on the supposition that the reservoir is kept constantly full. When the reservoir is not kept full the only available energy is the expansive energy, less the work done in overcoming P_0 through the volume $V_0 - V_1$. This is graphically represented by the curvilinear triangle $N01$ in Figs. 203 or 204, and is most conveniently given by the formula

$$\text{Available Energy} = U_1 - U_0 - (P_1 - P_0)V_1.$$

284. *Cycle of Mechanical Operations in a Pneumatic Motor—Mechanical Efficiency.*—Motors operating by the pressure of an elastic fluid may be described generally as Pneumatic Motors. They are either supplied from an accumulator, as in hydraulic motors of the same class, or they may be heat-engines serving as the means by which heat energy is utilized. In either case the mechanism of the motor is the same, and consists of a chamber which expands to admit the fluid and contracts to discharge it, with a proper kinematic chain for utilizing the motion of the chamber pair.

In water-pressure engines the contraction to expel the water from the chamber is not considered, because all pressures are reckoned above the atmosphere, and the pressure in the accumulator is so great that small differences of pressure may be disregarded. With elastic fluids it is commonly different: the "exhaust" of the chamber must be taken into account.

Returning to Fig. 203, suppose that the piston has reached the end of its stroke, the cylinder is then filled with fluid of a certain pressure p_2 which may be supposed known. Let now a valve be opened allowing the cylinder to communicate with the atmosphere, or with a reservoir containing fluid at a lower pressure p_0. The fluid in the cylinder then rushes out into the reservoir, and the pressure in the cylinder speedily subsides to p_0; the fluid expands in this process, but its expansive energy is wasted in producing useless motions in the air which afterwards subside by friction. After subsidence let the piston be moved back by an external force applied to it which supplies the energy necessary to overcome the "back" pressure p_0. The fluid is discharged from the chamber, and so long as the communication with the exhaust reservoir is open the pressure remains constantly p_0. We represent this on the diagram by drawing a horizontal line bb_0 the ordinate of which is p_0. The work done in overcoming back pressure is $144p_0V_2$ and is represented on the diagram by the rectangle ba; this is so much subtracted from the energy exerted by the motor.

Thus the volume of the chamber goes through a cycle of changes alternately expanding and contracting. During expansion energy is exerted, the corresponding mean pressure p_m is the "mean forward pressure." During contraction work is done, and the corresponding mean pressure is the "mean back pressure." The difference between the two is the "mean effective pressure" which measures the useful work done, as shown by the equation

$$\text{Useful work} = (p_m - p_0)144V_2,$$

and is graphically represented by the area of the closed figure $L12bb$.

In most cases the moveable element of the chamber pair divides the chamber into two parts, one of which expands while the other contracts, and conversely: the motor is then described as "double-acting." The force acting on the moving piece is then the difference between the forward pressure in one chamber and the back pressure in the other, and when the stress on the parts of the machine is to be considered this is the effective pressure upon which the stress depends (p. 230). For all other purposes, however, the back pressure is to be taken as just explained.

If the pressure p_1, p_0 in the supply and exhaust reservoirs be given, and also the form of the expansion curve, the only waste of energy in this process arises from incomplete expansion. Imagine the expansion curve prolonged to the point *o* where it meets the back pressure line, and suppose the stroke lengthened so as to reach this point, then additional work would be done by the fluid which would be represented graphically by the area of the curvilinear triangle *2ob*. This area represents energy lost by unbalanced expansion, and to avoid it the expansion must be "complete," that is, the fluid must be allowed to expand till its pressure has fallen to p_0, the pressure, in the exhaust reservoir, a condition seldom fulfilled in practice, because the loss by friction and other causes becomes disproportionately great. Leaving this out of account, a pneumatic motor is capable of exerting only a certain maximum amount of energy, quite irrespectively of the nature of its mechanism, but dependent only on the pressures between which it works and the nature and treatment of the fluid. A motor which reaches this maximum power may be described as *mechanically* perfect, and the ratio of the actual useful work done to the theoretical maximum may be described as the MECHANICAL EFFICIENCY of the motor.

In practice the back pressure is greater than p_0 the pressure in the exhaust reservoir itself, the excess being due to the resistance of the passages connecting it with the cylinder. It depends on the speed of piston, the density and nature of the fluid together with the dimensions and type of the passages. No satisfactory formula has been found for it, but its value must be supposed known in each individual case. In Fig. 205 the ordinate of the horizontal line 33

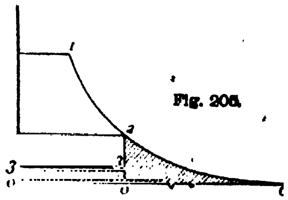


Fig. 205

represents the actual back pressure p_3 while the other lines are the same as in Fig. 203: then the shaded area 03320 represents

W_0 the waste work at exhaust due to incomplete expansion and excess back pressure for a given terminal pressure p_2. It is given by the formula

$$W_0 = \frac{n}{n-1}\left\{P_2V_2 - P_0V_0\right\} - (P_2 - P_3)V_2$$

$$= P_2V_2\left\{\frac{1}{n-1} - \frac{n}{n-1}\left(\frac{p_0}{p_2}\right)^{\frac{n-1}{n}} + \frac{p_3}{p_2}\right\},$$

which becomes if $n = 1$,

$$W_0 = P_2V_2\left\{\log_e.\frac{p_2}{p_0} - \frac{p_2 - p_3}{p_2}\right\}.$$

The waste at exhaust may also frequently be conveniently expressed by an equivalent pressure p upon the piston. Dividing by V_2 we find

$$p = \frac{p_2}{n-1}\left\{1 - n\left(\frac{p_0}{p_2}\right)^{\frac{n-1}{n}}\right\} + p_3$$

$$= p_2\left\{\log_e\frac{p_2}{p_0} - 1\right\} + p_3 \qquad (n = 1).$$

For a given value of the expansion index n this is independent of the initial pressure and of the nature of the fluid. Hence for given values of the pressure-fractions p_0/p_2, p_0/p_3, the fractional loss at exhaust is smaller the higher the initial pressure, a very important principle which will frequently be referred to further on.

The waste at exhaust here considered is, at least in condensing steam engines, the principal *mechanical* loss in pneumatic motors, apart from leakage, but there are also some minor losses by a portion of the fluid being retained in the "clearance" space of the chamber after the exhaust is completed, and by the "wire drawing" due to the resistance of the passages connecting it with the supply reservoir. These, however, are details which cannot be considered here. The theoretical maximum is clearly the same as the store of energy in the fluid used (already found in previous articles), which for brevity will be denoted by $\mathcal{A}$. The consumption of fluid (neglecting clearance and leakage) is one cylinder full, at the terminal pressure, in each stroke.

285. *Pneumatic Pumps.*—A pneumatic like an hydraulic motor may be reversed by applying power to drive it in the reverse direction, and the machine thus obtained is a Pump which takes fluid at a low pressure and compresses it into a reservoir at high pressure.

The cycle in the pump is the same as the cycle in a motor, but the operations take place in reverse order. As the chamber expands fluid is drawn in from the low-pressure reservoir and energy is exerted on the piston by the original "back" pressure: as the chamber contracts the fluid is compressed till it reaches the pressure p_1, when a valve opens and admits it to the high-pressure reservoir. There is, however, this important difference, namely, that the process of unbalanced expansion in the motor cannot be reversed; and therefore, if the pump is to operate on the same weight of fluid, the volume of the working cylinder must be enlarged so that the expansion curve may start from o. If this be supposed, the compression curve will, for the same fluid treated in the same way, be identical with the expansion curve of the motor. If there were no unbalanced expansion the motor would be exactly reversible, and the condition of a motor being mechanically perfect may therefore be described by saying that it must be mechanically reversible. The difference of working of the valves in pumps and motors has already been referred to in Art. 269.

The work done in pumping air into a reservoir is the same apart from resistances as the available energy $U - U_0$ found in a previous article (p. 554), but, for reasons to be explained hereafter, it is generally advisable to express it in terms of the volume of air used at atmospheric pressure, the formula for compression to P atmospheres absolute then becomes

$$\begin{aligned} \text{Work done} &= 7400 V_0 (P^{\frac{2}{7}} - 1) \qquad (n = 1{\cdot}4), \\ &= 2116 V_0 \log_e P \qquad (n = 1), \end{aligned}$$

where V_0 is the volume of atmospheric air consumed, and, as before, the work done per cubic foot depends solely on the pressure. Air pumps are still more frequently employed for the purpose of exhausting a chamber, in which case the atmosphere is the high-pressure reservoir into which the low-pressure air in the chamber is forced. The formula for $U - U_0$ is, with change of sign, directly applicable to this case, V being the volume of low-pressure air and P the pressure expressed as a fraction of an atmosphere. In either case if the pressure in the chamber or reservoir is not maintained constant the formula must be modified as before explained. Examples are given at the end of the chapter.

In all pneumatic motors a pump is required to replace the fluid in the supply reservoir. Unless the motor be a heat engine this pump must be driven by external agency, and the whole process is one of storage, transmission, and distribution of energy, a subject briefly considered further on.

286. *Indicator Diagrams.*—The pressure existing in the chamber of a pneumatic machine may be graphically exhibited by means of an instrument called an Indicator. In steam engines especially its use is indispensable to enable the engineer to study the action of the steam.

PLATE XI.

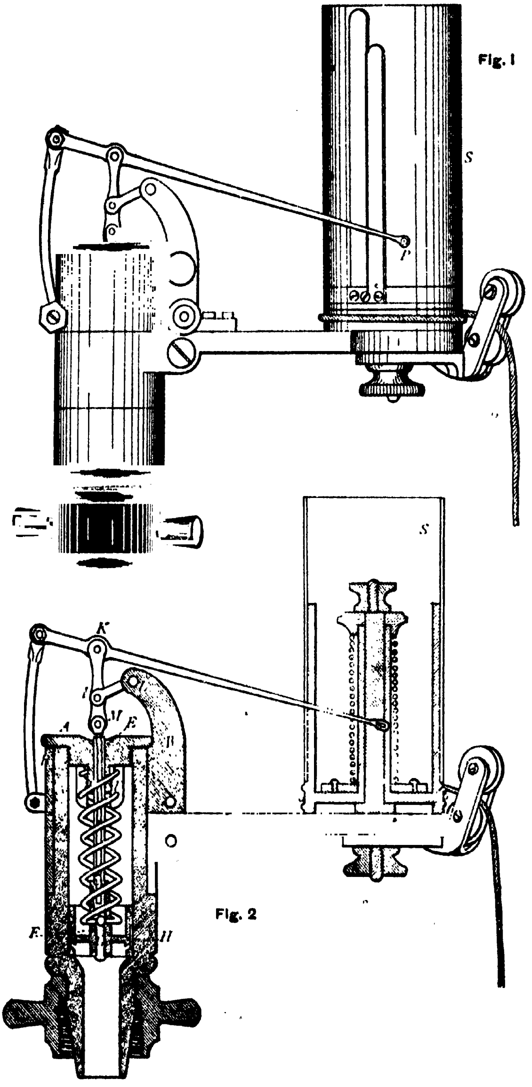


Fig. 1

Fig. 2

To face page 559.

Figs. 1 and 2, Plate XI., show an indicator in elevation and section. *S* is a drum revolving on a vertical axis, *A* is a cylinder communicating with the steam cylinder, the pressure in which is to be measured. *P* is a pencil connected by linkwork with a small piston *H* so as to move with it up or down in a vertical line. The piston is pressed down by a spring which measures the pressure, while the drum, by means of a cord passing over pulleys and connected with the steam piston, revolves through arcs exactly proportional to the spaces traversed by it. A card is folded round the drum, and as the engine moves a curve is traced by the pencil upon it which shows the pressure at each point of the stroke. In practice many precautions are necessary to secure accuracy in the diagram: the more so the higher the speed, because the friction and inertia of the parts of the indicator, together with unequal stretching of the cord and inaccuracy in the reducing motion connecting the drum with the steam piston, may give rise to serious errors. To diminish the effect of inertia the stroke of the indicator piston is made short and multiplied by linkwork.

In the example shown (Crosby's patent) the spring applied to the drum to keep the cord tight has a tension which increases as the drum rotates from rest. This increase compensates for the inertia of the drum, and is said to give a more nearly uniform tension of the cord.

Fig. 206 shows an indicator diagram taken in this way from the high pressure cylinder of a compound engine.

BB is the atmospheric line drawn on the card by the indicator

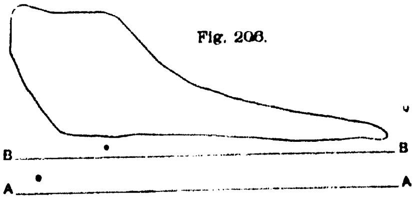


Fig. 206.

pencil when the cylinder communicates with the atmosphere. *AA* is the vacuum line laid down on the diagram at a distance below *BB*, which represents the pressure of the atmosphere, as found by the barometer, reckoned on the scale of pressures. Then on the same scale any pressure shown by the indicator is the absolute pressure when measured from *AA*.

The figure drawn is a closed curve bearing a general resemblance to the diagram (Fig. 203), which was drawn to represent the cycle of operations of a motor. The principal difference is that the corners of the theoretical diagram are rounded off in the actual diagram, an effect principally due to the valves closing gradually instead of instantaneously. Also at the end of the return stroke a certain amount of steam is retained in the cylinder and compressed behind the piston, causing the very considerable rounding off observable at the left-hand corner.

In every case the mean effective pressure may be determined graphically by measuring the area of the diagram and dividing by the length of the stroke. This, with the number of revolutions per 1′ determines the horse-power for an engine of given dimensions, and the consumption of steam in cubic feet per 1′ for each horse-power thus "indicated" can be found. The *weight* of steam used, however, cannot be found without measurement of the feed water used, because the steam always contains an unknown amount of water mixed with it.

287. *Brake Efficiency.*—The indicated horse-power (I.H.P.) determined as in the preceding article, is subject to a deduction consequent on the friction of the mechanism of the engine, and the power actually delivered is the Brake Horse-Power (B.H.P.), which can, at least theoretically, be measured by a suitable dynamometer, and which in small motors actually is frequently measured by a "friction brake" (p. 279). The ratio of the two is the frictional or brake efficiency. The term "mechanical efficiency" is commonly employed with reference to the external (frictional) mechanical waste alone, but the internal mechanical waste considered in Art. 284 may also properly be included in the meaning of the word.

The external waste by friction of mechanism, as will be seen on reference to page 259, may be represented by a pressure f on the piston given by the formula

$$f = el + f_0,$$

where l is the load on the engine reduced to unit of area of the piston and e, f_0 are constants. To the remarks made on the page cited, it may be added that recent researches * show that these constants, though nearly independent of the load, increase with the initial pressure of the steam, that is, they are greater for high ratios of expansion than for low. They also increase with the speed, but no definite law of increase with initial pressure and speed has been discovered. In many types of non-condensing engine the friction is independent of the load, that is, the co-efficient e is the zero and the friction pressure f_0 may reach 3 or 4 lbs. per sq. inch.

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* *Manual of the Steam Engine*, by R. H. Thurston. Part I., second edition, p. 560. Wiley & Sons, New York, 1892.

In any case the pressure f is equivalent to an increase in the back pressure, and the lowest value which the terminal pressure p_2 can have consistently with economy is

$$p_2 = p_3 + f.$$

Hence also the external and the internal mechanical wastes are subject to nearly the same laws. The most important part of the external waste is approximately constant, and may be included, if we please, with the corresponding part (W_0) of the internal waste calculated in Art. 284.

SECTION II.—THERMODYNAMIC MACHINES.

288. *Cycle of Thermal Operations in a Heat Engine.*—So far all that has been said applies equally well to all pneumatic motors, though its most important application may be to the case where the fluid serves as the means whereby mechanical energy is obtained through the agency of Heat. We now go on to consider very briefly the principles which apply especially to heat engines.

In heat engines the pump necessary to replace the fluid in the supply reservoir, or discharge it from the exhaust reservoir, is worked by energy derived from the working cylinder, so that the engine is self-acting. Now, if the condition of the fluid were the same in the pump as it is in the working cylinder, as much energy would be required to drive the pump as is supplied by the motor, or in practice, more; a necessary condition therefore that any useful work should be done is that, by the agency of heat, the condition of the fluid should be changed so that its mean density, while being forced into the supply reservoir, shall be greater than when doing its work in the working cylinder. Hence, the fluid must be heated in the supply reservoir, and cooled in the exhaust reservoir, and therefore in every heat engine, in addition to the cycle of mechanical operations, there is a cycle of thermal operations consisting of an alternate addition and subtraction of heat; the heat in question being supplied by a body of high temperature and abstracted by a body of low temperature.

In non-condensing steam engines the pump is the feed pump which supplies the boiler with the fluid in the state of water; in the boiler heat is supplied which converts it into steam of density many hundred times less than that of water. The pump is in this case very minute, and requires a trifling amount of energy to work it. In condensing engines we have, in addition, the air pump.

In air engines the compressing pump is generally a conspicuous part of the apparatus and requires a large fraction of the power of the motor to drive it; because the changes of density due to the alternate heating and cooling are comparatively small.

289. *Mechanical Equivalent of Heat.*—Heat and mechanical energy are mutually convertible; a unit of heat corresponding to a certain definite amount of mechanical energy which is called the "MECHANICAL EQUIVALENT" of heat.

The statement here made is the First Law of the Science of Thermodynamics, and it shows that quantities of heat may be expressed in units of work, and, conversely, quantities of work in units of heat. In dealing with questions relating to heat and work, a common unit of measurement must be selected. In most cases the thermal unit is adopted, and quantities of work reduced to such units by division by the mechanical equivalent of heat. Until recently the numerical value of the equivalent was taken as 772 in British units, but it is now recognized that this is somewhat too small. In this work the value 780 will be employed, which is just one per cent. greater, and quantities of work in foot pounds are therefore reduced to thermal units by division by 780. Thus the horse-power of 33,000 ft. lbs. per minute becomes 42·3 thermal units per minute or 2538 per hour.

In heat engines the cycle of thermal operations consists of an alternate addition of heat (Q) and subtraction of heat (R), so that, if W be the useful work,

$$W = Q - R,$$

that is, the work is done at the expense of an equivalent amount of heat which disappears during the action of the engine. In steam engines this has been tested experimentally by measuring the heat supplied in the boiler and the heat discharged from the condenser. The difference should be, and in fact is found to be, the thermal equivalent of the work done by the engine. The ratio W/Q is usually called the "absolute," or sometimes, for reasons we shall see presently, the "apparent" efficiency of the engine, but would be much better described as the Co-efficient of Performance. It is always a small fraction: in the best steam engines, for example, it seldom exceeds ·18 losses connected with the furnace and boiler not being included. Supposing as on page 577, A the theoretical maximum for a pneumatic motor working between the given limits of pressure, the ratio W/A, which we will call e, is the "mechanical" efficiency.

290. *Mechanical Value of Heat.*—In stating the first law of thermodynamics nothing is said about the temperature at which the heat is used. In other words, the mechanical equivalent of heat is just the same, whether the temperature be low or high. Yet common experience tells us that the value of heat for mechanical purposes depends very much on this circumstance. The heat discharged from the condenser of a condensing steam engine, or with the exhaust steam of a non-condensing engine, is of little value for the purposes of the engine. So obvious is this fact that the first attempts at connecting the work done by a heat engine with the heat supplied to it may be partly described as attempts to show that temperature, not quantity, was equivalent to energy, heat being supposed as indestructible as matter.

It is now known, however, that difference of temperature is not in itself energy, but merely an indispensable condition that heat may be capable of being converted into work. The power of a heat engine depends on difference of temperature, being greater, the greater that difference is; but in all cases only a fraction of the heat supplied is converted into mechanical energy.

In the converse operation of converting mechanical energy into heat it is possible, by employing it in overcoming frictional resistances, to obtain an amount of heat equal to the energy employed, but such processes are always irreversible. The only way of converting heat into work is by means of a heat engine in which the rejection of heat at low temperature is as essential as the supply of heat at high temperature.

Difference of temperature is wasted if heat be allowed to pass from a hot body to a cold one without the agency of steam, air, or some other body, the density of which is changed by its action. When once wasted it cannot be recovered, a fact of common experience which is expressed in other words by a second thermodynamic principle.

SECOND LAW.—Heat cannot pass from a cold body to a hot one by a purely self-acting process.

By a "self-acting" process in this statement is meant any process which is of the nature of a perpetual motion being independent of any external agency. By the employment of mechanical energy drawn from external bodies, heat may be made to pass from a cold body to a hot one, the amount of energy required being greater, the greater the difference of temperature. And the method sometimes employed of raising steam, without the use of a furnace, by means of heat derived from the exhaust steam condensed in a solution of caustic soda, shows that energy derived from chemical affinity may serve the purpose. But, if no energy is employed, no heat will pass.

Difference of temperature must therefore be carefully utilized, and since the smallest difference of temperature is sufficient to cause heat to pass from a source into the air or steam which exerts energy, it at once follows that the process of conversion of heat into work will be most efficient if all the heat be supplied while the fluid has the temperature of the source of heat, and all the heat rejected while it has the temperature of the body which subtracts heat. These are the conditions of maximum efficiency, and if they are satisfied it is possible to show that a mechanically perfect motor (p. 576) supplied with heat Q will exert the energy

$$U = Q \cdot \frac{T_1 - T_2}{T_1},$$

T_1, T_2, being the temperatures of addition and subtraction of heat, reckoned from the "absolute" zero, a point 460° below the ordinary zero of Fahrenheit's scale. This is true whatever be the nature of the heat engine employed for the purpose, and no more heat can be converted into work under any circumstances. An engine which satisfies these conditions may be described as "thermally perfect."

If two bodies be at the same temperature heat may be made to flow in either direction from one to another, the actual direction being determined by a difference which may be made as small as we please: that is, the process is *reversible*. Hence the conditions of maximum thermal efficiency may also be described by saying that the cycle of thermal operations must be "thermally reversible." And the condition that an engine may be both mechanically and thermally perfect may be completely described by stating that the engine is reversible.

Whichever way we adopt of stating the result it follows at once that a unit of heat has a certain definite MECHANICAL VALUE given by the equation

$$M = 780 \cdot \frac{T_1 - T_0}{T_1},$$

where T_1, T_0 are the temperatures between which it can be used. When ckoned in thermal units M is also often called the AVAILABLE HEAT.

If, instead of the whole amount of heat Q being supplied at the me temperature T_1, the fractions q_1, q_2, q_3, ... are supplied at the veral temperatures T_1, T_2, T_3, ..., the temperature of abstraction of eat remaining the same, the mechanical value of the whole is the um of the mechanical values of each of the several parts taken parately. On expressing this principle algebraically it will be found at the mechanical value of the whole is now in thermal units

$$M = \frac{T_m - T_0}{T_m} \cdot Q,$$

where T_m is the average temperature of supply given by the equation

$$\frac{1}{T_m}=\frac{q_1}{T_1}+\frac{q_2}{T_2}+\dots.$$

In many cases the whole or a part of the heat is supplied at a uniform rate as the temperature rises or falls, as, for example, when a mass of hot air is employed as a source of heat by cooling it at constant pressure. The exact value of the mean temperature of heat so supplied may be found by integration, but unless the change of temperature is excessive, the mean in question is very approximately the arithmetic mean of the highest and lowest temperatures. If then a quantity of heat Q be supplied at a uniform rate as the temperature rises from T_2 to T_1, the part of that heat mechanically available will be

$$M=\frac{\frac{1}{2}(T_1+T_2)-T_0}{\frac{1}{2}(T_1+T_2)}\cdot Q,$$

a useful formula which we shall have occasion to use presently.

291. *Available Heat of Steam.*—When steam is formed from water supplied to a boiler the temperature of the boiler is connected with the pressure by a perfectly definite law, so that when the pressure is known the temperature can be found, and conversely. The results are well known, and given by a table which, being generally accessible, need not be reproduced here.

The process may be separated into two parts—(1) the raising of the feed water from T_0, the temperature at which it enters, to T_1, the temperature of the boiler; and (2) the formation of steam at the constant temperature T_1. The quantity of heat supplied during the first stage is approximately T_1-T_0, and the rate at which it is supplied is approximately uniform. Its mechanical value is therefore, putting $T_2=T_0$ in the general formula given above,

$$M_0=\frac{T_1-T_0}{T_1+T_0}(T_1-T_0)=\frac{(T_1-T_0)^2}{T_1+T_0}.$$

During the second stage, if the steam formed be saturated and free from moisture, the quantity of heat supplied is commonly called the "latent heat of evaporation," and is given for each pound of steam by the well known formula

$$L_1=966-\cdot71(t_1-212^\circ),$$

where t_1 is the temperature Fahrenheit. Since the whole of this is supplied at the temperature $T_1(=t_1+460)$, the corresponding mechanical value is

$$M_1=L_1\cdot\frac{T_1-T_0}{T_1}.$$

The mechanical value of the whole heat supplied is now M_0+M_1.

In using this formula the lower temperature T_0 must correspond to p_0 the pressure in the condenser as shown by the vacuum gauge, or assumed for the purposes of the calculation. In a non-condensing engine T_0 must correspond to the pressure of the atmosphere, which in this case is the exhaust reservoir, that is, it must be supposed $212+460$ or 673.

In perfect engines the mechanical value of the heat supplied is also the available energy of the fluid used, which is thus obtained from the temperatures of supply and rejection of heat without the necessity of knowing the form of the expansion curve, which always must be such that its area, as in preceding articles, represents the energy in question. The available energy is therefore given by the formula

$$M=(T_1-T_0)\left(\frac{T_1-T_0}{T_1+T_0}+\frac{L_1}{T_1}\right).$$

A close approximation, however, to the available energy may be obtained by considering the form of the expansion curve (see Appendix). This leads to the very simple formula

$$M=P_0v_0\log_e\frac{p_1}{p_0},$$

where P_0v_0 is the product of the pressure and the volume of dry saturated steam at the lower limit of pressure p_0, a quantity found in thermal units by a formula given further on or from a table of the properties of saturated steam.

If the steam be superheated $\theta°$, the additional heat supplied will be $\frac{1}{2}\theta$ thermal units nearly, and the rate of supply will be approximately uniform. The corresponding mechanical value will therefore be, putting $T_2=T_1+\theta$ in the general formula,

$$M_2=\tfrac{1}{2}\theta\cdot\frac{T_1+\frac{1}{2}\theta-T_0}{T_1+\frac{1}{2}\theta}.$$

The whole available energy is now $M_0+M_1+M_2$, but the increase is relatively small, the actual economy due to superheating being not due to this cause but to a reduction in cylinder condensation, as will be further explained presently.

292. *Thermal Efficiency.*—If an engine be mechanically perfect the work done per unit of heat will be simply the mechanical value, if the conditions of maximum efficiency are satisfied. In general, however, some of the heat will be supplied at a lower temperature than the source of heat, and some will be abstracted at a higher temperature than that of the refrigerator. When this is the case difference of temperature is wasted, and there is a corresponding loss of thermal efficiency. If the temperature is known at which the air or steam is, while it is being supplied with a certain quantity of heat, or while a certain quantity of heat is being abstracted, the mechanical value of that heat can be found corresponding to that temperature. This quantity represents the work actually done since the engine is supposed mechanically perfect, and the same calculation being made for all the heat supplied or abstracted, the total actual work will be known. Dividing this by the total quantity of heat the actual work (A) per unit of heat will be known. The ratio

$$k = \frac{A}{M}$$

may be described as the "THERMAL EFFICIENCY" of the engine.

Thermal losses may be perfectly definite and practically unavoidable in the type of heat engine under consideration, and may then properly be taken into account in calculating the mechanical value of the heat supplied. Such is the case, for example, in steam, the available energy of which was found in the last article. The heat supplied by the furnace gases is in the first instance at a much higher temperature than that of the boiler, but no use is made of the difference, and it is therefore supposed to be all supplied at the boiler temperature. Again, the portion $T_1 - T_0$ of this heat passes by conduction from temperature T_1 to a lower temperature, which gradually increases from T_0 to T_1 after the feed-water has entered the boiler. If it had been supplied at temperature T_1 its mechanical value would have been

$$M = (T_1 - T_0)\frac{T_1 - T_0}{T_1} = \frac{(T_1 - T_0)^2}{T_1},$$

and, therefore, would have been increased in the proportion $T_1 + T_0 : T_1$. As, however, the supply of heat at rising temperatures cannot practically be avoided, the available energy is considered to be that which remains after deduction of the corresponding loss. A part of the loss may be regained by use of a properly constructed feed water heater, but the resulting gain is most conveniently estimated independently.

The standard of comparison in heat engines therefore is not always an ideally perfect engine, but is fixed with reference to the result which could be attained in an engine of that type if all its working arrangements were perfect.

In practice the engine will not be either mechanically or thermally perfect; its efficiency (W/M) will then be the product (ek) of the mechanical efficiency (W/A) and the thermal efficiency (A/M). The efficiency thus calculated is estimated relatively to an engine which is mechanically and thermally perfect, and may be described as the "relative" or "true" efficiency, as distinguished from the "absolute" or "apparent" efficiency defined in a former article.

To estimate the efficiency of a heat engine without any reference to the temperatures between which the heat can be used is very misleading. The true efficiency of the best condensing steam engines is about 65 per cent., instead of 18 per cent. as it appears to be merely from the quantity of heat used. The standard of comparison is, however, for reasons which have just been pointed out, generally to some extent conventional, and consequently varying estimates of the efficiency may be made.

293. *Compound Engines.*—The working fluid may be discharged from one contracting chamber into a second which simultaneously expands. In many cases an intermediate reservoir is employed, which receives the fluid from the first chamber and supplies it to the second; the two chambers are then virtually separate, and form two distinct motors, the power of which can be separately calculated. The sum of the two is the power of the compound motor; it is necessarily the same as if the fluid had been used with the same expansion curve between the same extreme pressures in a single chamber; except that the frictional resistance of the passages between the chambers and the intermediate reservoir represents a certain loss of energy in the compound motor which does not occur in the simple one. When there is no intermediate reservoir there is no distinct period of admission or expansion in the low-pressure chamber, but the power may still be determined graphically for each chamber, and the results added. The process of compounding may be carried further by the employment of triple and quadruple expansion.

In every case the energy of the fluid is the same, and cannot be affected by the mechanism employed to utilize it, unless its density or elasticity be altered by contact with the sides of the chamber in which it is enclosed. In steam engines, however, the action of the sides of the cylinders has great influence by condensing steam as it enters the cylinder. The liquefied steam is re-evaporated towards the end of the stroke as the temperature of the steam falls, but the process is nevertheless a very wasteful one. The action is greater the greater the degree of expansion employed, because the range of temperature is greater, and the gain by expansion is thus in great measure neutralized or even converted into a loss. By employing two cylinders instead of one the expansion is divided into two parts each of moderate amount, and liquefaction may be diminished. Moreover for constructive reasons the excessive expansion necessary to obtain the full advantage of high-pressure steam cannot be carried out in a single cylinder. Compound engines are therefore being used more and more wherever economy of fuel is a consideration, and in marine practice have almost superseded the simple engine.

The principal losses in steam engines are (1) a mechanical loss due to incomplete expansion, and (2) a thermal loss due to liquefaction. One of these cannot be diminished without increasing the other; but considerable economy may be effected by the use of a "steam jacket," by the employment of superheated steam, and by compounding.

294. *Useful Work of Steam.*—The relation between the pressure (P) and the volume (v) of dry saturated steam is expressed by the equation

$$Pv^{1\cdot0646} = \text{Constant},$$

from which is readily derived the formula

$$\log (Pv) = 1\cdot7882 + \cdot0607 \log p,$$

which gives in thermal units the value of Pv for 1 lb. of dry saturated steam of pressure p lbs. per sq. inch. The logarithms are here common, not hyperbolic. In the formula for W_0 given in Art. 285 (p. 557), the value to be used for P_2V_2 can be obtained by calculating P_2v_2 for the terminal pressure p_2, and then multiplying by x_2 the dryness fraction of the steam at release. The index n of the expansion curve may for this purpose be taken as $10/9$, and we thus obtain for the waste work at exhaust the formula

$$W_0 = x_2P_2v_2\left\{9 - 10\left(\frac{p_0}{p_2}\right)^{\frac{1}{10}} + \frac{p_0}{p_2}\right\}.$$

The remaining losses may conveniently be expressed as a fraction $1 - k$ of the available heat (M) of the steam for the given boiler and vacuum pressures p_1, p_0. The useful work of 1 lb. of steam is then

$$W = kM - W_0.$$

The value of the co-efficient k depends mainly on thermal losses, of which the principal is cylinder condensation, but it also includes leakage and the minor mechanical losses already referred to. Thus, in compound engines, k is diminished by the losses by clearance and leakage in all the cylinders and by wire drawing between the cylinders. The quantity $(1 - k)M$ may conveniently be described as the "missing work," representing, as it does, mainly, losses which cannot be detected by the indicator alone, but only by measurement of the feed-water.

In a given engine W, W_0, and M can be derived from data furnished by experiment, and hence k can be found. Examples of this calculation for engines of various types will be found in the author's work on the *steam engine,* from which it appears that unless there is some special cause of waste, the "missing work" is from 20 to 30 per cent. of the available energy; that is, the value of k is from ·7 to ·8.*

By assuming values of k, and x_2 the dryness fraction of the steam at release, the consumption of steam in lbs. per I.H.P. per hour can be determined for given pressures; for since one horse-power is 2538 thermal units per hour,

$$\text{Lbs. per I.H.P. per hour} = \frac{2538}{W}.$$

The fraction x_2 is less variable than k, and may generally be assumed at ·8, unless there be some special cause of waste; and thus in the most economical engines commonly occurring in practice, the consumption of steam will be found approximately by writing $k = ·8$, $x_2 = ·8$ in the preceding formulæ. But where cylinder condensation is excessive, as, for example, is the case in small engines running at a low speed, the consumption may be double the amount thus given. On the other hand it may be somewhat less when superheated steam is used, mainly because cylinder condensation may in this way be greatly reduced.

295. *Efficiency and Performance of Steam Engines.*—We have already described the ratio W/M (p. 567) as the efficiency of the steam; it is given by the formula

$$\frac{W}{M} = k - \frac{W_0}{M}.$$

If expansion be carried to the greatest extent which can in any case be advantageous (Art. 287), p_2 will be a given quantity, and W_0 may be taken as constant. The efficiency for a given vacuum is then greater the higher the boiler pressure; that is, an increase in the boiler pressure has not only the effect of increasing the energy of the steam theoretically available, but it renders it possible to utilize a greater fraction of it. On the other hand, an improvement in the vacuum increases the available energy; but as it also increases W_0 in a much greater proportion, the efficiency is lowered. This is a necessary consequence of the low pressure of the steam requiring large cylinders and great friction for a given power. The energy theoretically available from heat employed at temperatures much below 212° can only be made use of without great waste by means of some fluid which is more volatile than water.

In non-condensing engines the fraction W_0/A may be and generally is small; their efficiency therefore may be as much as ·75, or, in special cases, more.

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* *The Steam Engine considered as a Thermodynamic Machine*, third edition, p. 322, Spon, 1895.

The efficiency of an engine is not generally greatest when expansion is carried to the extreme limit fixed by the back pressure and by friction, because the value of 1 - k is greater than it would be if a smaller expansion had been employed; this is specially the case in single-cylinder engines working at a moderate speed. The expansion which can be usefully employed in practice is further limited by considerations of cost; interest on capital, as Professor Thurston has pointed out, being a "waste" which ought to be taken into account.

The general question of steam engine economy is far too large and important to consider in detail in the present work, but the foregoing observations may be of service in drawing attention to the principal points to be studied.

296. *Reversed Heat Engines.*—A heat engine like an hydraulic motor may be reversed, and then becomes a machine for drawing heat out of cold bodies and supplying it at a higher temperature just as a pump takes water from a low and discharges it at a high level. Most heat engines occur in their reversed form, being employed as "refrigerating", or, to use the phrase employed in Germany, "cold" machines in the artificial production of ice, or the maintenance of a low temperature in a chamber for the preservation of articles of food. If the heat engine be perfect the reversal will be exact, the same thermodynamic machine, or as for brevity we might perhaps describe it, the same THERMO being a heat motor or a heat pump according to the direction in which it is driven. As in hydraulic machines, however, the reversal in practice will not be perfect, and certain constructive differences between the motor and the pump will generally be rendered necessary by the different conditions under which they work. The refrigerating machines most in use are the air machine, which operates by the compression and subsequent expansion of atmospheric air, and the ammonia-compression machine. The first of these, which is a reversed air engine, we shall have occasion to refer to presently. The second, which is much employed in the manufacture of ice, will now be discussed in illustration of the foregoing remarks.

In making ice by this method, the water to be frozen, originally, of course, at the atmospheric temperature (T_1), is contained in chambers forming divisions of a refrigerating tank filled with brine, at a temperature below the freezing point, by which it is first cooled to 32° and finally frozen. The heat thus received by the brine, together with that which leaks into the tank from surrounding bodies, is then abstracted by the evaporation of liquid anhydrous ammonia contained in a coil of piping immersed in the tank. The liquid in question is highly volatile, its vapour having a pressure of over four atmospheres at the temperature 32° F. As fast as it is formed the resulting ammonia gas is drawn into a double-acting compressing pump, by means of which its pressure is raised; the temperature at the same time rising above that of the atmosphere. When the pressure has reached a certain limit, ranging from 8 to 12 atmospheres, a valve opens, and the gas passes into a second coil of piping surrounded by circulating water of atmospheric temperature, by which it is condensed once more into liquid. To carry out the process perfectly it would now be necessary to admit the liquid into an expansion cylinder, where its pressure would gradually fall while driving a piston. A portion of the liquid would then evaporate, and the temperature would be reduced till it had fallen to T_0, the temperature of the evaporating coil in the brine tank. This part of the process not being practicable, the high-pressure liquid is actually allowed to rush through a small connecting pipe into the coil, thus completing a continuous cycle. The difference this makes will be considered further on; for the present we suppose the expansion cylinder to exist, and to be connected with the crank shaft by which the compressing pump is driven.

If now R be the heat abstracted from the refrigerating tank at temperature T_0 and U the energy exerted in driving the crank shaft, the heat transferred to the circulating water by the condensing coil will be

$$Q = U + R;$$

the final result of the process being that a quantity of heat R passes from the temperature T_0 to the higher temperature T_1 by the agency of a certain amount of mechanical energy U, which is converted into heat in the process. Further, assuming the temperatures T_1 and T_0 of the coils to differ very slightly from the temperatures of the atmosphere and the tank, every step of the process will be exactly reversible, and when reversed the machine becomes a heat motor, of which U must be the mechanical value of the heat Q and R the heat rejected. Hence the relations between U, Q, R must be the same in the two cases, that is if U be the mechanical energy necessary to abstract the heat R,

$$U = Q \cdot \frac{T_1 - T_0}{T_1} = \frac{T_1 - T_0}{T_0} \cdot R.$$

The quantity of heat

$$R = \frac{T_0}{T_1 - T_0} \cdot U$$

may be described as the REFRIGERATING VALUE of the energy U.

In the actual machine the expansion cylinder is omitted, and the energy required to drive the crank shaft is correspondingly increased; while a portion of the liquid ammonia is none the less evaporated as it rushes into the evaporating coil without drawing heat from the brine, so that the heat abstracted is not increased. If then R' be the heat actually abstracted from the freezing water for a given amount of energy U, R' will be less than R from this cause, as well as from leakage of heat and other losses. The ratio R'/R is the efficiency of the machine, which in this case includes the friction of the mechanism, and in good machines of this class appears to be about 40 per cent. But, as in motors, the standard of comparison is to some extent conventional, because it is possible to make various practical estimates of the "refrigerating value" of the energy employed.

The foregoing sketch, necessarily very brief, of the action of thermodynamic machines is all that can be attempted in the present work. We now pass on to consider more particularly the transmission of energy by elastic fluids and the flow of gases through pipes and orifices.

SECTION III.—TRANSMISSION OF ENERGY, FLOW OF GASES.

297. *Internal Energy. Internal Work.* The distinction between internal work and external work was pointed out in Art. 92, p. 186, and the corresponding distinction between internal and external energy of motion in Art. 133, p. 268. These distinctions are principally important in fluids, because the extreme mobility of their parts renders internal motions, of great magnitude, of common occurrence. We have already seen in Chapter XIX. how energy is dissipated by the internal action of liquids; in gases the same dissipation occurs, and is even more important.

In liquids the absorption of energy is almost completely irreversible, but in gases it is not so. We may have internal energy as well as internal work: the greater part of the expansive energy of a gas being due to internal actions.

The state of an elastic fluid is completely known when its pressure and volume are known, but these quantities are capable of any variation we please within wide limits, provided only that we have unlimited power of adding or subtracting heat. If, however, a third quantity, the temperature, be considered, it will be found that the three are always connected together by a certain equation depending on the nature of the fluid, so that when any two are given the third is known. For example, in the so-called "permanent" gases, such as dry air, the equation is very approximately

$$PV = c \, . \, T,$$

where T is the temperature reckoned from the "absolute" zero, as in Art. 290, and c is a constant which for pressures (P) in lbs. per square foot and volumes (V) in cubic feet per lb. has, for dry air, the value 53·2. The "state" of the fluid is completely known if any two of these three quantities are known, but not otherwise.

To produce a given change of *state* a certain definite amount of work must be done in overcoming molecular resistances; this is the internal work, and is the same under all circumstances. But in gaseous fluids, the molecular forces being reversible, may tend to give rise to the change of state, and then we have internal energy instead of internal work. Taking the first case: if the change be at constant volume, this internal work will be the total work done; but in general the volume changes, and in consequence external work is done, the amount of which depends not merely on the change of state, but also on the way in which that change is carried out. The total amount of work is the sum of the internal and external work: it is done by the agency of heat energy supplied from without, so that we write

Heat Expended = Internal Work + External Work,

the three quantities being expressed in common units.

An important application of this equation is to questions relating to the formation of steam, but this we must pass over, our present object being to consider the flow of gases through pipes and orifices, for which purpose the equation is written

Expansive Energy = Internal Energy + Heat Supply,

or, in other words, of the whole expansive energy of the fluid, a part is derived from internal molecular forces, and a part from heat supplied from without.

If no heat is supplied from without the expansive energy is equal to the internal energy: this case is called "adiabatic" expansion, obtained by writing $n = 1 \cdot 4$ in the formulæ of Art. 281. More generally, it is shown in treatises on thermodynamics that the internal work done in changing the temperature of a lb. of air from T_1 to T_2 is $I_2 - I_1$, where

$$I = K_v \, . \, T = 2 \cdot 5 \; PV,$$

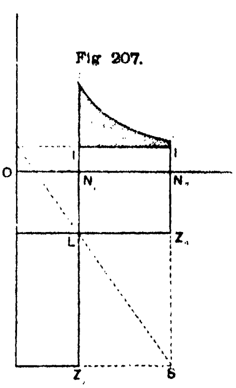
K_v being the specific heat at constant volume, which is $2 \cdot 5 \, c$. Hence when the temperature falls from T_1 to T_2 the internal energy supplied by the fluid is $K_v(T_1 - T_2)$ and the equation becomes for a heat supply Q

$$E = 2 \cdot 5 \, c(P_1V_1 - P_2V_2) + Q.$$

This is the fundamental equation from which all cases may be derived.

If the heat be supplied to a permanent gas at a uniform rate as the temperature falls, it may be shown that the law of expansion is $PV^n =$ constant, as supposed in Art. 281, and this is generally permissible with sufficient approximation. The expansion index n then depends upon the proportion which Q bears to E. If $Q = E$ the expansion is hyperbolic, and the whole of the expansive energy is derived from heat supplied from without. The manner in which the expansive energy (E) depends on the heat supply (Q) is well seen by the annexed diagram (Fig. 207). Let, as before, the ordinates of the point 1 represent the pressure and volume before expansion and those of the point 2 after expansion; 1, 2 being the expansion curve. Set downwards N_1Z_1, N_2Z_2, each equal to 2½ the corresponding pressure ordinates, and complete the rectangles OZ_1, OZ_2. Then complete the rectangle Z_1Z_2, and draw the diagonal SL to meet the vertical through O. Finally through the intersection draw II horizontally; then the rectangle IN_2 will be found to be the difference of the rectangles OZ_1, OZ_2, and therefore represents the internal energy exerted during expansion. Thus the area $12II$ (shaded in the figure) represents the heat supply: which will depend not only on the points 1, 2, that is, on the change of state of the air, but also on the form of the expansion curve, that is, on the way in which the change takes place.

Fig 207.



298. *Transmission of Energy by Compressed Air.*—A reservoir of compressed air furnishes a supply of energy which may be transmitted by pipes to distant points and distributed at pleasure. The losses which occur in the pipes by leakage and friction will be discussed further on; the present article will be devoted to the consideration of the process of compression and expansion.

The volume of 1 lb. of air at the atmospheric pressure is

$$V_0 = \frac{cT_0}{2116} = \frac{T_0}{40} \text{ nearly,}$$

where T_0 is the absolute temperature. The work done in compressing 1 cubic foot to a pressure of P atmospheres without gain or loss of *heat, and forcing it into a reservoir,* is (Art. 285, p. 558)

$$\text{Work done} = 7400(P^{\frac{2}{7}} - 1),$$

while the temperature will rise to

$$T_1 = T_0 . P^{\frac{2}{7}}.$$

If the temperature could be prevented from rising the work done would be reduced to 2116 $\log_e P$; but this can only be effectively done by the injection of water in the form of spray into the compressing cylinder. No form of water jacket appears to have any considerable effect in the short space of time occupied by the working stroke. After the compression is complete the air may be cooled on its way to the reservoir by passing it through pipes exposing a large surface to the external application of cold water; an operation which is conducive to economy, for otherwise the hot air in the reservoir will lose heat by radiation and conduction, and the pressure will be reduced. Let us suppose the air thus cooled at constant pressure to temperature T, the work done in forcing it into the reservoir will not be reduced, the only difference is that a part of the admission work will be done outside the reservoir in compressing the cooling air at constant pressure, the total amount remaining the same. Hence, in the absence of spray-injection, the work done per cubic foot of air drawn from the atmosphere is always nearly the same, being given by the above formula, and this conclusion would be correct if the air were heated instead of cooled before entering the reservoir.

The compressed air is now conducted by pipes to a corresponding motor at any distance. The air-motor consists of a working cylinder and piston with valves attached, as in the case of a steam engine. Assuming the expansion adiabatic and complete the energy exerted, per cubic foot of *compressed* air consumed is, as shown on page 554, $7400(P - P^{\frac{2}{7}})$, whatever the temperature. Now if T be the temperature, the density of the compressed air is greater than that of the atmosphere in the ratio PT_0/T, and therefore we obtain by division

$$\text{Energy exerted} = 7400\frac{T}{T_0}\left(1 - \left(\frac{1}{P}\right)^{\frac{2}{7}}\right),$$

a general formula giving the available energy of an air motor per cubic foot of air drawn from the atmosphere by the compressing pump. For reasons already stated the whole of this will not be utilized (p. 555). On the other hand, the expansion has been supposed adiabatic, although there can be little doubt that the cooling of air below the atmospheric temperature is greatly hindered by the condensation of vapour mixed with it, and by drawing heat from external bodies.

Subject to these remarks the efficiency of transmission will be

$$\text{Efficiency} = \frac{T}{T_0}\left(\frac{1}{P}\right)^{\frac{2}{7}} = \frac{T}{T_1}.$$

(1) Let the air after compression be cooled to T_0, and supplied without re-heating to the motor, the efficiency is now $1/P^{\frac{2}{7}}$, and therefore diminishes rapidly as the pressure increases. The loss is due to change of temperature, and may be greatly diminished by compounding the compressing pump so as to compress the air in two or more stages; the air being thoroughly cooled between each stage. Compression by stages is necessary for constructive reasons when the pressure is very high, and, when properly carried out, is economical. It has of late been introduced for economical reasons.

(2) If the air is re-heated after transmission before entering the motor the efficiency will be increased as the formula shows. It is true that heat will be spent in raising the temperature of the air, but the corresponding gain of work in the motor cylinder is proportionally very large. If $T > T_1$ more energy will be exerted in the motor cylinder than is necessary to drive the compressing pump, the whole arrangement operating as a heat-engine. If $T < T_1$ the arrangement operates as a reversed heat-engine, being, in fact, a well-known form of refrigerating machine. The theory of the process considered in this light is given in the author's work on the Steam Engine already cited, in which the principles of thermodynamics are explained at length.

299. *Steady Flow through a Pipe. Conservation of Energy.*—Referring to Fig. 174, p. 475, suppose that the reservoir is closed, and that it contains an elastic fluid at high pressure which is flowing through the pipe. Unless the change of pressure be very small, difference of level may be disregarded as relatively unimportant (p. 550), and we have only to consider differences of pressure, while, on the other hand, we must now remember that, when the pressure changes, energy is exerted by expansion as well as by transmission. The energy transmitted from the reservoir *to* any point where the pressure is P and volume V is P_0V_0, where the suffix indicates the state of the fluid in the reservoir. Of this the amount PV is transmitted through the point, and the difference $P_0V_0 - PV$ together with the expansive energy E is employed in generating the kinetic energy which the gas possesses in consequence of the velocity u with which it is rushing through the pipe at the point considered. Thus, if the motion be steady,

$$\frac{u^2}{2g} = 3{\cdot}5(P_0V_0 - PV) + Q,$$

where Q is the heat (if any) supplied during the passage from the reservoir to the point. If no heat be supplied,

$$\frac{u^2}{2g} + 3{\cdot}5\,PV = \text{Constant},$$

an equation which may also be written

$$\frac{u^2}{2g} + K_p \,.\, T = \text{Constant},$$

where K_p is the specific heat at constant pressure. If we have to do with any elastic fluid other than a permanent gas, $3{\cdot}5PV$ must be replaced by $I + PV$, where I is the internal energy, and if the question be such that the elevation of the point considered has any sensible influence, the term z must be added as in the corresponding case of an incompressible fluid.

The equation for a compressible fluid, however, is much more general than that for an incompressible fluid, because the internal energy is taken into account, and consequently any energy exerted in overcoming frictional resistances is replaced by an equivalent amount of heat generated. It follows that the equation is true whether there be frictional resistances or whether there be none, provided that the internal motions have time to subside and be converted into heat by friction, and provided that none of the heat thus generated is transmitted to external bodies.

It sometimes happens that we have to consider cases where a quantity of heat Q is supplied to a permanent gas during its passage from a point 1 to a point 2, we shall then have the equation

$$Q = K_p(T_2 - T_1) + \frac{u_2^2 - u_1^2}{2g},$$

an equation which is true, however great the variations of pressure or temperature are, and whether or not there are frictional resistances.

300. *Velocity of Efflux of a Gas from an Orifice.*—The most important applications of the equation for the steady flow of a gas are to the discharge of air or steam from an orifice and to the flow of air through long pipes.

In the first case the frictional resistances are small and are consequently neglected. It will be desirable to give a method of treating the question which is independent of the general equation.

In Fig. 208a $o12k$ represents the expansion curve for a small portion of the gas as it rushes out of the reservoir A (Fig. 208b) in which it is confined through a small orifice into the atmosphere. The jet contracts at issue to a contracted section kk, nearly as in the case where the fluid is incompressible, and then, in general expands again in some such way as is shown in the figure. The velocity through the contracted section may be denoted by u, and the pressure there by P. The area of the contracted section is connected with the area of the orifice by the equation

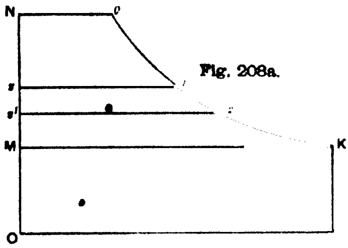


Fig. 208a.

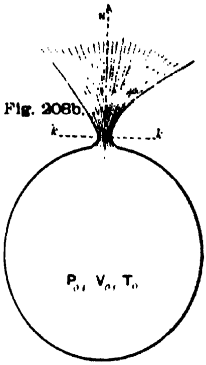


Fig. 208b.

$$A = k . A_0,$$

as on page 473, k being a co-efficient.

Each small portion of the fluid expands from the state represented by the point o on the diagram to that represented by K; in some intermediate state it will be represented by a point 1 on the expansion curve, and immediately after by 2, a point near to 1. Let u_1, u_2 be the corresponding velocities, then

$$\frac{u_2^2 - u_1^2}{2g} = \frac{P_1 - P_2}{w} = V . \delta P,$$

where w is the mean density and V the mean specific volume represented graphically by the mean of s_1, s_2. Hence $V . \delta P$ is represented by the area of the strip cut off by these ordinates. Dividing the whole area into strips, the area of each strip represents the corresponding change in $u^2/2g$, so that the total area represents the final value of this quantity. We have then

$$\frac{u^2}{2g} = \text{Area } N_{oK}M = \int_P^{P_0} V dP = h.$$

The quantity h thus found and graphically represented is the "head" due to difference of pressure, as fully explained in Art 283.

Assuming the expansion curve $PV^n = \text{Constant}$, as before,

$$\frac{u^2}{2g} = \frac{n}{n-1}(P_0V_0 - PV) = \frac{nc}{n-1}(T_0 - T).$$

Now, if the expansion be adiabatic $n = 1{\cdot}4$, and $nc/(n-1)$ is equal to K_p, so that the result might have been written down at once from the general equation of the preceding article.

Employing the notation of Art. 281, but replacing the suffix 1 by the suffix 0, the velocity of efflux is given by the formula

$$\frac{u^2}{2g} = \frac{n}{n-1} \cdot P_0 V_0 (1 - \mu r).$$

301. *Discharge from an Orifice.*—The weight of gas discharged per second from an orifice of contracted area A is now found from the formula

$$W = \frac{Au}{V},$$

where V is the specific volume of the gas at the instant of passing through the contracted section, and therefore supposing A unity the weight per unit of area is given by

$$W^2 = 2g \cdot \frac{n}{n-1} \cdot \frac{P_0 V_0}{V^2} (1 - \mu r).$$

For V we now write rV_0 and finally obtain

$$W^2 = 2g \cdot \frac{n}{n-1} \cdot \frac{P_0}{V_0} \cdot \frac{1 - \mu r}{r^2}.$$

In applying this formula μ must be supposed known and r calculated from it by the equation on p. 552.

It will be found on examination that as μ diminishes from unity W increases to a maximum value and then diminishes again to zero. That is, if the pressure in the throat of the jet at the contracted section be diminished the discharge does not increase indefinitely, but reaches a maximum and then decreases. On substitution for r in terms of μ it will be seen that for a given pressure (P_0) in the reservoir W is greatest when $\mu^{\frac{2}{n}} - \mu^{\frac{n+1}{n}}$ is greatest.

This will be found to be the case when

$$\mu = \left(\frac{2}{n+1}\right)^{\frac{n}{n-1}}.$$

The expansion is adiabatic, and the values of n with the resulting values of μ for maximum discharge are shown in the annexed table.

NATURE OF GAS.	VALUE OF n.	VALUE OF μ.
Dry Air,	1·4	·528
Superheated Steam,	1·3	·546
Dry Saturated Steam,	1·135	·577
Moist Steam,	1·1	·582
	1	$\frac{1}{\sqrt{e}} = {\cdot}6$

The discharge is therefore a maximum when the external pressure is from ·5 to ·6 the pressure in the reservoir. For dry air it will be found on substitution that the maximum discharge per second per unit of contracted area is

$$W_m = \frac{3\cdot 9\, P_0}{\sqrt{P_0 V_0}} = \frac{P_0}{1\cdot 87\sqrt{T_0}}.$$

and for dry steam

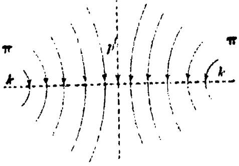
$$W_m = \frac{3\cdot 6\, P_0}{\sqrt{P_0 V_0}}.$$

The pressure P_0 was originally supposed expressed in lbs. per square foot, but it may now be taken as lbs. per square inch in the numerator of these fractions, in which case W_m will be the discharge per square inch.

The diminution of the discharge on diminution of the external pressure below the limit just now given, is an anomaly which had always been considered as requiring explanation, and M. St. Venant had already suggested that it could not actually occur. In 1866 Mr. R. D. Napier showed by experiment that the weight of steam of given pressure discharged from an orifice is really independent of the pressure of the medium into which the efflux takes place; and in 1872 Mr. Wilson confirmed this result by experiments on the reaction of steam issuing from an orifice.*

The explanation lies in the fact that the pressure in the centre of the contracted jet is not the same as that of the surrounding medium. The jet after passing the contracted section suddenly expands, and the sudden change of direction of the fluid particles gives rise to centrifugal forces which cause the pressure to increase on passing from the surface of the jet to the interior on the principle explained on page 478. This will be better understood by reference to the annexed figure (Fig. 209) which shows a longitudinal section of the jet at the point where the contraction of transverse section is greatest. The particles describe curves the radius of curvature of which increases from a small minimum value at the surface k, to an infinite value at the centre. The pressure p increases from that of the medium (π) at k to a maximum p' at the centre, the increase being very rapid at first and afterwards more gradual. The problem is therefore far more complicated than we have supposed,

Fig. 209.



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* *Discharge of Fluids*, by R. D. Napier. Spon, 1866. *Annual of the Royal School of Naval Architecture for* 1874.

each small portion of the jet having its own pressure and (consequently) its own velocity and density.

The results of experiment however suggest that an approximate solution may be obtained by the assumption of a mean pressure in the throat of the jet, with a corresponding mean velocity; this mean pressure being that which gives maximum discharge in every case in which that quantity is greater than π. At lower pressure it is to be assumed equal to π.

Adopting this hypothesis we see that whenever steam is discharged into the atmosphere from a boiler the pressure in which is greater than, about, 25 lbs. per square inch absolute, or 10 lbs. above the atmosphere, the formula given above for maximum discharge is to be used. If we assume the mean value 252 for $\sqrt{P_0V_0}$ this gives $p_1/70$ for the weight discharged from an orifice per square inch of effective area per second, the pressure p_1 being the absolute pressure in the boiler expressed in lbs. per square inch. Contraction and friction must be allowed for by use of a co-efficient of discharge, the value of which however is more variable than that of the corresponding co-efficient for an incompressible fluid. Little is certainly known on this point.

302. *Flow of Gases through Pipes.*—Returning to the general equation, we have now to examine the case where air or steam flows through a pipe of considerable length. As in the case of water, the frictional resistances are then so great that most of the head is taken up in overcoming them. The velocities of the fluid are therefore not excessive, and the value of $u^2/2g$ varies comparatively little.

Now, in the equation

$$\frac{u^2}{2g}+K_pT=\text{Constant},$$

the numerical value of K_p is about 184, and therefore a variation of temperature of a single degree will correspond to a great change in $u^2/2g$; it may therefore be assumed that the temperature remains very approximately constant provided only that the difference of pressure at the two ends of the pipe is not too great compared with its length.

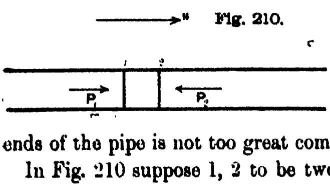

Fig. 210.

In Fig. 210 suppose 1, 2 to be two sections of the pipe at a distance Δx so small that, in estimating the friction, the velocity may be taken at its mean value u; then the force required to overcome friction is

$$R=f.s.\Delta x.u^2,$$

where s is the perimeter and f the friction per square foot, as on page 484. Replacing this by a new co-efficient f', as on page 486,

$$R = f'\frac{w}{2g}\,.\,s\Delta x\,.\,u^2,$$

in which equation w means the weight of unit-volume of the gas. Now, it was pointed out on page 493 that surface friction was a kind of eddy resistance, and that in the case of water it was proportional to the density. This leads us to suppose that in fluids of varying density, not f but f' is a constant quantity. Replacing w by its equivalent $1/V$, we obtain, suppressing the accent of f,

$$R = f\,.\,\frac{s\Delta x}{V}\,.\,\frac{u^2}{2g}.$$

We now apply the principle of momentum which will be expressed by the equation

$$(P_1 - P_2)A = W\,.\,A\,\frac{u_2 - u_1}{g} + R,$$

where W is the weight of gas flowing through the pipe per unit of area per second, and the suffixes refer to the two sections in question, the area of which is A. Now, the motion through the pipe being steady, W is the same throughout, so that

$$\frac{u}{V} = W = \text{Constant}.$$

By substitution for W and writing H for $u^2/2g$, an equation is obtained which, when written in the differential form, becomes

$$-V\,.\,\delta P = \delta H + f\,.\,\frac{\delta x}{m}\,.\,H,$$

m being the hydraulic mean depth.

Next, if T be the temperature, which, as remarked above, is sensibly constant,

$$P = \frac{cT}{V} = W\,.\,\frac{cT}{u}; \quad \therefore\ \delta P = -W\,.\,\frac{cT}{u^2}\,.\,\delta u.$$

Substitute again for W and u, we then find

$$-V\,.\,\delta P = \frac{1}{2}cT\,.\,\frac{\delta H}{H}.$$

On substitution for $V\,.\,\delta P$, the value of which has just been given, the differential equation becomes integrable by dividing by H, and we obtain on performing the integration

$$\frac{1}{2}cT\left\{\frac{1}{H_0} - \frac{1}{H}\right\} = \log_e\frac{H}{H_0} + f\,.\,\frac{l}{m},$$

where l is the length of the pipe, and H_0, H the values of $u^2/2g$ at entrance and exit respectively. In application of this equation the term containing the logarithm is small as compared with the rest, and may generally be omitted: also

$$\frac{H}{H_0}=\frac{p_0^2}{p^2},$$

a ratio which is known if the pressures are the data of the question. In the case of steam cT is to be replaced by the nearly constant product PV, which is to be taken from a table for its quantity so as to obtain a mean value according to the pressure considered.

The value of the co-efficient in small pipes not more than 3 inches diameter is about the same as in the case of water, namely ·007, but it appears to diminish much more rapidly as the diameter of the pipe increases. In pipes 12 inches diameter and upwards, with velocities from 10 to 25 f.s., Professor Unwin gives the value ·003 as the result of a reduction of a large number of experiments made on the resistance of the Paris air mains.*

The equation just found must not be applied to cases in which the difference of pressure is too great compared with the length of the pipe. The friction is then not great enough to prevent the velocity from becoming excessive; the temperature then sensibly falls, instead of remaining constant as supposed in the calculation. An equation can be found which takes account of the fall of temperature when necessary, but in such cases as commonly occur in practice, the supposition of constant temperature is sufficiently approximate. When the difference of pressure is small, the equation will be found to reduce to the hydraulic formula for flow in a pipe. This case will be considered presently.

The head is given by the formula

$$h=PV.\log_e\frac{p_0}{p}=PV\log_e r,$$

and the power expended in forcing the air through is Wh or $PAu\log_e r$ ft.-lbs. per square foot of sectional area per 1″.

The efficiency of the process of compression and expansion has already been considered when compressed air is used for the transmission of energy, and it need only be added that the question of leakage is one of great importance. In some cases the method has proved a failure from this cause. It is probably always more difficult to render the joints of a pipe tight under air pressure than under steam pressure; but experiments by Professor Riedler on the Paris mains showed that the loss may be made very small, not exceeding one per cent. per mile per hour.

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* *Proceedings of the Institution of Civil Engineers*, vol. cv., p. 192.

303. *Flow of Gases under Small Differences of Pressure.*—When the differences of pressure are small and no heat is added or subtracted, a gas flows in the same way as a liquid of the same mean density. In the case of air the mean specific volume is found from the equation

$$V = \frac{cT}{P} = \frac{T}{40},$$

the units being feet and pounds, the mean pressure that of the atmosphere, and the temperature measured on Fahrenheit's scale from the absolute zero. At 59° this gives $V = 13$ cubic feet, but the actual volume will vary slightly from variations in the mean pressure.

The small differences of pressure with which we have now to do are commonly measured by a syphon gauge in inches of water. One inch of water is equivalent to a pressure of 5·2 lbs. per square foot.

If now ΔP be the difference of pressure in lbs. per sq. ft., i the corresponding number of inches of water, the head due to it will be, as in Art. 300, $V\Delta P$, and therefore

$$h = V\Delta P = \frac{T}{7{\cdot}7} \cdot i.$$

The velocity due to this head, or, what is the same thing, the volume discharged per sq. ft. of *effective* area per second in the absence of frictional resistances, is in cubic feet

$$u = \sqrt{2gh} = 2{\cdot}89 \sqrt{Ti},$$

and the weight-discharge in pounds per second

$$W = \frac{u}{V} = 155{\cdot}6 \sqrt{\frac{i}{T}}.$$

At 59° one inch of water gives a head of 67·5 feet and a discharge of 65·9 cubic feet, or 5·07 lbs. per second; but at 539° the head is 130 feet and the discharge 91·3 cubic feet, or 3·67 lbs., results which show that the effect of a given difference of pressure is entirely different according to the temperature of the flowing air. This is a point which must always be borne in mind in applying hydraulic formulæ to the flow of gases. Frictional resistances are taken into account by the employment of a co-efficient as in hydraulics, and, as elsewhere explained, there is reason to believe that the values of these co-efficients are the same, except so far as they may be dependent directly or indirectly on the co-efficient of contraction (p. 473). Co-efficients of contraction are more variable in air than in water, but their average value does not differ widely in the two cases, and may provisionally be assumed the same.

In particular, it is well established that the formula for the discharge of a pipe in cubic feet per second (p. 488),

$$Q = k\sqrt{\frac{h'}{l}} \cdot d^{\frac{5}{2}},$$

applies to air at the low velocities here considered, with the same value of the co-efficient k as in water, that is (d in feet) about 40. The head h' is calculated, as just explained, according to the temperature of the air, for a given difference of pressure.

It is sometimes necessary to consider the flow of some gas other than atmospheric air. In approximately permanent gases this is easily done if we know the density of the gas. For example, the density of common coal gas is about ·43, air being unity. The value of c in the formula $PV = cT$ is then proportionately increased, but in other respects the formulæ are unaltered, the index of the adiabatic curve and the constants 2·5, 3·5 which depend on it remaining unaltered. The formula for small differences of pressure may also be employed for the non-permanent gases, such as steam, with a corresponding modification.

Pneumatic machines in which the variation of pressure is small are analogous to hydraulic machines, and most of what was said in the last chapter is applicable to them. The common fan, for example, is a centrifugal pump, the lift of which is the difference of pressure reckoned in feet of air, that is, at ordinary temperatures, about 67 feet for each inch of water. The speed of periphery is $\sqrt{gh}$ in feet per second, where h is the lift increased, as explained in the case of the pump, on account of frictional resistances and the curving-back of the vanes. Some remarks on the influence of the form of the vanes on the efficiency of a fan will be found in the appendix (note to p. 542).

Fans are employed to produce a current of air for the purpose of ventilating a mine, ship, or structure of any kind. In mines they are often 30 feet in diameter or more. The pressure required is here small and the speed moderate. They are also used to produce a forced draught in torpedo boats, or the blast of a smithy fire. The pressure is then 5 inches of water and more, corresponding to a lift of 300 feet and upwards. The speed of periphery is consequently excessive, and for the comparatively great pressures required for a foundry cupola or a blast furnace, it is necessary to resort to some other sort of blowing machine.

304. *Varying Temperature. Chimney Draught.*—If the temperature of the flowing air is varied by the addition or subtraction of heat, its density will be altered during the flow, and it is then necessary to know the mean density, in order that we may be able to calculate the "head" due to a given difference of pressure as measured by the water gauge.

In Fig. 211 OX, OY are axes of reference parallel to which ordinates are drawn as usual to represent volumes and pressures. A given difference of pressure ΔP is represented by the difference ab of a pair of ordinates. The original volume of the air is represented by $a1$. Suppose now that in flowing through a passage of any description the air is heated, as for example in passing through a furnace, the volume increases greatly while the pressure falls slightly; this will be represented by the curve 12, terminating at a point 2. The form of the curve will depend on the law of heating, and will be very different according to the state of the fire: if the bars of the grate be blocked by clinker and the surface of the fire be free from special obstruction, most of the frictional resistance and corresponding fall of pressure will occur before the air is heated, and the curve will slope rapidly near 1 and slowly afterwards; while, conversely, if the fire be covered with fresh fuel and the grate bars clear, the reverse may be true. After being heated let the air pass through a boiler tube, by which heat is abstracted, till it reaches the chimney: the volume then diminishes greatly while the pressure falls slightly, as shown by the curve 23, terminating at a point 3, such that $b3$ represents the volume of the air in the chimney. The form of the curve 23 will depend on the law according to which the tube abstracts heat. The area of the whole figure $a123b$ represents the "head" due to the whole difference of pressure ΔP, and it will now be obvious that this head will vary according to circumstances which cannot be precisely known. Thus the mean density cannot be found except by empirical formulæ derived by direct experience, and consequently applicable only to the special cases for which they have been determined. It has hitherto been most usually assumed in the case of a furnace and boiler that the mean density was that of the air in the chimney, which amounts to supposing that the forms of the curves 12 23 are such that the area of the rectangle $a3$ is equal to the area of the whole figure. This is the supposition employed by Rankine, and in many cases it appears to lead to correct results.

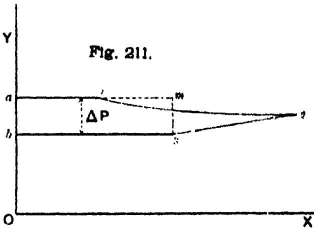
Fig. 211.

In every case of the flow of heated air it must be carefully considered what the mean density will probably be. Its value can often be foreseen without difficulty. It is only in the case of long passages, where the air suffers great frictional resistance while being heated or cooled, that it is uncertain what value to adopt.

The draught which draws air through a fire may be produced artificially or by the action of a chimney. In the latter case there is a difference of pressure within and without the chimney at its base due to the difference of weight of a column of air of the height of the chimney at the temperature of the chimney and at that of the atmosphere. Radiation causes the temperature of the air to be less in the upper part of the chimney and so diminishes the draught, the frictional resistances have the same effect. If these be disregarded the draught in inches of water will be

$$i = 7{\cdot}7 \left\{ \frac{1}{T_0} - \frac{1}{T} \right\} l,$$

where T_0 is the temperature of the atmosphere, T that of the chimney, while l is the height of the chimney in feet. The temperatures are here reckoned from the absolute zero.

If, for example, the temperature of the chimney be 539° F., and that of the atmosphere 59° F., the height of the chimney required for a draught of 1 inch of water will be about 141 feet, or in practice more on account of friction and radiation.

The effect of this draught in drawing air through a furnace or through passages of any kind will vary according to the circumstances which have just been explained.

EXAMPLES.

FIRST SERIES (SECTION I.).

1. Find the store of energy in the reservoir of a Whitehead torpedo. Capacity 5 cubic feet. Pressure 70 atmospheres.

Ans. If $n=1$ 2,420,000 ft.-lbs., or 3,130 thermal units.
$n=1·4$ 1,092,000 ,, or 1,414 ,,
Ratio of results $=$ ·45.

2. In the last question the air is supplied to the torpedo engines by a reducing valve so that the pressure in the supply chamber remains constantly at 13 atmospheres: find the available energy.

Ans. If $n=1$ 1,900,000 ft.-lbs.
$n=1·4$ 1,346,000 ,,

NOTE.—The difference between these results and the preceding is the effect of wire-drawing (resistance of valve). The supply chamber is supposed small.

3. Air is stored in a reservoir the pressure in which is maintained always nearly at 10 atmospheres: find the store of energy per cubic foot of air supplied from the reservoir.

Ans. If $n=1$ 48,700 ft.-lbs.
$n=1·4$ 35,700 ,,
Ratio $=$ ·733.

4. A chamber of 100 cubic feet capacity is exhausted to one-tenth of an atmosphere; find the work done, assuming $n=1$.

Here if the chamber be imagined to contract, compressing the air still remaining in it, the energy exerted will be due to the pressure of the atmosphere, and the difference between this energy and the work done in compression will be available for other purposes. In exhausting this is reversed. *Ans.* 142,000 ft.-lbs.

5. Find the mechanical efficiency of an engine so far as due to incomplete expansion (ratio r): assuming the expansion hyperbolic.

Ans. If R be the ratio of complete expansion,

$$\text{Efficiency} = \frac{1 + \log_e r - \frac{r}{R}}{\log_e R}.$$

6. In the last question obtain numerical results for a condensing engine, taking the back pressure at 2 lbs. and boiler pressure 60 lbs.

Ans. Ratio of expansion,	1	2	5	10,
Efficiency,	·284	·48	·72	·87.

7. Find the comparative mechanical efficiencies in a condensing and a non-condensing engine. Back pressure in condensing engine 2, in non-condensing 16. Boiler pressure 60 and 100. Ratio of expansion 5 in both cases.

The engines must here be supposed to have the same lower limit of pressure of 2 lbs.; and the result for the non-condensing engine includes the loss by the actual back pressure being 16 lbs. *Ans.* ·72, ·46.

8. Find the loss by wire-drawing between two cylinders from one constant pressure of 60 lbs. to another constant pressure of 40 lbs. Expansion hyperbolic. *Ans.* ·405 PV.

9. One vessel contains A lbs. of fluid at a given pressure P_A, and a second B lbs. of the same fluid at a lower pressure P_B. A communication is opened between the vessels and the fluid rushes from A to B; find the loss of energy.

The loss here is the difference between the energy exerted by A lbs. expanding from V_A to V, and the work done in compressing B lbs. from V_B to V: where V_A, V_B are the specific volumes of the fluid in A and B, and V that of the fluid after equilibrium has been attained, found from the formula

$$V=\frac{AV_A+BV_B}{A+B}.$$

Hence the loss is very approximately

$$\text{Loss}=\frac{AB}{A+B}\cdot\frac{(V_B-V_A)(P_A-P_B)}{2}.$$

10. In a compound engine the receiver is half the volume of the high-pressure cylinder, and at release the pressure in the cylinder is 25 lbs. per square inch, while that in the receiver is 15 lbs. per square inch: find the loss of work per lb. of steam. Obtain the results also when the receiver is double instead of one-half the cylinder.

Ans. Case I., 1638 ft.-lbs.
Case II., 3873 ,,

11. In a condensing engine find the mean effective pressure and the consumption of steam in cubic feet per I.H.P. per minute at the boiler pressure: being given, back pressure 3, boiler pressure 60 lbs. per square inch (absolute), ratio of expansion 5.

Ans. Mean effective pressure = 28·33 lbs. per square inch.
Consumption of steam = 1·62 cubic feet per minute.

12. If the volume of 1 lb. of dry steam at the boiler pressure be taken in the preceding question as 7 cubic feet and the liquefaction during admission 20 per cent.; find the weight of steam consumed in lbs. per I.H.P. per hour. *Ans.* 1·75.

13. Find the H.P. necessary to compress 100 cubic feet of air per minute to a pressure of 7 atmospheres (absolute), the air being drawn from the atmosphere at temperature 60° and forced at constant pressure into a reservoir. Suppose the compression (1) adiabatic, (2) isothermal.

Ans. Work per lb. of air = 92·2 thermal units.
,, ,, = 68·8 ,,
H.P. = 16¾ or 12½.

14. In the last question suppose the compression carried out in two stages, the air at each stage being cooled at constant pressure after adiabatic compression: find the work done per lb. of air. *Ans.* 79·2 thermal units.

EXAMPLES.

SECOND SERIES (SECTION II.).

1. Find the mechanical value of a unit of heat, the limits of temperature being 600° and 60°; 300° and 100°; 400° and 212°.

Ans. 393, 203, 169 ft.-lbs.

2. The limits of temperature in a heat engine are 350° and 60°; find the thermal efficiency when two-thirds of the whole heat supplied is used between 300° and 100°, one-sixth between 200° and 100°, and one-sixth between 250° and 100°. *Ans.* ·658.

3. In question 6, First Series, on account of a gradual increase in the liquefaction the thermal efficiency at the several ratios of expansion mentioned is assumed as ·9, ·85, ·7, ·5; find the true efficiency *Ans.* ·256, ·408, ·504, ·435.

4. In a compound engine the pressure of admission is 100 lbs. per square inch, the steam is cut off at one-third in the high-pressure cylinder, the ratio of cylinders is 2½; the back pressure is 3 lbs. per square inch, the large cylinder 40 inches diameter, and the speed of piston 400 feet per minute. Find the H.P., neglecting wire-drawing and sudden expansion. *Ans.* 567.

5. In the last question suppose that the engine has a very large intermediate reservoir, and that the cut-off in the low-pressure cylinder is ·5; find the pressure in the reservoir, neglecting wire-drawing, also the loss per cent. by sudden expansion at exhaust from the high-pressure cylinder, and the percentage of power developed in the two cylinders.

Obtain the results also for a cut-off of one-third in the low-pressure cylinder.

Ans.	Cut off ½.	Cut-off ⅓.
Pressure in reservoir,	26·7	40
Loss by sudden expansion per cent.	·8	·7
Percentage of power in high-pressure cylinder,	46·5	32·4
,, ,, low-pressure ,,	52·6	57·6

6. Compare the efficiencies of the simple and compound engine, assuming the liquefaction the same at the best ratio of expansion, which is 5 in the simple engine and 7 in the compound engine, while in the latter 5 per cent. of the work is lost by wire-drawing between the cylinders. Back pressure and boiler pressure in both cases 3 lbs. and 84 lbs. respectively.

Ans. Gain by compounding 2½ per cent.

7. In question 5, instead of supposing the whole expansion represented by a single hyperbolic curve, assume that at the end of the stroke in the high-pressure cylinder the steam is dry, while at the end of the stroke in the low-pressure cylinder the steam contains 10 per cent. water. Obtain the required result for the cut-off ·5 and find the weight of steam used (exclusive of jacket steam) in lbs. per I.H.P. per hour. Also obtain the results when the steam at the end of the stroke in the high-pressure cylinder contains 30 per cent. water, all other data remaining the same.

Ans.	Case I.	Case II.
Pressure in reservoir,	22·5	14·9
Percentage of power in high-pressure cylinder,	55	37·5
,, ,, low-pressure ,,	55	62·5
Lbs. of steam per I.H.P. per hour,	13	16·5

8. The available heat of a pound of coal is 10,000 thermal units; find the consumption of coal per I.H.P. per hour in a perfect heat engine working between the limits 600° and 60°.

9. The temperature of the atmosphere is 70° and that of a tank in which ice is being made 26°. Find the H.P. necessary to drive a perfect ice-making machine, per ton of ice per hour. Latent heat of water = 142; specific heat of ice = ·5. *Ans.* 14½.

10. Air is heated at constant volume till its temperature is raised from 70° to 300°, then expanded to three times its volume at constant temperature. Find the mean temperature of supply. *Ans.* 247° F.

11. In the last question suppose the air subsequently to expand adiabatically till its temperature has fallen to 70°, and then to be compressed at constant temperature till the original pressure is reached. Deduce the co-efficient of performance, and verify your calculation. *Ans.* Co-efficient = ·25.

12. Air at a pressure of 1,000 lbs. per sq. inch and a temperature of 539° expands to 6 times its volume without gain or loss of heat; find the pressure and temperature at the end of the expansion. *Ans.* $p=81$, $t=27°$.

13. In the last question suppose the air at the end of the expansion to have a pressure equal to 1½ times that given by the adiabatic law, and heat to be supplied at a uniform rate as the temperature falls; find the index of the expansion curve and the work done during expansion. Compare the heat supplied with the work done and find the specific heat. (See page 575.)

Ans. $n=1·174$. Specific heat = ·228.
Work done = 82,000 ft.-lbs. Ratio = ·575.

14. Find a formula for the useful work done per lb. of steam in thermal units with a vacuum of 1·41 inches of mercury absolute, a back pressure of 1½ and a terminal pressure of 4 lbs. per sq. inch; assuming $x_1 = ·8$, $k = ·8$ (p. 569).

Ans. $W=\frac{4}{5}M-54$.

15. By means of the formula of the preceding question deduce the consumption of steam and the efficiency for the series of pressures stated below.

Boiler Pressure.	Pounds of Steam, per I.H.P. per hour.	Efficiency.
350	9·77	·662
180	11·3	·642
84	13·7	·619
60	15·1	·605
20	22·5	·535

16. Find a formula and deduce numerical results as in the last two questions, assuming a terminal pressure of 8 lbs. per square inch and $k = ·7$, all other data remaining the same. (Compare pages 570, 571.)

EXAMPLES.

THIRD SERIES (SECTION III.).

1. Air is contained in a vessel at a pressure of 25 lbs. per sq. inch and temperature 70°. What will be the velocity with which the air issues into the atmosphere (pressure 15 lbs. per sq. inch)? Also find the discharge and the head.

Ans. $h=13{,}420$; $u=930$ ft. per second.
$W=34\cdot26$ lbs. per sq. inch of orifice per minute.

2. In the last question find the initial pressure corresponding to maximum discharge for all external pressures less than that of the atmosphere. Find this discharge.

Ans. Pressure $=28\cdot5$ lbs. per sq. inch.
Discharge $=39\frac{1}{3}$ lbs. per sq. inch per minute.

3. What weight of steam will be discharged per minute from an orifice 2 inches diameter, the absolute boiler pressure being 120 lbs. per square inch? Co-efficient of discharge ·7. *Ans.* 227 lbs.

4. Air flows through a pipe 6 inches diameter and 4,000 ft. long; the initial pressure is 20 and the final pressure 15 lbs. per sq. inch; temperature 70°; find the velocities and the discharge. $4f=\cdot03$.

Ans. Velocity at entrance $=39$ feet per second.
exit $=52$ feet „
Discharge $=4$ lbs. per sq. ft. $=\cdot78$.

5. In the last question find the loss of head and the H.P. required to keep up the flow. *Ans.* $h'=8{,}124$ feet. H.P. $=11\frac{1}{2}$.

6. Steam at 50 lbs. rushes through a pipe 3 inches diameter and 100 feet long with a velocity at entrance of 100 feet per second; find the loss of pressure. $4f=\cdot03$. *Ans.* 1·6.

REFERENCES.

For descriptive details and illustrations of the mechanism of steam engines the reader is referred amongst other works to

THURSTON. *History of the Growth of the Steam Engine.* International Scientific Series. Kegan Paul.
YEO. *Steam and the Marine Steam Engine.* Macmillan.
RIGG. *Practical Treatise on the Steam Engine.* Spon.

APPENDICES.

APPENDIX.

NOTES AND ADDENDA.

Notes marked [1890] . . . [1900] *have been added on reprinting at the dates mentioned.*

I.—STATICS OF STRUCTURES.

RANKINE'S treatise on *Applied Mechanics* appeared in 1858. The sixth edition is quoted in the following notes by the letters A.M.

PAGE 2. "The word STRESS has been adopted as a general term to comprehend various forces which are exerted between contiguous bodies or parts of bodies, and which are distributed over the surface of contact" (A.M., p. 68). It appears from this that RANKINE'S use of the word is confined to internal forces, but by some writers it is employed for all forces, whether external or internal. Ties and struts are, however, defined as in the text (A.M., p. 132).

PAGE 3. The total load on the platform of a timber bridge carrying an ordinary roadway may be assumed as 250 lbs. per sq. ft., of which 120 represents the weight of a closely packed crowd, and the remainder is the weight of the roadway and platform. The weight of a timber roof (slate or tile) is from 12 to 24 lbs. per sq. ft. The travelling load on railway bridges is commonly estimated at 1 ton per foot-run.

[1900.] On account of the continued increase in the weight of trains, the foregoing estimate of the travelling load on railway bridges has long ceased to be adequate. The weight of heavy locomotives now ranges from 70 to 100 tons and the equivalent uniform load for a span of 70 to 100 feet is estimated by Mr. Farr (*Proc. I.C.E.*, Vol. CXLI., p. 12) at $1\frac{1}{2}$ to 2 tons per foot-run. For short spans it is still greater, the load being concentrated on a limited number of axles.

PAGE 14. The diagram of forces for a funicular polygon under a vertical load was (probably) first given by ROBISON in his treatise on *Mechanical Philosophy*, Vol. I. Dr. Robison died in 1805, and this work is a collection of his papers published in 1822.

PAGE 20. In the Saltash bridge the compression member of each girder is a tube of elliptical section 15 feet in breadth, 8 feet in depth. A pair of chains, one on each side, carry the platform.

PAGE 20. A. Of the various methods of constructing a parabola the most convenient is that in which a curve is drawn through the intersections of a set of lines radiating from a point, with a set of equidistant lines drawn parallel to a fixed line: the radiating lines being drawn so as to cut off equal intercepts on another fixed line. It can easily be proved that this curve is the funicular polygon proper to a uniform load without introducing any properties of the parabola.

[1892.] The foregoing remark applies to purely graphical methods, but it is much simpler to plot the products AK, BK (p. 39) for equidistant positions of K.

PAGE 21. Let P be the vertical tension of the chain at the point P, then, since $dy/dx = P/H$, where H is constant,

$$\frac{d^2y}{dx^2} = \frac{1}{H} \cdot \frac{dP}{dx} = \frac{w}{H}.$$

This equation is equally true if w vary according to any law, and is therefore the general differential equation of a cord or linear arch under any vertical load. Particular cases are:—

(1) *The Common Catenary.* Here if m be the weight of a unit of length of the cord, ds an element of arc,

$$w = m \cdot \frac{ds}{dx} = m\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

the equation then becomes, if $H = m \cdot c$,

$$\frac{d^2y}{dx^2} = \frac{1}{c}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Divide by the right-hand member, multiply by dy/dx, and integrate, then

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{y}{c} \text{ or } \frac{d^2y}{dx^2} = \frac{y}{c^2},$$

an equation which, by integration and a proper determination of the constants, gives for the form of the curve

$$y = \frac{c}{2}\left(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}\right).$$

(2) *The Catenary of Uniform Strength.* Here, if T be the tension of the chain at P,

$$T = m\lambda = w\lambda\frac{dx}{ds},$$

where λ is the length equivalent to the stress (p. 80),

$$\therefore \frac{d^2y}{dx^2} = \frac{T}{H\lambda} \cdot \frac{ds}{dx} = \frac{1}{\lambda} \cdot \left(\frac{ds}{dx}\right)^2.$$

Integrating by the same process as before we find

$$y = \lambda \log_e \sec \frac{x}{\lambda}$$

as the equation to the curve.

In ordinary cases there is very little difference between the catenary and the parabola, and these curves therefore are not of much interest.

If the form of an arch be not such as corresponds to the distribution of the load on it, a horizontal force will be necessary for equilibrium, and the investigation of the magnitude of this force is a problem of wider application. Let p be the intensity of this force per unit of length of a vertical ordinate, then H is no longer constant, but is given by

$$\frac{dH}{dy} = p, \text{ also } \frac{dP}{dx} = w \text{ and } \frac{dy}{dx} = \frac{P}{H},$$

three equations from which p can be found for any distribution of load and form of arch. This is the general problem of the linear arch. For examples see A.M., p. 199. If $p=\text{const.}$, $w=\text{const.}$, we obtain the ellipse as the proper form of arch to sustain the pressure of a great depth of earth.

PAGE 30. On reciprocal diagrams of forces in general the reader is referred to a memoir by CLERK MAXWELL in the *Transactions of the Royal Society of Edinburgh* for 1870.

The notation used in the text was suggested by HENRICI in the course of a discussion on a paper by CROFTON read before the Mathematical Society in 1871. The figure in the text was drawn at the time by the writer to illustrate the method. The notation was afterwards given by BOW in the treatise referred to.

PAGE 33. It is convenient to have a general term for the tendency to separate into parts due to the action of external forces on a structure or part of a structure. The term "straining action" used in the text is taken from Ch. II., Part III., of a treatise on *Shipbuilding* (London, 1866), edited and in great part written by RANKINE. By some writers this tendency to separate would be called "stress," and for a simple thrust or pull there is no objection to doing so (A.M., p. 182). In more complex cases a separate word is preferable, as the conception is very different. (Comp. p. 295.)

PAGE 44. In some of his engines, before the introduction of cast iron, WATT employed a timber beam trussed with iron rods, forming a Warren girder in two divisions with diagonals inclined at about 30° to the horizontal. This is perhaps the earliest example of such a construction. (ROBISON, Vol. II., p. 14.)

PAGE 53. See Plate VIII., p. 463.

PAGE 56. The method here detailed is given by RANKINE in his work on Civil Engineering (p. 242), who ascribes it to LATHAM. If M be the bending moment, F the shearing force, w the load per foot-run, we have the equations

$$\frac{d^2M}{dx^2}=\frac{dF}{dx}=w,$$

which are the symbolical expression of the method. They may be used to find by integration the bending moment and shearing force at any section due to a given load, the constants of integration being found by considering that the bending moment is zero at two points, which must be known if the problem is determinate. (See Art. 38, p. 77.)

PAGE 65. See Ch. II., Part III., of the work on *Shipbuilding*, cited above.

PAGE 66 [1900]. On travelling loads the reader is referred in addition to the works cited at the end of the chapter, to the paper by Mr. Farr already mentioned in the note to p. 3.

PAGE 70. The properties of funicular polygons were first thoroughly investigated by CULMANN, who based upon them a complete system of graphical calculation. In the semi-graphical methods employed in this treatise the integral calculus, trigonometry, and even, to a great extent, algebra, are replaced by geometrical constructions, but arithmetic is still used, and certain steps of the various processes are conducted by numerical calculations." For example, in Ch. II., the supporting forces of a loaded beam are found by the ordinary process of taking moments. In the modern purely graphical methods every step is taken graphically, whatever the calculation be. For example, the displacement of a vessel at a given draught, or her stability at a given angle of heel, would be found without the use of arithmetic.

The pressure of other matter, and the amount of illustration required, have prevented the writer from making any considerable use of these methods in this treatise. At present they can hardly be considered suitable for an elementary work, though, if graphical calculations were introduced into our schools, the case might be different. A full account of them will be found in the treatises referred to in the text (p. 74).

PAGE 72. The property of the funicular polygon expressed by the equation $Hy = M$ follows immediately by comparing the equations

$$\frac{d^2y}{dx} = \frac{w}{H}; \quad \frac{d^2M}{dx^2} =$$

of which one gives the form of the polygon for a given load, and the other the bending moment due to the same load.

Another fundamental property is that any two sides of the polygon must meet on the line of action of the load on that part of the polygon which lies between the two sides. When the load is vertical, and represented by a curve, as in Fig. 36*a*, p. 62, this is equivalent to saying that any two tangents to the curve of moments must intersect on the vertical through the centre of gravity of the area of the curve of loads between the corresponding ordinates. (See p. 328.)

The funicular polygon, considered as a line of transmission of stress, will be again referred to in the notes to Ch. XVII.

PAGE 79. The theory of *linear* arches is merely an introduction to the theory of arches in general. Arches are of two kinds—(1) the stone or brick arch; (2) the metallic arch. In either case the theorem of the text is of equal importance. In a blockwork arch the linear arch corresponding to the load shows the direction and position of the resultant of the mutual action between the blocks, and must therefore (p. 331) fall within the middle third of the arch ring. (A.M., p. 258.)

PAGE 82. See CLERK MAXWELL's memoir referred to above (p. 599).

PAGE 86. For the effects of changes of temperature, see Ch. XVIII., p. 460.

PAGE 88. One of the most remarkable suspension bridges which have been constructed is the East River Bridge at New York, opened in May, 1883. The principal opening of this bridge is 1,600 feet span, the platform 85 feet wide, and 135 feet above the water. Cables, four in number, each of 145 square inches *net* area, constructed of 19 steel wire ropes, each containing 278 wires. Estimated strength of wire, 170,000 lbs. per square inch.

II.—KINEMATICS OF MACHINES.

PAGE 93. Referring to Figs. 1, 2, Plate II., p. 111, it seems clear that the sector pair *CD*, Fig. 1, differs kinematically much more from the turning pair *BA* than it does from the sliding pair *CD* of Fig. 2. The writer, therefore, would have been disposed to classify the three lower pairs as the "oscillating pair," the "turning pair," and the "screw pair." This, however, would have probably involved more considerable alterations in REULEAUX's nomenclature than would have been justified in a general elementary treatise.

[1892.] Turning pairs are not unfrequently distinguished into lever pairs and rank pairs as in the classification of crank chains (p. 112).

[1900.] In a paper read before the Royal Society in 1895, and published in heir *Transactions* (Vol. 187, p. 15), Professor Hearson proposes a classification of nechanisms based on the combination of "turning," "swinging," and "sliding" notions. A new notation is introduced of a very simple and expressive character, which may probably be found of great service in descriptive mechanism. It is pointed out that the three-slide or "wedge" chain (p. 111) may be bent into a cylinder and then becomes a screw. Other important changes are proposed in the nomenclature and methods employed by REULEAUX, which are worthy of careful attention. Had the author been writing a new book advantage would have been taken of these researches. It may be hoped that Professor Hearson may pursue a subject which has been far too much neglected, and the study of which, in the author's judgment, may be expected to lead to important results.

PAGE 95. The three incomplete lower pairs are considered by REULEAUX as higher pairs. The writer here follows GRASHOF (*Theoretische Maschinen-Lehre*, Band II.).

PAGE 100. Diagrams of velocity are considered generally by CLERK MAXWELL (*Matter and Motion*, p. 28). The application to mechanism is, so far as the writer is aware, new.

The construction of curves of position and velocity of a piston has, for many years past, formed a regular part of the course of instruction at Greenwich, and formerly at South Kensington.

PAGE 103 [1895]. In a letter which appeared in *Engineering* of June 14th, Mr. Archibald Sharp calls attention to a construction for the acceleration of a piston due to Professor Klein of Lehigh University, U.S.A. This construction, published in 1891, escaped the author's notice when revising this book in 1892; it is much simpler and more useful than that given in the text. (Ex. 11.)

Referring to Fig. 48, page 100, imagine a circle described on DP, the connecting rod, as diameter and a second circle with centre P the crank pin, and radius PT. Let EE be the points of intersection of the two circles, then the chord EE, produced if necessary, will cut the line of centres BDO in a point Z, such that when the crank rotates uniformly

$$\frac{OZ}{OP} = \frac{\text{Acceleration of Piston.}}{\text{Acceleration of Crank Pin.}}$$

For if the chord EE cut the rod DP in N, and OM be drawn perpendicular to PT to meet PT in M,

$$NM = OZ \,.\, \cos\phi.$$

Now if OP represent the acceleration of the crank pin P, PM will represent the resolved part of that acceleration in the direction of the rod DP; and if f be the acceleration of the cross head D, $f \,.\, \cos\phi$ will be the resolved part of that acceleration along DP; and therefore

$$f \,.\, \cos\phi - PM = \text{Length of rod} \times (\text{Ang. Vel.})^2,$$

in which equation the Ang. Vel. is that of the connecting rod when the angular velocity of the crank is supposed unity. But the angular velocity-ratio of the rod and crank is, as is well known, PT/PD, and considering the circle on PD as diameter,

$$PN \, . \, PD = PE^2 = PT^2;$$

whence it is clear that

$$f \, . \cos\phi - PM = PN,$$

and

$$f = OZ.$$

[1900.] Referring again to the figure, and employing the usual notation as on page 100,

$$\frac{PT}{PD} = \frac{PT}{PO} \cdot \frac{PO}{PD} = \frac{\cos\theta}{\cos\phi} \cdot \frac{\sin\phi}{\sin\theta} = \frac{1}{n} \cdot \frac{\cos\theta}{\cos\phi}$$

and

$$PM = PO \, . \cos(\theta + \phi);$$

therefore, supposing the acceleration of the crank pin unity so that f is the acceleration-ratio,

$$f \cos\phi - \cos(\theta + \phi) = \frac{1}{n} \cdot \frac{\cos^2\theta}{\cos^2\phi};$$

$$\therefore \; f = \cos\theta - \frac{1}{n} \cdot \frac{\sin^2\theta}{\cos\phi} + \frac{1}{n} \cdot \frac{\cos^2\theta}{\cos^3\phi}.$$

This is the exact formula for piston acceleration which may also be obtained by differentiation of the exact formula for the velocity. It can be put in several different forms, of which the present appears the simplest. It is arrived at in a different way on p. 229. Since

$$\cos\phi = \sqrt{1 - \frac{1}{n^2}\sin^2\theta},$$

an approximation can be obtained by the binomial theorem in powers of $1/n$. The first three terms of the series give as a second approximation

$$f = \cos\theta + \frac{1}{n}\cos 2\theta + \frac{\cos 2\theta - \cos 4\theta}{4n^3}.$$

Since $4n^3$ would rarely be less than 100 in any practical case it is clear that for all ordinary purposes the first approximation obtained by supposing $\cos\phi$ unity is amply sufficient, a point we shall have occasion to mention hereafter. (See note to p. 396.)

Page 105. In Owen's air compressor two such mechanisms (Fig. 50) are placed face to face with the guide A and block D common, a steam piston is connected with d and the air-pump piston with the corresponding point d of the other mechanism. The object is to adapt the pressure of the steam to the varying pressure of the air during compression.

Page 108. Stannah's pump has been introduced since the publication of Reuleaux's work. The example there given is a mechanism used in the polishing of specula.

Page 111. The double-slider mechanism, with sliding pairs and turning pairs alternating, is common in collections of mechanisms, but is not often found in practice. It is omitted in Reuleaux's enumeration. The example given (Rapson's slide) and Stannah's pump were pointed out to the writer by Mr. (now Prof.) Hearson.

Page 120 [1892]. This article (Art. 56) was numbered 53 in former editions and placed earlier (p. 112). A new article (55, pages 118-120) has been added, partly in order to introduce the conception of an instantaneous centre at once instead of postponing it to Ch. VII., and partly to explain the connection between diagrams of velocity such as are here considered, and graphical methods in common use based directly on the properties of the centre.

PAGE 161. The propositions relating to centrodes have long been known, and are, perhaps, stated as clearly by BELANGER in his excellent treatise on kinematics (*Traité de Kinématique*, Paris, 1864) as by REULEAUX himself. In the author's opinion it is the conception of a kinematic chain which constitutes REULEAUX's great contribution to the theory of mechanism. It is virtually a complete reconstruction of the whole theory of machines, while the centrodes are only a method of stating results which was already known. Kinematic formulæ such as are employed by REULEAUX to indicate the component elements of a mechanism, in the same manner as a chemical formula shows the composition of a substance, may be regarded as indispensable, if it be attempted to proceed with the study of descriptive mechanism. (See above Note to p. 93.)

[1892.] In the first edition of this work the word *centrode* was spelt *centroid* —a term now very generally appropriated to the centre of mass of a body.

PAGE 170. The author has ventured on the introduction of the terms "driving pair," "working pair." They are simply the natural adaptation of the well-known phrases "driving point" and "working point" to REULEAUX's theory.

PAGE 173. The term "multiple chains" has also been introduced by the author.

III.—DYNAMICS OF MACHINES.

The impossibility of a perpetual motion and the practical application of the principle of work were well understood by SMEATON and others of our great engineers of the last century. Smeaton's papers, read before the Royal Society in 1759-82, were long regarded as an engineering text-book by his successors. The language in which their ideas are expressed, however, were not regarded as consistent with NEWTON's teaching, and this circumstance perhaps concealed the real importance of the ideas themselves. At any rate, although the term "energy" was proposed by YOUNG, no considerable use was made of them by students of mechanical science until the publication by PONCELET, in 1829, of the *Introduction à la Mécanique Industrielle*, a work which has had a great influence on the study of mechanics. The third edition of this work (Paris, 1870), published after PONCELET's death, will be quoted by the abbreviation *Méc. Ind.* Poncelet's methods were explained, and considerable additions made to the theory of machines, by MOSELEY in his *Mechanical Principles of Engineering* (London, 1843).

PAGE 182. This method was probably employed for the first time by WATT in his expansion diagram. See ROBISON's *Mechanical Philosophy*, Vol. II. It is given by PONCELET (*Méc. Ind.*, p. 66).

PAGE 184. The terms "statical" stability, "dynamical" stability, in relation to vessels, were introduced by MOSELEY (*Phil. Trans.*, 1850). They have been criticized by OSBORNE REYNOLDS, perhaps not without justice, but are too firmly rooted to be displaced.

PAGE 185. "Force is an action between two bodies, either causing or tending to cause, change in their relative rest or motion" (A.M., p. 15). The distinction between internal work and external work is due to PONCELET (*Méc. Ind.*, p. 30).

PAGE 187. The language in which writers on mechanics have expressed the distinctive character of frictional resistances has been severely criticized by REULEAUX in his notes to his work on the Kinematics of Machines (*Kennedy's translation*, p. 595). The author by no means supposes that he can escape this universal censure, for the difficulty of expressing abstract principles in a form to which no objection can be made is almost insuperable. As, however, CLERK MAXWELL remarks in reference to a different question, the language in which a truth may be expressed is less important than the truth itself. Friction always causes energy to disappear, and is never a source of mechanical energy except indirectly through the agency of thermal energy. In mechanics this is a distinction of such fundamental importance that it even justifies, in the author's opinion, the use of such phrases as "loss of energy."

The extension of the term "reversible" from a machine to the resistances which are overcome by the machine has been ventured on, though with some hesitation. The old term "active" can hardly be considered suitable.

PAGE 187. "Envisagé sous ce point de vue, le principe de la transmission du travail comprend implicitement toutes les lois de l'action réciproque des forces, sous un énoncé qui en facilite infiniment les applications à la Mécanique industrielle, qu'on pourrait nommer la *Science du travail* des forces. Dès le premier pas des jeunes élèves dans l'étude, cet énonce, en effet, se présente à eux comme une sorte d'axiome évident par lui-même, et donc la demonstration leur semble superflue aussitôt qu'ils ont bien saisi ce qu'on entend par *travail mécanique*, et qu'il leur est clairement démontre que ce travail, reduit en unités d'une certaine espèce est dans les arts, l'expression vraie de l'activité des forces" (*Méc. Ind.*, p. 3). This passage from PONCELET is quoted to show how clearly it was seen, even before the discovery of the conservation of energy in its complete form, that the principle of work ought to be regarded as fundamental, and not merely as a deduction from certain equations.

[1895.] In his interesting work on the development of dynamics, Professor Mach traces the ideas of PONCELET to HUYGHENS, and expresses his conviction that the difficulties which the conception of Work encountered were due to unimportant historical circumstances. The development of dynamical science might have proceeded on different lines, and the Principle of Work might have been regarded as fundamental at a much earlier date. There can be on question that it was a great misfortune to engineering science that such was not the case. The absence of due recognition of a principle which is actually forced on all those engaged in mechanical operations was the principal cause of the difference which for so long a period existed, and still does exist to some extent between the mechanics of the engineer and the mechanics of the schools.

An American translation of the book here referred to (*Die Mechanik in ihrer Entwickelung*, Prague, 1883,) appeared at Chicago in 1893 under the title of *The Science of Mechanics* (London: Watts & Co.). See especially pages 178, 248-251, 272.

PAGE 189. The modifications made here in the old statement of the principle of work, as applied to machines, are necessary consequences of REULEAUX's conception of a kinematic chain.

PAGE 193 [1892]. The author has little faith in the utility of "definitions" as applied to such a conception as that of a machine, and the remarks here made —which have been somewhat amplified in the present edition—must not be understood as an attempt at constructing one.

PAGE 194 [1895]. A living agent works to best advantage when exerting a certain effort at a certain speed during a day's work of a certain length. The best effort and speed depend obviously on the strength and training of the individual as well as on the kind of work done. Some examples may here be given for a day's work of 8 hours.

	EFFORT. (Lbs.)	SPEED. (Feet per 1'.)	POWER. (Ft.-Lbs. per 1'.)
MAN.—Without a Machine, . .	33	160	5280
Working a Crank, . .	22		3520
HORSE.—Direct Traction, . .	130	250	32,500
Working a Machine, .	100	160	16,000

It will be observed that the power of a horse on this estimate when directly employed is little less than the conventional horse power of 33,000 ft.-lbs. per minute introduced by WATT, it is said, as the result of experiments on the work done by the powerful dray horses employed in London breweries. RANKINE's estimate (*Steam Engine and other Prime Movers*, page 89,) of the power of an ordinary horse is much less, being 26,000 ft.-lbs. per 1', and he also gives smaller values for the work of men.

When working at best speed the power during a whole day's work is a maximum, but all living agents can work at a much greater speed for a short interval developing from 3 to 5 times as much power as when the work is continuous. Thus strong men working a fire engine at two-minute intervals can develop half-a-horse power or even more. Empirical formulæ have been constructed showing the relation between the power exerted at given effort and speed for a given time with that developed under the most favourable circumstances, but they can hardly be considered as satisfactory.

PAGE 196. To avoid misapprehension, it may here be stated that in this, as much as in the preceding section, the object is to explain and to verify the principle of work: not in any sense to demonstrate it.

PAGE 198. Except in the use of the word "kinetic" instead of "actual," the statement here is in the form given by RANKINE (A.M., p. 500). The author is entirely of (the late) Mr. W. R. Browne's opinion that this is the best form and has always used it himself. The idea of energy being stored in a body in motion perhaps first appears clearly in MOSELEY'S treatise.

PAGES 203-207 [1892]. Art. 103 on oscillations has been re-written with additions. The formula for the length of the simple equivalent pendulum on page 206 was printed incorrectly in the first edition—an error corrected in the second. Various other changes and additions have been introduced in the second half of this chapter for the sake of clearness and to make it harmonise better with the rest of the book. The infinite series, by which the time of vibration of a pendulum is given, will be found in most treatises on the kinetics of a particle. See, for example, Price's *Infinitesimal Calculus*, Vol. III., p. 549.

PAGE 212. The construction by means of which curves of crank effort are obtained was given by PONCELET, but it does not appear that any such curves were actually drawn until they were given by ARMENGAUD in his treatise on the Steam Engine. In Fig 97, to save room, the curves are placed half above and half below the base, but otherwise the figure is that of ARMENGAUD; it is far the most convenient form for applications.

PAGE 220. The stress due to centrifugal action on the rim of a wheel is given by a formula (p. 288) which may be written in the simple form $V^2=g\lambda$, where λ is the length due to the stress (p. 80). A velocity of 80 feet per second gives a length of only 200 feet, or about one-fifth of the stress cast iron would safely bear in tension. The inequality of distribution produced by inextensible arms tying together opposite points on the rim of the wheel doubtless increases the maximum stress; but the principal reason for the low limit required for safety is the alternate bending backwards and forwards of the arms as energy is alternately stored and restored by the wheel. The speed is occasionally increased to 100 feet per second. The author is indebted to Prof. Unwin for the information that when the wheel is in segments the speed should be limited to 40 feet per second.

PAGE 222. The method here given occurred to the author many years back; but it is believed to have been previously published in *Engineering*.

PAGE 226 [1900]. As regards the effect of the varying speed of the crank on the inertia-pressure of a piston, it will be seen on reference to page 102 that the part of the piston acceleration due to this cause is

$$f_1=\left(\sin\theta+\frac{1}{2n}\sin 2\theta\right)\frac{dV_0}{dt},$$

while the part considered in this article is

$$f_2=\left(\cos\theta+\frac{1}{n}\cos 2\theta\right)\frac{V_0^2}{a}.$$

Evidently the average relative importance of these parts is measured by the ratio of dV_0/dt, the tangential acceleration of the crank pin to V_0^2/a the radial acceleration. If now we write $q\,.\,V_0$ for ΔV where q is a co-efficient as on page 220, and $m\,.\,T_0$ for Δt where T_0 is the period and m a fraction, the mean value of dV_0/dt for the change considered will be

$$\frac{dV_0}{dt}=\frac{q}{m}\cdot\frac{V_0}{T_0}=\frac{q}{m}\cdot\frac{V_0^2}{2\pi a},$$

and therefore

$$\text{Ratio of accelerations}=\frac{q}{2\pi m}.$$

A change of 10 per cent. in the speed taking place in one-tenth of a revolution, or of 25 per cent. in one quarter of a revolution, would make q/m unity, and it is evident that when an engine is running steadily this is an extreme case. In general the value of q/m must be much less. It is true that the maximum value of dV_0/dt is greater than the mean, and under certain conditions it may be very much greater; but on the whole it may be said that the average error of the supposition that the crank rotates uniformly is not of much importance. But if the maximum dV_0/dt occurs near the middle of the stroke, as it often will, where f_2 is small and f_1 greatest, the inertia-curve for that part of the stroke will be completely altered.

PAGE 225 [1892]. The investigation here given of the effect of the angular motion of a connecting rod has been added to the present (1892) edition. Art. 111 (page 231) on pumping engines has also been added to render the chapter less incomplete.

PAGE 229 [1900]. The investigation of the exact value of the inertia-pressure of a connecting rod is given now for the first time. On the representation of a curve of crank effort by a Fourier series, see Note to p. 395.

PAGE 240. The Friction Circle was defined and its use explained by RANKINE in his treatise on *Millwork and Machinery*, p. 428.

PAGE 265 [1895]. The adoption of metric measures in engineering practice was recommended by a committee of engineers 40 years ago, and in 1868 a Bill passed its second reading in the House of Commons by a large majority for rendering it compulsory within three years. In June of the present year a committee of the House has reported in favour of its compulsory introduction in two years. The inconvenience of a change to the present generation of engineers would be very great, but it is probable, even in the absence of compulsion, that the pressure of foreign competition may render it inevitable before very long. From the point of view of abstract science this change, however, is only part of that which is desirable or even necessary: for the metric system just as much as our own is a *gravitation* system of measurement, that is the unit of force instead of being derived from the unit quantity of matter with due regard to the units of time and space is taken as the force with which the unit quantity of matter is drawn to the ground at a given point on the earth's surface.

As explained in the text this renders it necessary to dissociate the unit of inertia from the unit quantity of matter, and to use the words "weight," "pound," "kilogramme," etc., in a double sense since they are applied indiscriminately to forces and to quantities of matter. Most modern writers on mechanics when using gravitation measure seek to avoid ambiguity as regards the term "weight" by confining the use of the word to the force of gravitation and employing the term "mass" to signify the quantity of matter determined by weighing as well as the inertia measured by the quotient W/g. It may be questioned whether the ambiguity thus introduced is not more misleading than the original, and the term "weight" has therefore been used in its old meaning throughout this work. It might be avoided perhaps by calling the quotient W/g the "Inertia" of the body, but the author has not felt at liberty to introduce a new term in this connection.

In the absolute system of measurement the unit of force is dissociated from the unit quantity of matter, and so taken that the units of inertia and quantity of matter become identical. To do this it is only necessary to take as a unit of force the force necessary to generate unit velocity in unit time in the unit quantity of matter. A special name is then given to the unit of force which is now entirely independent of gravitation. In the C.G.S. system this unit is called the Dyne.

The system possesses undoubted advantages on the score of clearness and precision, but the practical difficulty of dissociating the unit of force from the unit of quantity of matter would be very great in any case. In the C.G.S. system the difficulty is greatly aggravated by the smallness of the units chosen, the force called 1 kilogramme in gravitation metric measure being no less than 981,000 dynes. *No force commonly occurring in practice therefore can be expressed in dynes without multiplication by some large power of 10 subject to a great liability to error in the index. It is probable, however, that some modified form of the C.G.S. system may ultimately be found which is* capable of being practically used. Through the agency of the electrical engineer some of its nomenclature is becoming well known.

PAGE 268. The distinction between internal and external kinetic energy is pointed out by RANKINE (A.M., p. 508).

PAGE 271. On Governors in general the reader is referred to a paper by CLERK MAXWELL in the *Proceedings of the Royal Society*, No. 100, 1868. A full account of the principles of construction of centrifugal regulators will be found in *Theoretische Maschinen-Lehre*, Band III., Leipzig, 1879, von. Dr. F. Grashof.

PAGE 280 [1900]. The internal balance of forces in machines at rest considered in Art. 142 is only complete in machines which are actually or virtually self-contained (compare p. 193). In motors or machines driven from a motor the balance is rarely perfect because the connection between the driving pair and the working pair is not sufficient to completely close the circuit. Thus in the vertical engine taken as an example, suppose the engine to be employed in driving a screw; the resistance to the rotation of the screw arises from water outside the vessel and disconnected from it: the moment of crank effort is therefore unbalanced and tends to heel the vessel over: while any variation in that moment in the course of the revolution will furnish a periodic couple tending to produce vibration.

PAGE 284. The utility of balance weights, sufficiently heavy to neutralize completely the horizontal forces, is by no means universally admitted. The vertical forces introduced are very great (Ex. 17, p. 291), and, should they synchronize with the period of vertical oscillation of the engine on its springs, most dangerous results might follow.

PAGE 285 [1900]. The investigation given in Art. 144A of the effect of the inertia of a connecting rod on the primary and secondary inertia-forces has been added to this edition.

PAGE 288 [1900]. As a simple example of a self-balanced engine may be taken the case of a 3-cylinder engine: one cylinder *A* being between the other two, B_1 and B_2, but on the opposite side of the crank shaft. The centre lines of the three cylinders are all in the same plane, but crank *A* is at 180° to cranks B_1, B_2. For simplicity let the cranks B_1, B_2 be of equal length, but in any proportion to crank *A*: the ratio of connecting rod to crank being the same in all. Evidently in such an arrangement the rods will always be parallel and all the reciprocating parts similarly situated. If they be similar and the weights for *A* and B_1, B_2 jointly, be inversely as the cranks; the centre of gravity of the whole must always lie in the axis of the shaft, and there can be no alternating forces. And if the weights of B_1, B_2 are inversely as the distances apart of the cranks there can be no alternating couples. The engine therefore is completely balanced so far as regards the *inertia* forces, and would commonly be described as "self-balanced," but the balance does not include the forces mentioned in the note to p. 280.

IV.—STIFFNESS AND STRENGTH OF MATERIALS.

Page 317 [1892].* The calculation here given (Art. 161) relating to beams of uniform strength, presupposes that the transverse section of the beam varies slowly, a condition which of course is very far from being satisfied near the ends of the beam. Any attempt to take into account the variation of section, would, however, only lead to results of much greater complexity without any corresponding increase in their utility. The forms obtained are purely ideal, being incapable of being practically used without the addition of material necessary to provide against straining actions other than bending.

Page 332. If an elastic solid or, more generally, a set of connected pieces of perfectly elastic material, be under the action of any number of forces P_1, P_2, ..., and any number of couples M_1, M_2, ..., in equilibrium, the value of U must be

$$U=\tfrac{1}{2}\Sigma Px+\tfrac{1}{2}\Sigma Mi,$$

where x_1, x_2, ..., are the displacements of the points of application of the forces and i_1, i_2, ..., the angular displacements of the arms of the couples. For if the forces gradually increase from zero, always remaining distributed in the same way, each part of the load (P) will exert the energy $\frac{1}{2}Px$, since the space moved through (x) must clearly be proportional to P. The same argument applies *mutatis mutandis* to couples. Hence the whole energy exerted must be given by the above formula, and this is always represented by the energy stored up in the system when the parts are perfectly elastic.

Now, imagine the solid immoveably fixed at three or more points, and let one of the forces P_1 be increased by a small quantity δP_1, all the other forces retaining their original magnitudes. The effect of this is that the points of application of all the forces move through certain small spaces (δx), and the arms of all the couples through certain small angles (δi). The total additional work done will be

$$\delta U=\Sigma P\delta x+\Sigma M\delta i.$$

But, on differentiating the value of U on the supposition that P_1 alone varies, we find

$$2\delta U=\Sigma P\delta x+\Sigma M\delta i+x_1\,.\,\delta P_1,$$

and therefore by substitution

$$\delta U=x_1\,.\,\delta P_1.$$

A similar equation is derived by supposing one of the couples to vary, and we obtain the general equations

$$\frac{dU}{dP}=x;\quad \frac{dU}{dM}=i,$$

that is, the displacements are the partial differential co-efficients of U with respect to the forces.

The forces to be considered are partly weights or other loads of known magnitude, and partly arise from the stress between the bounding surfaces (real or ideal) of the solid and external bodies. The boundary forces must be consistent with statical equilibrium, but subject to this condition are determined by equations found by differentiating the function U. In particular, when the bounding surface is fixed, the partial differential co-efficients of U with respect to the corresponding forces must be zero. The value of U is then, in most cases (perhaps always), a minimum, as stated in the text.

It appears then that whenever the elastic potential can be found and expressed in terms of the external and boundary forces acting on the system, the necessary equations for determining the boundary forces and the deflection produced by the external forces can all be found by differentiation of U and by the conditions of statical equilibrium. As an example, take the case of a beam loaded in any way and fixed at the ends. Let the beam be AB (Fig. 28), and let the notation be as on pages 40, 41, then (page 331)

$$U=\int\frac{M^2}{2EI}dx.$$

Substitute for M by the formula on page 41, and integrate between the limits l and o, we find

$$2EI\,.\,U=\tfrac{1}{3}(M_A^2+M_AM_B+M_B^2)l$$
$$+\int_0^l m^2dx+\frac{2M_A}{l}\,.\int_0^l m(l-x)dx+\frac{2M_B}{l}\,.\int_0^l mx\,dx.$$

The integrals are most conveniently expressed in terms of, S the area of the curve of moments (m), z the distance of its centre of gravity from A, and $\bar{y}$ the height of its centre of gravity above AB. The formula then becomes, dividing by 2,

$$EI\,.\,U=\tfrac{1}{6}(M_A^2+M_AM_B+M_B^2)l+S\frac{M_A(l-z)+M_Bz}{l^2}+S\bar{y}.$$

The potential is thus expressed in terms of the load on the beam and the bending moments at its ends. The latter may have any values we please consistently with statical equilibrium, and the partial differential co-efficients of U with respect to M_AM_B will be the slopes at the ends. In particular, if the ends are fixed horizontally,

$$2M_A+M_B+6\frac{l-z}{l^2}\,.\,S=0,$$

$$2M_B+M_A+6\frac{z}{l^2}\,.\,S=0,$$

equations which determine M_AM_B, and express that the function U is then a minimum. In the particular case of a symmetrical load

$$M_A=M_B=-\frac{S}{l}.$$

The value given on page 326 for the particular case of a uniform load will be found to agree with this result.

The potential for a continuous beam may be immediately deduced, by addition of the potentials for each span taken separately, in terms of the bending moments at the points of support. The theorem of three moments (page 330) for the case of supports on the same level, then follows at once by differentiating with respect to the moment at the middle point of support.

In all cases, differentiation of U with respect to any portion of the external load will give the deflection at the point where that load is applied.

In applying this method care must be taken that the supporting forces, in terms of which the potential is expressed, are independent: if they are not, then the equations of statical equilibrium will be conditions subject to which U will be a minimum. To take a simple example, suppose a perfectly rigid four-legged table standing on four similar elastic supports and loaded in any way, then

$$U=n(P_1^2+P_2^2+P_3^2+P_4^2),$$

where P_1, P_2, P_3, P_4 are the parts of the whole load resting on each leg, and n is some multiplier. Here the forces P are partly determined by three statical equations for equilibrium of the table, and only one additional equation is found by making U a minimum.

This method was explained and applied to a number of examples in some paper by the author, which appeared in the *Philosophical Magazine* for 1865; the demonstrations there given, however, were insufficient. The author at that time supposed it to be new, but it had already been given in a memoir by M. L. F. MÉNABRÉA. *Comptes Rendus*, vol. xlvi. (1858), page 1056.

PAGE 338. The lateral disturbance is here supposed small. With a larger disturbance the pillar would return even if the value of W were equal to $2EI/l^2$, and with a greater value would bend over into a position of equilibrium given by the formula

$$W = \left(\frac{\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\right)^2 \cdot \frac{2EI}{l^2},$$

where θ is the angle subtended by the circular arc into which the pillar is bent.

PAGE 340. When the pillar is absolutely straight and homogeneous and of uniform transverse section, the lateral deflection due to an actual deviation a is given by the formula

$$a + \delta = \frac{a}{\cos ml} = a \cdot \sec\frac{\pi}{2}\sqrt{\frac{p}{p_0}},$$

and the formula for the effect of deviation becomes

$$\left(\frac{f}{p} - 1\right)\cos\frac{\pi}{2}\sqrt{\frac{p}{p_0}} = \frac{qa}{nh}.$$

In any actual example, however, this formula would not be exact any more than that given in the text. Each particular example will have its own formula. The result of all such formulas, however, must be nearly the same for a small deviation. Further, a great proportional change in the deviation, always supposing it small, produces little change in the crushing load, and this probably explains why experiment gives tolerably definite values of the crushing load although its precise amount must depend on accidental circumstances.

PAGE 339 [1890]. The method here adopted of proving EULER'S formulæ is to assume the curve in which the pillar bends to be a curve of sines, and then to show that the sectional area is constant, a process which is the converse of that employed in the first edition of this work. Being more simple, the demonstration has been placed in the text instead of being relegated as before to the Appendix. It is worth remarking that the co-efficient of elasticity of flexion employed in them is not, strictly speaking, Young's modulus (E) but $E - p$ where p is the intensity of the stress on the cross section. This, though theoretically interesting, is of no importance in practice, because of the extreme smallness of the ratio p/E in all practical cases. The case where one end of the pillar is fixed and one rounded was first, it is believed, correctly treated by GRASHOF in 1866.

In a paper published in the *Proceedings of the Cambridge Philosophical Society*, Vol. IV., Part II., GREENHILL has determined the greatest height of a vertical pole which is consistent with stability, that is the greatest length of pole of given diameter which will stand upright without bending over laterally at the summit. Let $\lambda = E/w$ be the length due to a stress E for a given material, E being as usual Young's modulus. This method of measuring a stress is explained on page 80. Then for a pole of uniform transverse section of radius a the greatest height is given by the simple formula

$$h = 1\cdot 26\sqrt[3]{\lambda a^2}.$$

in which all the quantities are given in the same units. For a pole of radius a at the base diminishing in section uniformly to zero at the summit so that the longitudinal section is triangular the same formula serves, but the co-efficient is $\sqrt[3]{7 \cdot 63}$, or 1·97 instead of 1·26. For a pole of pinewood 6' diameter at the base the greatest height is about 90 feet in the first case and 140 in the second. The stress (wh) on the transverse section at the base must, as appears from what is said in the text, be much less than the crushing stress (f) of the material, a condition which would generally be satisfied. These formulæ are of considerable theoretical interest and are applied in the paper cited to questions relating to the growth of trees: it must be remembered, however, that the effect of wind pressure is neglected, a circumstance which limits considerably the practical application of the formulæ.

PAGE 347 [1890]. A formula, corresponding to EULER'S formula for pillars, has been obtained for the collapse of a flue of unlimited length, by LÉVY and HALPHEN. This formula as quoted by GREENHILL in a letter which will be found in the *Engineer* for February 1888 is

$$p = \frac{E}{4}\left(\frac{t}{a}\right)^3,$$

where t is the thickness, a the radius, both reckoned in inches, while E as usual is Young's modulus, and p the collapsing pressure. The corresponding form of Gordon's formula deduced as on page 340 will be

$$p = \frac{f \cdot \frac{t}{a}}{1 + \frac{a^2}{ct^2}},$$

where the "theoretical" value of the constant c is

$$c = \frac{E}{4f}.$$

This formula may be expected, with suitable values of the constants, to give the collapsing pressure of a flue, the length of which is so great as to have no sensible influence on its strength. In short lengths the strength is greater as described in the text.

Thrust and Torsion.—When it is a question of strength only this case is dealt with on the principles explained in Chapter XVII. GREENHILL has, however, pointed out that, if the unsupported length of the shaft be too great, it is necessary to consider its stability. Let P be the end thrust on the shaft, T the twisting moment, then the greatest unsupported length consistent with stability is given by the formula

$$\frac{\pi^2}{l^2} = \frac{P}{EI} + \frac{T^2}{4E^2I^2}.$$

Let P_0 be the greatest load which by EULER'S formula this length of shaft would carry considered as a pillar, and let T_0 be the greatest twisting moment consistent with strength, f being the co-efficient of resistance to twisting, then the formula may be written, supposing d the diameter,

$$P_0 = P + \frac{T^2}{T_0^2} \cdot \frac{f}{E} \cdot \frac{\pi}{16} f \cdot d^2,$$

or using p_0, p to represent the stress per square inch of section

$$p_0 = p + \frac{1}{4} \cdot f \cdot \frac{T^2}{T_0^2} \cdot \frac{f}{E}$$

As f/E is necessarily a very small fraction and T/T_0 is fractional this shows that here can be very little difference between p and p_0, so that unless these quantities hemselves be small, that is the unsupported length of the shaft very great, the wisting makes no sensible difference in the stability of the shaft. The formula, herefore, in ordinary cases, though theoretically interesting, is not of practical value. It will be found with numerical applications in a paper read by Professor Greenhill before the *Institution of Mechanical Engineers* in 1883.

PAGE 358 [1900]. The formula here given for the torsion of a tube of non-circular section has been inserted in the present edition chiefly in order to give some idea of the reason why the maximum stress in shafts generally occurs at points on the circumference nearest the centre, and as leading directly to a formula for the torsion of shafts which is exact for elliptic sections. It is not probable that it is exact for any other form of section, and in cases like that of a rail where the curvature of the profile is partly concave it can hardly be even approximate.

The corresponding formula for the rigidity of a tube under torsion is given on page 422, and also appears now for the first time.

PAGE 360. The formulæ given in different books for the moment of resistance of a shaft of rectangular section exhibit considerable discrepancies. COULOMB, to whom the formula for a circular section is due, supposed that in every case

$$T = f_s \cdot \frac{I}{r_1},$$

where I is the polar moment of inertia and r_1 is the outside radius. In a rectangular section of sides a and b this gives

$$T = \tfrac{1}{6} f_s ab \sqrt{a^2 + b^2},$$

which for a square section of side b becomes

$$T = \cdot 2357 f_s \cdot b^3$$

If these results were correct it would appear that a shaft of given sectional area was stronger the more unequal the sides were, a result quite contrary to experience. In a memoir on torsion published in the *Mémoires de l'Institut* for 1856, BARRÉ DE SAINT VENANT investigated the question thoroughly, and obtained the results given in the text.

RANKINE (A.M., page 358) gives $\cdot 281 f b^3$ as the result of SAINT VENANT'S calculations without further explanation. This value is greater than that given by COULOMB'S hypothesis, and is certainly too large.

[1890, 1892.] That the resistance of a shaft to torsion was not proportional to the ratio I/r had long been recognized, and prior to the acceptance of ST. VENANT'S results the formula

$$T = \tfrac{1}{3} f_s \cdot \frac{b^2 d^2}{\sqrt{b^2 + d^2}}$$

originally given by CAUCHY was much employed. This formula agrees with COULOMB for a square section, the result being about 11 per cent. greater than that given by ST. VENANT. The error diminishes as the inequality of the sides increases and vanishes when the ratio (n) of the sides is very small. The corresponding ratio of strengths of a rectangular and a circular section of the same area is

$$\frac{T}{T_0} = \cdot 8353\sqrt{\frac{2}{n + \frac{1}{n}}},$$

a result given in former editions of this work after modification, by replacing the constant factor ·8353 by ·738, so as to make it agree with St. Venant for square sections. The error is then very small for moderate values of β; but in the present [1892] edition it has been thought advisable to give St. Venant's own formula, the maximum error of which is estimated by him as 4 per cent.

[1895.] The error of St. Venant's formula in practical cases may be diminished, as pointed out in the text, by a slight modification of the constants. The formula in common use by the best German technical writers at the present time appears to be

$$T = \frac{2}{9} fab^2,$$

which for values of β greater than ·5 may be considered as a rough approximation, but for small values of β gives values of T which are much too small. The investigation given by Bach in his treatise referred to further on is unsatisfactory as the distribution of stress assumed is not shown to be consistent with the corresponding warping of the section. From a letter in *Nature* (June 1888) by Mr. Dewar it appears that Rankine's value was obtained by taking the angle of torsion as a measure of the strain produced.

Page 367 [1892]. The formula for the distribution of shearing stress on a section has in this [1892] edition been put in a more simple and general form, and its true interpretation pointed out. See also Note to page 416 further on. In an excellent paper, which will be found in the *Transactions of the Institution of Naval Architects* for 1890, the late Professor P. Jenkins has applied this formula to investigate the effects of longitudinal shearing stress in a vessel. Professor Jenkins was a former distinguished student of the Royal Naval College, and the author is happy to have this opportunity of expressing his regret at his premature decease.

Page 388 [1895]. The subject of vibration has of late acquired increased importance in consequence of experimental investigations which have been made on the vibration of vessels and girder bridges. A brief explanation of the most important points relating to it has therefore been added to the present edition in the articles which follow in the text.

On reading Herr Schlick's interesting paper of 1894 the author felt some doubt whether the values of the constants quoted in the text were for *complete* or *single* vibrations. In a letter dated April 13th, 1895, Herr Schlick kindly informed him that *complete* vibrations were meant, and at the same time pointed out that the result would depend very much on the allowances made in calculating the moment of inertia I. In the method actually adopted the sections of angle irons and similar pieces were supposed concentrated in their centre lines, no allowance was made for reduction of area by rivet holes, and no account was taken of bilge stringers, keelsons, etc.

It will be observed that the constant is 23 per cent. greater for a torpedo-boat destroyer than for a merchant vessel, showing that the distribution of the weight is a leading element in the question as might be anticipated from what is said in the text. Unequal distribution and concentration of the weight in the neighbourhood of the nodes increase the frequency and more than compensate for the influence of the causes pointed out which tend to reduce it.

To the examples given in the text on the effect of synchronism may be added the case of a locomotive traversing a girder bridge. In order to balance the reciprocating parts heavy counter-balance weights are necessary attached to the driving wheels, and these weights produce periodic vertical forces of great magnitude (Ex. 17, p. 291), the frequency of which is equal to the number of revolutions per 1″ of the wheels. When the speed reaches a certain limit experience shows that a bridge over which the locomotive is passing vibrates greatly, an effect due to synchronism between the period of a revolution and the period of free vibration of the structure.

The strength of a screw shaft to resist the combination of thrust and torsion to which it is subject is diminished by centrifugal action, a question which has been discussed by Professor Greenhill.

PAGE 393 [1892]. In a paper published in the *Transactions of the Institution of Naval Architects* for 1892, Mr. Yarrow conclusively showed that the vibration of torpedo boats is almost entirely due to the reciprocating parts of the engines and can be got rid of by a proper system of balancing.

PAGE 395 [1900]. The statement made in the text (Art. 203) depends on the mathematical proposition that any periodic function of θ may be fully expressed by a series of sines or cosines of θ, 2θ, 3θ, ... This series is called Fourier's Series, a full and clear account of which will be found in Professor Byerly's *Treatise on Fourier's Series* (Boston, U.S.A., 1895). In its complete form a Fourier's Series consists (1) of a constant term representing the average value of the function, (2) of a series of sines, (3) of a series of cosines, but it generally can be simplified by considering the nature of the function. In the case considered in this article the function is the acceleration-ratio of a piston that is (p. 602)

$$f = \cos\theta + \frac{1}{n}\cdot\frac{\sin^2\theta}{\cos\phi} + \frac{1}{n}\cdot\frac{\cos^2\theta}{\cos^3\phi},$$

a formula true for all values of θ. This is unchanged when θ is changed into $-\theta$, so that the series can contain no sines, and only the first term is changed when θ is changed into $\pi-\theta$, so that there can be no cosines of odd multiples of θ. Hence the Fourier Series for f must be

$$f = \cos\theta + A\,.\,\cos 2\theta + B\,.\,\cos 4\theta + \ldots.$$

The co-efficients A, B... of a Fourier Series are found by a process of integration when the integrations can be performed, or mechanically by an instrument of the nature of a planimeter, or calculated approximately. In the present case in the note already cited, the series is given to a second approximation, but the first approximation used in the text is always sufficient.

If instead of the acceleration-ratio we consider the velocity-ratio, the first approximation to which is

$$\frac{V}{V_0} = \sin\theta + \frac{1}{2n}\sin 2\theta \quad \text{(p. 102)},$$

we find that it is true for all values of θ and therefore is itself the Fourier Series to the same approximation. For values of θ between 0 and π the ratio of the crank effort (R) to the piston pressure (P) is given by the same formula (p. 191), but this is now no longer the Fourier Series because it is not true for all values of θ. For values of θ between π and 2π the steam pressure is transferred to the other side of the piston and the formula is

$$\frac{R}{P} = -\sin\theta - \frac{1}{2n}\,.\,\sin 2\theta.$$

This discontinuity in the ordinary formulæ for the crank effort is one of the principal reasons why graphical methods are almost exclusively used in such questions. To find the form of the Fourier Series, consider that the crank effort is always in the same direction and its average value is $2P/\pi$; the series therefore contains a constant term. Also it is unaltered when θ is changed into $2\pi - \theta$, so that the series can contain no sines and must consist of two series of cosines. The first of these representing $\sin\theta$ for values of θ between 0 and π and $-\sin\theta$ between π and 2π can contain only even multiples of θ. The second representing $\pm\sin 2\theta$ between the same limits contains only odd multiples of θ. The coefficients are found by integration. Hence omitting higher terms

$$\frac{R}{P} = \frac{4}{\pi}\left\{\frac{1}{2} - \frac{\cos 2\theta}{3} - \frac{\cos 4\theta}{15} + \frac{1}{n}\left(\frac{\cos\theta}{3} - \frac{\cos 3\theta}{5}\right)\right\}.$$

This formula gives R when P is constant. When P varies, the series contains sines as well as cosines; the odd multiples of θ as above representing the effect of obliquity. We have therefore

$$\frac{R}{P_m} = \frac{2}{\pi} + A\ .\ \cos 2\theta + B\ .\ \sin 2\theta\ .\ + \ldots,$$

where P_m is the mean pressure and A, B, etc., are constants.

The idea of employing a Fourier Series to represent a curve of crank effort is due to Professor Lorenz, and its advantage is that, being true for all values of θ, any number of cranks at any angles may be superposed, and a formula found for the combination. In his paper read before the Institution of Naval Architects, and published in their *Transactions* for the present year (1900), Professor Lorenz finds the general conditions that the co-efficients of $\cos 2\theta$, $\sin 2\theta$, may vanish so that the series may contain only terms of a higher order. These conditions are taken as being approximately the conditions for greatest uniformity of crank effort, and it is pointed out that they are not inconsistent with the conditions necessary for the balance of the inertia forces. Obliquity however is neglected, and, as the formula given above for a constant pressure shows, the effect of obliquity is often very important. The method being new and capable of many applications, it has been thought proper to notice it here.

Vibration in a vessel due to inequality in the turning moment on a screw shaft must not be confounded with the bending vibrations due to inequality in the action of the water on the blades of a screw. It is torsional, a kind of vibration which according to Schlick really does occur (*Trans. I.N.A.*, 1895, p. 292).

Page 396 [1900]. Since 1892 the subject of the vibration of vessels and the balancing of marine engines has been very extensively discussed in papers for the most part appearing in the *Transactions of the Institution of Naval Architects.* It has been placed on a solid foundation by the experimental researches combined with theoretical investigation of Herr Otto Schlick, some of whose earlier results are given in the text. Also may be mentioned papers by Mr. Mallock, Mr. Macfarlane Gray, Professor Dalby, and others. The author regrets that the limits of this work render it impossible to do more than allude to these researches.

Article 203A on the centrifugal whirling of shafts has been added to the present edition, as the subject has of late become one of considerable importance.

Page 403. A line of stress may be regarded as the geometrical axis of a curved rod which is in tension or compression, as the case may be, under the action of a load perpendicular to itself. The whole solid, therefore, may be conceived as made up of a set of rods, each of which is a rope of linear arch in equilibrium under a transverse load. Each rod transmits stress in the direction of its length. If there be no lateral stress the rods are straight, but otherwise they are curved. In a framework structure loaded at the joints, the bars of the frame may be regarded as lines of stress except at the joints where those lines assume complex forms. The tendency of modern science is to regard all force as being due to the transmission of stress through a medium of some kind, even in such cases as that of gravity, where no medium perceptible to our senses exists. All forces on this conception are represented by a system of lines of stress.

Page 405 [1892]. Except in beams of *I* section, the effect of shearing in increasing the maximum stress and strain due to bending is unimportant, even when the length is only two or three times the depth. This remark, however, does not hold good for materials such as timber, which have a relatively small resistance to longitudinal shearing. Timber beams not unfrequently give way by longitudinal shearing at the neutral surface.

Page 410. The theory of elastic solids has been much more fully treated with reference to practical application by Grashof, Saint Venant, and other continental writers than in any English treatise. The author is chiefly indebted to Grashof's work, *Die Festigkeitslehre* (Berlin, 1866), a new edition of which appeared in 1878. An attempt has been made in the present work to distinguish clearly between those parts of the subject which are necessarily true either exactly or to a degree of approximation which is capable of being exactly calculated, and those parts which depend on hypotheses more or less probable. The first are placed in the present chapter; the second in the chapter which follows.

[1892.] Since the publication (1884) of the present work, a part of the late Dr. Todhunter's *History of the Elasticity and Strength of Materials* has appeared. The book has been edited and put into its present form by Professor Karl Pearson, Vol. I. containing the early history of the subject, and an exhaustive account of all that has been done up to the year 1850 appeared in 1886. The first chapter of the second volume, entirely written by Professor Pearson, was published in 1889 as a separate work entitled the *Elastical Researches of Barré de St. Venant*. It covers the whole of St. Venant's labours on the subject of Elasticity, extending over a period of 35 years.

[1895.] The second volume of the *History* has since appeared, in which the work just mentioned is incorporated, and to which therefore references are made. The standard German treatise on the technical applications of the theory of elasticity is at present Professor Bach's *Elasticitat und Festigkeit*, Berlin, 1890, to which valuable work the author is indebted for information and corrections, especially in Chap. XVIII.

Page 412. Attempts have been made to prove by theoretical reasoning that, in a perfectly elastic isotropic material, the value of m is necessarily 4, and the demonstration is still considered valid by some authorities, while others consider that such reasoning simply shows that matter is not constituted in the way supposed in the demonstration. It is difficult to obtain material which is really perfectly isotropic, but all the experimental evidence at present goes to show that m may have various values.

Page 416. Some other points in the theory of bending may here be noticed:—(1) The effect of curvature is that a lateral stress p' must exist on the longitudinal layers given by the same equation as is used for thick hollow cylinders under internal fluid pressure (page 410), viz.,

$$\frac{d}{dr}(p'r) = p.$$

Replacing r by $R + y$, and p by the value given in the text we find

$$\frac{d}{dy}(p'r) = \frac{Ey}{R},$$

and therefore, by integration,

$$p'r = \frac{Ey^2}{2R} + \text{constant}.$$

Since p' is zero at the outer surface where $y = \pm \frac{1}{2}h$,

$$p' = \frac{E}{8R^2}(4y^2 - h^2) = p_1 \cdot \frac{4y^2 - h^2}{4Rh},$$

where p_1 is the stress due to the bending at the outer surface, and r is replaced by its mean value R. At the neutral surface p' is greatest, but even there has only the very small value

$$-p' = p_1 \cdot \frac{h}{4R}.$$

This lateral stress is therefore never great enough to have any perceptible influence on the elasticity of the layers.

(2) It has been stated on page 299 for the case of tension, page 311 for the case of bending, and page 355 for the case of torsion, that the distribution of stress on any transverse section is the same, however the straining forces are applied to a bar, provided only that their resultant be given in magnitude and position. This may be regarded as a general principle applicable in all cases. Any other distribution of stress produced on a transverse section by friction or other external forces applied directly to it will change with great rapidity on passing to transverse sections not directly exposed to such forces. It is, however, generally necessary to provide additional strength at these exceptional sections.

PAGE 416 [1895]. Assuming the transverse curvature circular the elevation (u) of the sides of the beam above the centre is given by the formula

$$u = \frac{b^2}{8mE} \cdot \frac{p}{y} = \frac{b^2}{8mE} \cdot \frac{M}{I} = \frac{3b^2}{2mEh^2} \cdot \frac{M}{A},$$

obtained on page 413, for a rectangular section.

Now the value of du/dx is evidently the difference of steepness of a central line traced on the side of the bar and the geometrical axis of the bar. Hence if Δq be the difference of shear at the side and at the centre, q_0 the mean shear over the whole section,

$$\Delta q = C\frac{du}{dx} = \frac{3b^2}{4(m+1)h^2} \cdot q_0.$$

For a square section assuming $m = 4$ and remembering that $\frac{3}{2}q_0$ is the mean shear along the neutral axis, we find that the difference between the shear at the side where it is a maximum and the shear at the centre where it is a minimum is one-tenth the mean. The maximum then exceeds the mean by about 5 per cent. This rough calculation is given for the purpose of illustrating the remarks in the text, but as the formula for u is not exact when shearing is taken into account, accurate numerical results can only be obtained by ST. VENANT'S calculations.

PAGE 417 [1895]. The formulæ for plates supported at the edges, and exposed to normal forces, have long been known in two or three simple cases, and are frequently quoted. Unfortunately these formulæ have not as yet been sufficiently verified by experiment and have therefore been omitted in the present work notwithstanding the importance of the question. Readers who wish to see the present state of the subject are referred to Professor Bach's small work, *Versuche über die Widerstandsfähigkeit ebener Platten.* Berlin, 1891.

PAGE 420. The lines of stress for a thick hollow cylinder under internal fluid pressure, and also under the action of tangential stress applied as in Ex. 6, p. 408, will be found to be equiangular spirals, the angle of the spiral depending on the proportion between the fluid stress and the tangential stress.

The verification given in the text is necessary because, otherwise, we could not be sure that the assumptions on page 418 were consistent with one another. This is very well shown by supposing the cylinder to rotate and obtaining a solution of the problem when thus modified, assuming the cylinder to remain cylindrical and employing the equation of verification. It will be found that the solution thus obtained can only be true if the stress on the transverse section varies according to a certain law. If the cylinder is long it appears that this must really be the case except very near the ends. The problem of a swiftly rotating circular saw appears not as yet to have been attempted; it is found by experience that a saw to run at high speed must be hammered so as to be "tight" at the periphery. The same difficulty occurs if the material of the cylinder be not isotropic.

PAGE 427. TRESCA's experiments are described in detail, with a great variety of interesting illustrations in a series of memoirs which have been separately published (*Mémoires sur l'Ecoulement des Corps Solides*). The example in the text is taken from the second memoir (Paris, 1869). It is to be remarked that the influence of time was not taken into account.

PAGE 435. The modulus of elasticity in compression is found to be less than that in tension in cast iron as well as wrought iron in about the same ratio. This circumstance, together with the equality of the moduli for bending and tension, leads us to conjecture that the effect is due to lateral bending which cannot be wholly prevented by the trough.

PAGE 436 [1892, 1895]. When a tube is thin and stiffened so as to prevent flexure as a whole, crushing may take place by local buckling. On the subject of buckling the reader is referred to a paper by Mr. J. A. Yates on the *Internal Stresses in Steel Plating due to Water Pressure* in the *Transactions I.N.A.* for 1891 (Vol. XXXII., page 190). Since that time the mathematical theory of buckling has been discussed in some papers by Mr. Bryan, which have appeared in the Proceedings of the Mathematical Society.

PAGE 438. The argument of Art. 224 applies equally to any case where stress is not uniformly distributed. In the hydraulic press cylinder the stress is never reversed, and the increase of strength is probably reliable.

PAGE 446 [1895]. BACH's experiments on plates here referred to are noticed on page 614. See also a paper on *Bulkheads*, by Dr. Elgar, in the *Trans. I.N.A.*, 1893.

PAGE 440. No formula of this kind is anything more than a formula of interpolation supplying the place of missing experiments. The author is led to make this remark by the elaborate manner in which such formulæ are discussed by some writers. The study of WÖHLER'S original memoir cannot be too strongly recommended to those interested in the subject.

PAGE 449 [1890]. The proposal (which has been partially carried out in practice) to employ a margin instead of a factor of safety in boilers was made by the late Mr. Sennett, formerly engineer-in-chief of the navy, and his views were endorsed on the discussion of his paper (*Transactions of the Institution of Naval Architects*, Vol. XXIX.) by Mr. A. C. Kirk and Mr. Marshall. To the remarks made in the text it may be added by way of caution that though theoretical reasoning and laboratory experiments may furnish valuable indications of the direction in which to move, yet any steps in the direction of lower factors of safety should be very gradual, and only taken by those who possess the widest knowledge and the greatest experience of nearly similar cases. It is impossible *a priori* to foresee all the circumstances which may influence the necessary margin of safety.

PAGE 456 [1892]. The values given for the resilience of timber in the two earliest editions of this work were much too large. When the yield-point of the ductile metals is regarded as the limit of elasticity, the resilience is given in Table III. on the following page.

V.—HYDRAULICS AND HYDRAULIC MACHINES.

PAGE 473. The standard experiments on the co-efficients of velocity and contraction in the case of orifices are those made by WEISBACH, and described by him in his treatise *Die Experimental Hydraulik* (Freiberg, 1855), to which the reader is referred for details. A short pipe projecting inwards is known as Borda's mouthpiece. The theoretical minimum value of the co-efficient of contraction (·5, see p. 498) is closely approached when the pipe is very thin and sharp-edged; otherwise the value is somewhat larger, say about ·55.

[1892.] Experiments by BAZIN and others on orifices of large size (7 inches diameter and upwards) give a co-efficient of discharge of ·6. MAIR, however, obtained the value ·61 in an orifice only 1 inch diameter.

PAGE 476. The use of the term "head" for the energy per unit of weight of a fluid is not free from inconvenience—the two things not being identical unless the datum level be at the surface of the fluid.

PAGE 477. The velocity of the water in any one of the ideal pipes is inversely proportional to the sectional area of the pipe. Now the form of the pipes depends solely on the form of the bounding surfaces, and it follows, therefore, that the velocities of all parts of the stream bear a fixed proportion to each other, depending only on the nature of the bounding surfaces. In the language of the theory of mechanism, the fluid forms a closed kinematic chain. The chain is closed by the pressure of the bounding surfaces, and when the velocity exceeds a certain limit the chain opens. Energy can then no longer be transmitted uniformly to all parts of the fluid, and is no longer uniformly distributed. When energy is unequally distributed, eddies are formed.

PAGE 477 [1895]. The steady motion of an undisturbed stream is one of the principal objects of study in treatises on analytical hydrodynamics, and to such works the reader is referred for an account of what is known on the subject. Some important points, however, can be rendered intelligible by methods of more limited scope, and the note which follows, taken from the *Philosophical Magazine* for February 1876, is therefore here introduced. The motion of the fluid is supposed to be in two dimensions, that is, all particles are supposed to be moving in directions parallel to a given plane—taken for convenience as vertical—and the motion in all planes parallel to the given plane is the same.

In Fig. 212 *AB*, *CD* are consecutive lines of motion, commonly described as

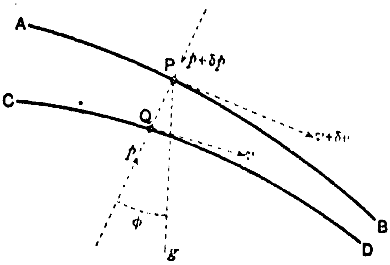

Fig. 212.

"stream lines" lying in the same vertical plane, and representing the section of an elementary stream which flows, as described in the text, without intermingling with the rest of the fluid, just as it would in a pipe of varying section. *P* and *Q* are particles moving in these lines with velocities v and $v + \delta v$, which at the instant considered are so placed that *PQ* the line joining them is normal to the stream. Then if h be the total head at *Q*,

$$h = z + \frac{p}{w} + \frac{v^2}{2g},$$

which remains by Art. 243 always the same as the particle moves. The total head at *P* is given by a similar equation, and the difference δh of the two is found by differentiation. Thus

$$\delta h = \delta z + \frac{\delta p}{w} + \frac{v \,.\, \delta v}{g}.$$

Now δz is the elevation of *P* above *Q*, that is,

$$\delta z = PQ \cos\phi,$$

where ϕ is as shown in the figure the angle *PQ* makes with the vertical.

But if we imagine a small cylinder described round *PQ* as an axis, and consider its equilibrium in the direction of the normal, it is clear that

$$\delta p \,.\, a = \frac{w}{g} \,.\, \frac{v^2}{\rho} \,.\, a \,.\, PQ - w \,.\, a \,.\, PQ \,.\, \cos\phi,$$

where a is the sectional area of the cylinder and ρ the radius of curvature of the stream lines at *P* or *Q*. Combining this with the preceding equation we get

$$\delta h = \frac{v^2}{g\rho} \,.\, PQ + \frac{v \,.\, \delta v}{g} = \frac{v \,.\, PQ}{g}\left(\frac{v}{\rho} + \frac{\delta v}{PQ}\right).$$

Now $v \,.\, PQ$ is constant, being the flow in the elementary stream, and δh is constant, being the difference of two quantities which are each of them constant. We conclude therefore that

$$\frac{v}{\rho} + \frac{\delta v}{PQ} = \text{constant}.$$

Each of the terms of this equation has a definite physical meaning for v/ρ is the angular velocity of the tangent to the stream line at P or Q, while $\delta v/PQ$ is the angular velocity of the line joining the particles P, Q, which line at the instant considered is perpendicular to the tangent, but does not remain so, having a different angular velocity. If a small square portion of the elementary stream be considered, having PQ as a centre line, the angular velocities in question are the angular velocities of the sides of the square, and their half sum is the angular velocity with which the square would rotate if it became suddenly solid. It is described as the molecular rotation, and the physical signification of the equation just obtained is that the molecular rotation for each fluid particle remains unchanged as the fluid moves. Moreover, it is proportional to δh, and therefore when the energy is uniformly distributed, as in Art. 244, the molecular rotation is zero. The motion of the fluid is then described as being "irrotational," and in the absence of hydraulic resistances of any kind no other motion than this can be produced from fluid originally at rest. In the particular case of a free vortex, considered on page 468, the equation becomes

$$\frac{dv}{dr}+\frac{v}{r}=0,$$

which when integrated gives

$$vr = \text{constant},$$

as stated in the text.

PAGE 478. In the flow of rivers it is well-known that it is the outer side of a bend, not the inner, which suffers erosion, so that the windings of the river have a constant tendency to increase in extent. The reason of this has been explained by J. THOMSON to be that the layers of water in contact with the bottom are greatly retarded, and hence have less centrifugal force than the upper layers. The excess pressure at the outer side of the bend is therefore partially unbalanced below, and an inward flow takes place carrying material with it from the outer side to the inner. This was verified by experiment. (*B.A. Report* for 1876, p. 31.)

PAGE 482 [1892]. The article here introduced on similar motions has been added to the present (1892) edition. The importance of the principle in hydraulics generally appears to have been first perceived by J. THOMSON, who recommended the employment of a triangular notch (p. 510). There are obvious objections on principle to the method of finding the discharge from a notch or orifice.

PAGE 484. That hydraulic resistances of all kinds are independent of the pressure is one of the best established laws of experimental mechanics, but how far this may be true at very high pressure is, of course, uncertain. In some books it is stated as confidently as if it were an observed fact that the friction of the skin of a vessel near the keel is greater than that near the surface on account of increased pressure, but there is no foundation for this assertion.

The explanation in the text of the diminution of friction in long surfaces is that given by FROUDE in his reports on surface friction, and also by BOURGOIS in his treatise referred to farther on.

PAGE 490. The formation of eddies by the meeting of different streams and the passage of water past solid bodies is familiar to all observers of the motions of fluids, and is described in the earliest treatises on Hydraulics. The way in which they absorb energy has long been understood: thus PONCELET says, "En général, la production des tourbillons est l'un des moyens dont la nature se sert pour éteindre ou, *plutôt dissimuler*, la force vive dans les changements brusques de mouvement des fluids" (*Méc. Ind.*, p. 571). The italics are the author's. The passage is too long to quote at length, but is worth studying throughout. The extent to which eddy motion may prevail throughout the mass of a fluid, often without any clear indication at the surface, was not understood till long afterwards.

The theory of simple systems of eddies has of late attracted much attention, but the extreme intricacy of the internal motions of fluids will probably long defy calculation in such cases as commonly occur in practice.

The particular case mentioned in the text (Fig. 180) is one observed by the author, in which conspicuous eddies were formed, one or two at a time, with great regularity.

Page 495. If the motion of water in a pipe or channel be supposed of the undisturbed kind (p. 467) and viscosity be taken into account (p. 479), it is possible to find the discharge due to a given head. In the case of tubes of very small diameter it was shown by Poiseuille that the flow actually does take place according to this law, and the co-efficient of viscosity was found. The loss of head is then proportional not to the $(\text{vel.})^2$, but to simple velocity.

In pipes of ordinary diameters through which water is flowing with ordinary velocities, the loss of head is, however, certainly, approximately as the $(\text{vel.})^2$, and, moreover, Boussinesq has shown that it is enormously greater than it would be according to the law for undisturbed flow with the co-efficient deduced by Poiseuille. The inevitable conclusion is, that the loss is mainly due to the formation of eddies. In the case of large rivers it is found by experiment that the velocity diminishes as the bottom is approached according to a law represented by the ordinates of a parabola, a result which is consistent with the law of undisturbed flow. Nevertheless, in this case also, the facts cannot be explained except by supposing that the resistance is due to eddies. With fluids, the viscosity of which is small, as in water, undisturbed flow only occurs at very low velocities in very small channels.

Although these facts were tolerably well established, it is only very recently that any attempt has been made to discover the connection which must exist between the viscosity of the fluid, its velocity, and the dimensions of the channel in which it flows, in order that the flow may or may not be undisturbed. This has at length been done by Osborne Reynolds, who has succeeded in connecting by a common law Poiseuille's experiment on capillary tubes and Darcy's experiments on full-sized pipes. For particulars the reader is referred to his paper published in the *Philosophical Transactions* (1883, Part III.). It need only here be mentioned that it is shown that the loss of head in a pipe may be expressed by a formula which, when stated in a simplified form sufficient for our present purpose, becomes

$$h' = 4f \cdot \frac{l}{d^{3-n}} \cdot \frac{v^n}{2g},$$

where n is an index depending on the nature and condition of the surface. When the surface is rough $n = 2$, and we get the formula already given on p. 487; this is the case for an encrusted pipe, but for a clean cast-iron pipe it falls off to 1·9, and in a lead pipe is 1·723. This falling off in the index in smooth surfaces is quite analogous to that already found by Froude in his direct experiments on surface friction (p. 485).

[1892.] A great variety of formulæ have been proposed for determining the loss of head in a pipe, and the results deduced by different investigators from the same experiments are by no means always consistent. It seems clear, however, that the law of velocity is that stated in the text, the index varying according to the nature and condition of the surface. Further, at a given velocity the loss of head in a pipe the length of which is a given multiple of the diameter, is less the greater the diameter, a fact pointed out by YOUNG in the early part of the present century. DARCY proposed the formula,

$$4f = \cdot 02\left(1+\frac{1}{d}\right) \quad (d \text{ in inches}),$$

and as DARCY'S experiments are recognized as the most important yet made on pipes of considerable size this formula has been much employed, and was given in former editions of this book. It does not appear, however, that this formula has any sufficient basis, and the HAGEN formula appears preferable, especially as it is more convenient for calculations notwithstanding the apparent simplicity of DARCY'S form. It will be seen that according to REYNOLDS the sum of the indices of the velocity and diameter should be equal to 3 or $x = y$ in the form of formula given in the text. In small pipes this may be true, but the conclusion has not as yet been verified by other investigators in pipes of some size.

The influence of temperature (page 486) was pointed out by GERSTNER and YOUNG, though the experiments relied on were apparently only on pipes of small diameter. REYNOLDS concludes that the effect diminishes as the velocity increases, becoming insensible at high velocities, a very important result should it ultimately be confirmed.

PAGE 503. The formula for eddy resistance is given in this form by PONCELET, and the reasoning in the text is essentially that employed by him (*Méc. Ind.*, p. 585). It is well suited to show the real nature of the law of hydraulic resistance (p. 472). All that is supposed in this law is, that the average velocities of the particles of fluid bear a fixed proportion to each other depending solely on the form of the bounding surfaces, as is actually the case in undisturbed motion. If the bounding surfaces are of invariable form the law should be accurately verified for a fluid absolutely devoid of viscosity. The causes of irregularity are explained on page 495 and elsewhere. A variation of 20 per cent. in the course of the same experiment was actually observed by FROUDE.

PAGE 504. In the absence of hydraulic resistances the motion of water past a submerged body is necessarily irrotational (p. 622), and in consequence the paths described by the fluid particles are definite curves, the forms of which depend only on the form of the solid and of the fixed bounding surfaces within which the fluid must be imagined to be enclosed. This question will now be briefly discussed, taking, as before, the case of motion in two dimensions only, and employing the method explained on the page just cited.

Fig. 213 shows a portion of a stream flowing between fixed boundaries *BB*, *BB*. One of the elementary streams, into which it may be supposed to be analysed, is shown in full, bounded by two stream lines, of which *LL* is one. The stream line 1, 1, 1, 1, 1 next the boundary *BB* is also shown in full, and parts of three others *aa*, *bb*, *cc*; the rest of the lines are omitted for clearness. If

e be the breadth of any elementary stream, v the velocity at the point considered, the flow in the stream is evidently

$$ve = \text{Constant}.$$

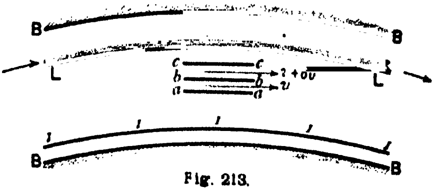

Fig. 213.

Now, in representing a stream by a set of lines, it is convenient to suppose them so drawn that the flow in all the streams is the same, and the constant in the foregoing equation is consequently the same for all streams. Differentiating on this hypothesis we obtain an equation

$$v.\delta e + e.\delta v = 0,$$

which connects the change of velocity δv on passing transversely from the stream ab to the stream bc, with the corresponding change of breadth δe. Now, in irrotational motion, we found (p. 607) that

$$\frac{v}{\rho} + \frac{\delta v}{e} = 0,$$

the breadth e being PQ in Fig. 212; hence by substitution we obtain

$$\delta e = \frac{e^2}{\rho},$$

a simple geometrical relation between the change of breadth of two consecutive elementary streams and their radius of curvature. The sharper the curvature the more rapidly does the breadth of the streams increase on moving away from the centre of curvature. The equation, moreover, shows that if two consecutive stream lines are given, all the rest can be found: for example, suppose one boundary line BB to be given, and also the stream line 1, 1, 1, 1, 1, which lies next to it; then e and ρ are known at every point, so that δe can be found. Thus, the second line reckoning from BB is determined, from which the third can be found, and so on. If both the boundary lines BB are given, it is clear from this reasoning that one and only one set of lines can be found which can represent the stream, and the whole motion is definitely determined. When the boundaries are arbitrarily chosen, the actual construction of the lines presents difficulties which are often insuperable, but a definite set of lines always exists, and hence the velocity of each particle of the fluid bears a fixed proportion to the velocities of all the others; the motion being, in fact, like that of a mechanism, or closed kinematic chain.

In particular cases the lines are known, and from these known cases any number of others can be obtained graphically by a method introduced by Rankine, which we have not space to explain. Rankine's methods in their application to ship-shaped forms have recently been considerably extended by Mr. D. W. Taylor in two papers which will be found in the *Transactions I.N.A.* for 1894, 1895.

The most obvious of these known cases is also that which is most important, namely, where the boundaries BB are parallel straight lines. The stream lines are now equidistant parallel straight lines, and the velocity uniform; this being the only motion within straight parallel boundaries which is irrotational.

Imagine now a horizontal casing of uniform rectangular section of indefinite length through which water is flowing, the motion being irrotational; and in the casing let a solid of any size or shape be placed, the sides of which are perpendicular to the flat sides of the casing between which it is fixed, so that the motion is still in two dimensions. Near the solid the stream lines will be curved, their form depending on the form of the solid, but at a great distance in front and behind they must become equidistant parallel straight lines, being evidently unaffected by its existence. Hence the particles of fluid after passing the solid necessarily return to their original straight line paths, and consequently to their original velocity. But as it is supposed that there is no dissipation of energy of any kind, the usual equation of steady motion shows that if the velocity is the same, the pressure must also be the same, and it appears that the solid produces no permanent change in the condition of the water; and therefore the water can have no longitudinal action on the solid. The resultant longitudinal force on the solid must therefore be zero; and this conclusion is equally true if the water in the casing be supposed at rest, and the solid move through it with uniform velocity in a straight line, for the relative motion is necessarily the same in the two cases.

The proposition that a body of any size or shape, moving through water with uniform velocity in a straight line, would experience no resistance in the absence of viscosity and eddies, paradoxical as it has often been thought to be, is thus definitely proved for motion in two dimensions. If a solid of revolution be placed within a cylindrical casing of circular section with its axis coinciding with the axis of the casing, it is easily understood that the same thing holds good. In fact the stream lines are plain curves, and the foregoing reasoning can be easily modified so as to apply to this case.

Nor does it appear that there can be any difference for any other form of solid placed in a cylindrical casing, unless it be skew-shaped like an ordinary screw propeller. If such a skew-shaped solid be placed in a stream and held fast so that it cannot rotate, the fluid after passing the solid will rotate, the resultant velocity of the fluid particles will be increased and their pressure consequently lowered. There will then be a resultant force on the solid in the direction of motion. The resistance in this case may be described as eddy resistance, and is given by the same formula,

$$R = k \cdot S \frac{V^2}{2g},$$

with a value of k, which in the present state of our knowledge can only be determined by experiment. The case differs from that given in the text only in the scale of the eddies produced, some of which are now so large that they may endure for a considerable period before being dissipated by fluid friction.

If the screw propeller be permitted to rotate freely, the resistance will be due to small scale eddies alone. If it be forcibly constrained to rotate by energy supplied from without, a longitudinal force will be produced which propels the vessel. The value of k is then a function of the ratio of the velocity of rotation to the speed of advance (the slip-ratio), which has been determined by experiment. Attempts have been made to calculate k: the simplest and most practically useful of these "theories of the screw propeller" is briefly noticed further on.

Page 507 [1895]. Experiments by Mr. Dines on the resistance of the atmosphere to the motion of a flat plate moving through it are described in Vol. XV. (1889) of the *Journal of the Meteorological Society*, p. 187. The plate was mounted upon a bell crank lever, pivoted on the end of a revolving arm about 29 feet long. The other end of the lever carried an adjustable weight, the centrifugal force of which measured the pressure. The pressure per square foot on a round or square plate was found to be about 1½ lb. per square foot at a speed of 20·86 miles per hour (30·63 f.s.). This is equivalent to 1 lb. per square foot at a speed of about 17 miles per hour, and differs little from Froude's estimate given by Sir W. H. White (*Naval Architecture*, 2nd ed., p. 491) for the pressure on a flat plate moving uniformly through still air. Assuming a density of the air of 13 cubic feet to the pound, this gives $k = 1{\cdot}35$, a value somewhat though not very much greater than the most probable value for water; an excess which may be attributed to the effect of the difference in viscosity and elasticity of the fluid on the formation and extinction of eddies. Some of Dines' experiments, however, give a somewhat smaller result. The value is independent of the size of the plate, but is somewhat greater (according to Dines about 13 per cent.) for a long narrow plate, as might be expected. For the case of a sphere, Dines gives a value which is four-ninths that for a flat plate. This corresponds to $k = {\cdot}6$, and appears somewhat large.

At low speeds, such as are here referred to, the resistance undoubtedly varies as the square of the velocity. The limit at which this ceases to be true on account of the compressibility of the air probably varies according to the form of the body. In the case of shot, according to Vallier, it is as low as 100 metres per second (330 f.s.), after which the resistance increases according to the 2·5 power of the velocity until the velocity of sound is approached. The value of k therefore continually increases, most probably because the compression of the air increases the plus portion of the co-efficient.

Beyond the velocity of sound (1100 f.s.) the resistance follows an entirely different law, being then closely represented by the ordinates of a straight line. This straight line law has been deduced by Vallier and others from extensive experiments made abroad at velocities up to 1000 metres per second; but it is also clearly apparent in the older experiments of Bashforth. For velocities in feet per second the resistance in lbs. of a spherical shot d feet diameter is given with tolerable approximation according to these experiments by the formula ($v > 1100$)

$$R = 3d^2(v - 800).$$

The resistance of an ogival-headed shot is of course much smaller; it depends on the angle of the ogive, but in the elongated shot experimented on by Bashforth was about two-thirds that of a sphere. In shot of recent type it is no doubt still less. If the hinder part of the shot were elongated instead of flat the resistance would be greatly reduced: in bullets this idea has been carried out by making them tubular with ends fined off both in front and rear. At these high speeds the resistance is mainly due to sound waves, which by the aid of photography have been rendered visible in bullets.

In a paper read at Chicago in 1893, and reproduced in the *Philosophical Magazine* for May 1894, Professor Langley shows that wind, however steady and uniform it apparently may be, is in fact a motion of an extremely complex character, each small portion of the fluid being in a state of pulsation. The velocity at any point of a wind current therefore goes through periodic changes of great magnitude, although the motion of a large body floating in the air may be perfectly uniform. It is believed that birds have the power of utilizing the internal energy corresponding to these periodic changes for the purpose of sustaining themselves, and even rising without visible movement of the wings.

Hence it is, most probably, that the pressure on small areas exposed to wind is much greater than that just given for motion through still air. The value commonly accepted for the pressure per square foot of a wind of V miles per hour is $V^2/200$. This corresponds to $k=2$, being 50 per cent. greater than before. According to the best authorities on hydraulics, as stated in the text, there is a corresponding increase in the case of water, but it is difficult to say how far this is due to irregularity in the stream and how far to errors in the experiments.

The pressure on a large area of 300 square feet has been shown by Sir B. BAKER to be only two-thirds that on a small area; that is, it is about the same as for motion in still air.

The relation between the pressures on an oblique and a normal surface, so far as is known, is the same in air as in water, but it must be remembered that the exposure of the surface will have an enormous influence. Thus, the pressure on a sloping roof will be much less if it rests on walls than if it is carried on pillars so that the air has free passage below.

PAGE 537 [1900]. The outward flow turbine was introduced by FOURNEYRON about 1828, and its theory given by PONCELET in 1838. The inward flow THOMSON turbine followed some 20 years later. A wheel sometimes less than 2 feet diameter in these machines replaces a slow-moving cumbrous water-wheel, and may be made to yield a very considerable power.

The efficiency of a turbine is in general not precisely a maximum when the conditions of no whirl at exit and no shock at entrance are satisfied. To explain this and some other points of more actual importance in the working of turbines, an approximate investigation of their efficiency is now added.

The losses considered are (1) the kinetic energy of flow on discharge from the wheel together with the loss by friction of pipes and passages, and (2) the kinetic energy of whirl on discharge together with the loss by "shock" on entrance. For simplicity Case I. of the text is taken in which the vanes are radial at entrance and the angles of the guide-blades and vanes so proportioned that, at a certain speed, the two conditions of no whirl and no shock are simultaneously satisfied. The notation is that of the text with a slight modification to be mentioned presently, and in addition the ratio r_2/r_1 is denoted by m.

The kinetic energy of flow is $u^2/2g$, and, as the velocity through the pipes and the passages of the wheel is proportional to u, in a turbine of this type where all the passages are always completely filled with water, the loss by surface friction will be $F \cdot u^2/2g$ where F as usual is a suitable co-efficient of resistance. The losses (1) are therefore $(1+F)u^2/2g$. Now $u=v_1 \cot\theta_1$ and it is convenient to express u in terms of v_1. These losses then become $\beta v_1^2/2g$ where

$$\beta=(1+F)\tan^2\theta_1$$

is a co-efficient which may be supposed known.

The whirl at exit is

$$v_2 = V_2 - u \cdot \cot\theta_2 = m(V_1 - v_1),$$

while the loss by shock at entrance is $(V_1 - v_1)^2/2g$, the losses (2) are therefore $(1+m^2)(V_1 - v_1)^2/2g$, and the total loss of head (h') is given by the equation

$$2gh' = \beta v^2 + (1+m^2)(V - v)^2,$$

in which formula the suffix 1 is suppressed, being no longer necessary.

For the useful work we have

$$\begin{aligned} gh &= V_1 v_1 - V_2 v_2 = V_1 v_1 - m^2(V_1 - v_1)V_1 \\ &= (1+m^2)Vv - m^2 V^2, \end{aligned}$$

the suffix 1 being suppressed as before.

Adding together the losses and the work we obtain the energy exerted represented by the fall or actual head H, so that

$$gH = \frac{1+m^2+\beta}{2} v^2 + \frac{1-m^2}{2} \cdot V^2.$$

The case considered in the text is that in which $v = V$. This gives

$$V^2 = \frac{2}{2+\beta} \cdot gH \quad : \quad \frac{h}{H} = \frac{2}{2+\beta}.$$

Hence the efficiency at this speed is $2/(2+\beta)$, and as this may be taken at 80 per cent. it follows that the value of β in well-designed turbines is about ·5.

At other speeds H remains the same, while v and V vary, and by differentiation

$$v\frac{dv}{dV} = \frac{m^2-1}{1+m^2+\beta} \cdot V.$$

As already remarked v is proportional to u the velocity of flow, and therefore to the delivery. Hence in an outward flow turbine the delivery increases with the speed, and in an inward flow, for which m is fractional, it diminishes. This important conclusion might have been foreseen; for reasoning as in the case of the reaction wheel (p. 531) the virtual head at discharge is increased in the outward flow and diminished in the inward flow, by the "head due to centrifugal force" (compare page 546) a quantity which increases rapidly with the speed. The influence on the working of the machine is most important, for, in consequence, the outward flow turbine is in practice unstable and the inward flow stable; and hence the value of m, which in a Fourneyron turbine should not exceed 1·25, may with advantage in a Thomson turbine be reduced to ·5. (See Unwin, "Hydraulics," *Encycl. Britannica*, page 529.)

Next let us see how the efficiency varies. Differentiating the equation for h, and then putting $v = V$

$$\frac{dh}{dV} = (1+m^2)V\frac{dv}{dV} + (1-m^2)V,$$

which on substitution for dv/dV gives

$$g\frac{dh}{dV} = -\frac{\beta(m^2-1)}{1+m^2+\beta} \cdot V.$$

The speed for which $v = V$ is that at which there is no shock at entrance and no whirl at exit, and we see that this is not the speed of maximum efficiency; in an outward flow turbine dh/dv is then negative and a lower speed is better, while in an inward flow it is positive, and the best speed is higher. The reason is that reduction of flow by a change of speed reduces the losses (1), thus more than compensating for the introduction of losses (2). If, however, δh be the change in h consequent on a change ΔV in speed from the value $\sqrt{gh}$ which gives $v = V$

$$\frac{\Delta h}{h} = -\frac{\Delta V}{V} \cdot \frac{\beta(m^2-1)}{1+m^2+\beta}.$$

from which it appears that the change of efficiency is only a small fraction of the change of speed. Thus to reduce the efficiency from 80 to 79 per cent. or to increase it to 81 per cent. a change of at least 7 per cent. in the speed is necessary. This agrees well with common experience, since it is well known that a change of 10 per cent. in the speed has little influence on the efficiency.

We might now proceed further and determine the actual speed of maximum efficiency from the equations just given, but the result would be of no practical importance, as the frictional resistances are not, and perhaps cannot be, determined with sufficient accuracy. It is, however, probably safe to infer that the best speed of an inward flow turbine is decidedly greater than that of an outward flow. The external resistances referred to on page 535 have the effect of lowering the speed of maximum efficiency.

The useful work (U) is wQh, and therefore proportional to the product vh. From the values of dv/dV and dh/dV given above it is easy to derive the value of $\Delta(vh)$, and hence at the speed for which $v = V$,

$$\frac{\Delta U}{U} = (1-\beta)\frac{\Delta V}{V} \cdot \frac{m^2-1}{1+m^2+\beta}.$$

The supply of available energy therefore increases with the speed in an outward, and diminishes in an inward, flow turbine; unless by obstruction of the supply the efficiency becomes less than two-thirds when the converse will hold good. But the rate of increase is small, and varies greatly at different speeds, while, for the reason stated above, it cannot be determined accurately.

A similar calculation may be made in the more general case (page 537) in which the vanes are not radial at entrance but inclined at a given angle. The results are similar though more complex, and may be expressed by replacing the constants β and m of the formulæ by other values dependent on this angle.

PAGE 538. The undisturbed motion of a perfect liquid within fixed boundaries is always *reversible*, that is, if every particle of liquid were imagined to be set in motion with the same velocity in the reverse direction, the motion would continue undisturbed. But if water be set in motion from rest this will generally not be the case. If, for example, we imagine a pipe connected with a tank by a mouthpiece in the form of the *vena contracta*, then, when water flows out of the tank, it will issue in a continuous stream with small loss of head; but if the motion be reversed most of the energy of motion of the water in the pipe will be wasted in the internal motions soon after entering the tank. The loss is not unavoidable, as will be seen on reference to the case of a trumpet-shaped pipe (Fig. 175, page 477), but may be rendered small by enlarging the pipe very gradually.

PAGE 540 [1895]. The second of the two approximate calculations of the efficiency of a centrifugal pump has been added to this article in the present (1895) edition, for the purpose of showing that when the chamber is properly designed radial vanes are not necessarily less efficient, and may be more efficient than curved back vanes. But it must not be supposed that the form is actually of little importance. The investigation is based on the supposition that the only loss during the passage through the fan is due to surface friction, as would be the case if the motion were steady and continuous. But it is by no means certain that steady continuous flow is possible under the circumstances, and if breaking up occurs the value of β might be greatly increased, and might depend on the speed. In a centrifugal pump with high lift the changes of velocity imposed on the water during the passage through the fan are enormously great and rapid, and the form of vane may be of great importance in facilitating or otherwise the tendency to break up. On page 536 it has already been suggested that it might be advantageous to curve the vanes, so that *vr* should change (in this case increase) uniformly from entrance to exit. It is only by systematic experiment that the best form and number of vanes could possibly be determined; any reasoning on this point must be very uncertain. The equation given in Art. 278, page 546, gives the steady flow through a pipe attached to a rotating casing, but it does not necessarily follow that steady flow is possible in a radiating current.

In recent designs of fans for blowing air the vanes are curved forwards instead of backwards, and it is quite conceivable that the tendency to break up may be diminished in this way. The RATEAU fan described in the *Engineer* for May 24th, 1895, is an example. It is said to give excellent results.

VI.—ELASTIC FLUIDS.

PAGE 562 [1900]. Some important experiments by Professor Osborne Reynolds and Mr. Moorby on the mechanical equivalent of heat are described by them in the *Philosophical Transactions* for 1897. The result obtained, omitting decimals, is 777 ft.-lbs., the unit of heat being such that the mean specific heat of water between 32° and 180° F. is unity. It is probable that this number will be generally accepted as the true value, but for the purposes of this work it has not been thought necessary to alter the provisional value given in the text.

PAGE 566 [1895]. If Δh be a small quantity of heat supplied at temperature T when raising the temperature of a lb. of water, the mechanical value of that heat as explained in the text is $\Delta h(T-T_0)/T$, and the total mechanical value of the whole heat supplied in raising the temperature of the feed water from T_0 to T_1 is

$$M_0=\int_{T_0}^{T_1}\frac{T-T_0}{T}\,.\,\Delta h.$$

If the specific heat of water is taken as unity this becomes

$$M_0=\{T_1-T_0\}-T_0\,.\,\log_e\frac{T_1}{T_0},$$

which is a formula very commonly used. The result is too small, because the specific heat of water increases as the temperature rises. If we adopt an approximation suggested by RANKINE we may take

$$\log_e\frac{T_1}{T_0}=2\frac{T_1-T_0}{T_1+T_0},$$

and on substitution the formula given in the text is obtained. The result is greater than before, and the error of the approximation partially compensates for the neglect of the excess specific heat of water. The formula in the text may therefore be preferred unless special tables of the "entropy" of water are available.

The second or mechanical formula for the available heat of steam

$$M = P_0v_0 \cdot \log_e\frac{p_1}{p_0}$$

given in the text is based on the fact explained in the author's work on the Steam Engine that the saturation curve is approximately midway between an hyperbola and an adiabatic curve starting from the same initial pressure. If then an hyperbola is traced starting from the *lower* pressure p_0 on the saturation curve the area of the hyperbola must be very approximately the same as that of the adiabatic curve. The result given by this formula is too small: the deficiency increasing as the pressure-ratio increases; but the error does not exceed 2 per cent. For pressure-ratio less than 5 it is insensible.

Page 575 [1892]. The facts relating to the transmission of energy by compressed air are much better known now than when this book originally appeared. The remarks made on the subject have therefore been re-written and amplified.

Page 578. If the fluid be supposed at rest, and elevation be taken into account, we obtain

$$K_pT + z = \text{Const.},\ \text{or}\ 3{\cdot}5\ PV + z = \text{Const.}$$

This gives the distribution of pressure and temperature of the atmosphere for "convective equilibrium" (Clerk Maxwell's *Theory of Heat*, 1st edition, p. 301). Energy is then uniformly distributed.

Page 579 [1900]. Until recently little of any importance had been added to the ordinary theory of efflux described in the text, the sketch given in Art. 300 remaining unaltered since it was originally written in 1884. Since 1895 the subject has attracted some attention, and it has been pointed out that the velocity through the section of maximum density when the critical ratio given by the table on page 580 is over-passed is the velocity of sound in the issuing fluid at the temperature of outflow. This observation, originally, it is believed, due to M. Hugoniot, can be easily verified by means of the formulæ given in the text. More recently experiments by M. Parentey, Mr. W. Rosenhain, and others have raised many points of interest. Rosenhain's experiments, described in a paper published in a recent volume (vol. 140, p. 99) of the *Proceedings of the Institution of Civil Engineers*, are chiefly devoted to the comparison of different types of orifice. They show that, as might have been expected, in trumpet-shaped nozzles the expansion down to atmospheric pressure takes place within the nozzle. In forms of orifice where this is not the case the expansion is extremely sudden and the jet subsequently is, for some distance cylindrical. The discharge is greater than the theoretical maximum, as calculated on the supposition of dry steam and adiabatic flow; the excess in trumpet-shaped nozzles with rounded inlet being apparently about 14 per cent. How far this excess can be accounted for by the water mixed with the steam or by the error of the hypothesis of adiabatic flow is uncertain.

Page 582. This formula for the flow of air in a long pipe was given by Unwin (*Min. Proc. Inst. C. E.*, Vol. XLIII.), and somewhat earlier by Grashof. It is a question of considerable practical interest. By comparison with experiment it has been shown that the co-efficient is given by a formula of the same form (Darcy's), as in the flow of water through pipes, an important verification of theoretical principles. The equation for the case where the temperature varies can be obtained without difficulty, but has not as yet been practically applied.

VII.—RESISTANCE AND PROPULSION OF SHIPS.

[1892.]

The importance of this subject is so great that though outside the intended limits of this work some information relating to it may be useful. The brief summary here given of the leading facts relating to it would, however, require expansion into two or three long chapters if anything like a full statement were attempted.

Submerged Bodies.—If a uniform current be flowing through a straight pipe or cylindrical casing of indefinite length, which it completely fills, and a solid of any size or shape be fixed within it, the particles of water after passing the solid return ultimately to the original straight line paths in which they moved before reaching the solid, unless the current be disturbed by the causes discussed at length in Ch. XIX. of this book. Each particle, after passing, has ultimately the same velocity and pressure as it had before reaching the solid, no permanent change being possible except such as may be produced (1) by viscosity (page 479) or (2) by eddies due to surface friction or other causes (pages 501-7). Hence it follows that the longitudinal resultant pressure upon the solid must be zero. The grounds on which these statements are made, the qualifications to which they are subject, are discussed on page 626.

If the water in the casing be at rest and the body move uniformly through it in a straight line parallel to the sides, the relative motion of solid and water is the same as when the solid is at rest and the water moves. We therefore conclude that the water will offer no resistance to the motion except such as may be due to hydraulic losses. And as the casing may be supposed of any size we please, this conclusion must be true for any case where a body is sufficiently deeply submerged.

In bodies of very small size, such as particles of a finely divided solid, the direct action of viscosity is the principal cause of resistance, but in bodies of the size of a ship or even of a model of a ship the direct action of viscosity is so small as to be negligible, and the resistance of a submerged solid is therefore practically due to exactly the same causes as produce loss of head in a pipe or passage.

Eddy resistance has already been discussed (page 502). On examination of Fig. 191 it will be seen that eddies are not formed immediately in front but behind the corners and in the rear. A solid, therefore, may be blunt ended in front without giving rise to eddy resistance, provided the shoulders are rounded off sufficiently, and a tail of sufficient length be attached behind. Eddy resistance may thus be reduced to a very small amount, and surface friction then becomes by far the most important cause of resistance in a deeply submerged body moving uniformly in a straight line.

Surface Friction has been discussed on page 485, and a table given of FROUDE'S results, which is directly applicable when the surface is plain. When the surface is curved the question is in principle much more complex, because the water glides over the different parts of the surface with a different velocity. Let q be the ratio which the velocity of gliding over a small area δS, bears to the speed of the solid (V), then $f . q^3V^3\delta S$ is the energy dissipated by surface friction per second, and $f . V^2\int q^3dS$ is the corresponding resistance, which is the same as that of a plane the area of which is

$$S_1 = \int q^3dS.$$

If the velocity of gliding was not altered by friction the co-efficient η would be on the average greater than unity. In short surfaces therefore S_1 is greater than the actual area of wetted surface, and is described as the Augmented Surface. The idea of an augmented surface is due to RANKINE, who based upon it a well-known formula for the resistance of ships. It has long been recognized that this formula is not of practical value, and the reason for its failure is simply that the velocity of gliding over a surface of any considerable length is so much disturbed by friction as to be far less than the calculation value, and the friction is correspondingly reduced. Calculations of the surface friction of vessels are therefore made as if the surface were plain, due regard being had to the length and nature of the surface in estimating the probable value of the co-efficient.

The area of wetted surface is calculated from the drawings of the vessel, but as the calculation is complex, and throws no light on the relation between the wetted surface, the displacement, and the draught of water, the formula

$$S = 2LD + \frac{35\Delta}{D}$$

may be used in which Δ is the displacement in salt water in tons, L the length, D the draught of water, both in feet. This formula in most cases gives a very fair approximation to the surface of the bare hull, as has recently been shown by Mr. Archibald Denny if the co-efficient 2 be replaced by 1·7. In this note, however, it will be used without this modification and then includes a certain margin for bilge keels, or similar appendages. If βBDL be the cubic displacement where B is the beam and β a co-efficient of fineness the formula becomes

$$S = L(2D + \beta B).$$

Law of Comparison.—When a careful estimate of the surface friction and eddy resistance of a solid is compared with the actual resistance the solid offers to uniform motion in a straight line, it will be found that in a submerged body of fair form the two are nearly the same, and this is also true for a vessel at low speeds. But in a floating body the difference at high speeds is very considerable, and increases rapidly with the speed. This difference is described as the Residuary Resistance of the vessel, and is mainly due to the formation of waves at the surface of the water, a cause which would operate even if there were no hydraulic resistances of any kind. It is no longer true as in a submerged body that the resultant pressure on the body in the direction of motion is zero. Hence so far as independent of hydraulic resistance the residuary resistance must be subject to the law of comparison stated in Art. 247 (page 483) of this book, so that in similar vessels at corresponding speeds under similar circumstances the residuary resistance must be in the proportion of their displacements. The most convenient way of expressing this principle is by taking

$$V = c \, . \sqrt{L}$$

where V is the speed in knots, L the length in feet, and c a co-efficient of speed. The law of comparison may now be expressed by saying that the residuary resistance when expressed in Pounds per Ton of displacement must be a function of c the speed co-efficient, which function must be the same in similar vessels. As a general principle in hydrodynamics this law of comparison had long been known, but FROUDE made it his own in its application to vessels by showing (1) that it was not applicable to surface friction, the resistance due to which is much more important on a small scale than the law would imply, and (2) that it was applicable to the residuary resistance. The experimental verification consists partly in the famous experiments made on the *Greyhound* and her model, and partly in the fact that it is now possible to predict the power required to propel an entirely new type of vessel by means of experiments made on a model of the vessel, a method systematically employed by FROUDE'S successors.

It must be distinctly understood that no particular law of speed is implied, but only a relation connecting the law of size with the law of speed. To illustrate this point, suppose the resistance to vary as the fourth power of the speed, then by the law of comparison the resistance is

$$R = k_1 \,.\, \Delta \,.\, v^4$$

where k_1 is a numerical co-efficient. Replacing v by its value $V/\sqrt{L}$, and remembering that Δ must be proportional to the *cube* of the linear dimension

$$R = k \,.\, l \,.\, V^4$$

where k is another co-efficient depending on the particular linear dimension (l) chosen which may be the length, beam, draught of water, or any linear combination of these quantities. If then the resistance vary at V^4 it must also vary in direct proportion to the linear dimensions of the vessel. Similarly, if the resistance vary as the square of the speed, it must also vary as the square of the linear dimension, that is, as the transverse section, and consequently eddy resistance pure and simple (page 503) satisfies the law. On the other hand surface friction does not satisfy it, for, referring to pages 495, 496, it will be seen that the general form of the formula is

$$R = \mu \frac{w}{62\cdot4} \,.\, S \left(\frac{V}{6}\right)^n$$

when μ the co-efficient is taken from FROUDE'S table for fresh water, in which the standard speed is 600 feet per minute, or approximately 6 knots. This may be written using the second formula for S

$$r = \frac{R}{\Delta} = f \,.\, \left(\frac{2L}{\beta B} + \frac{L}{D}\right) \,.\, v^2,$$

where f the co-efficient is not constant as it should be if the law of comparison applied, but is given by

$$f = \mu \,.\, \left(\frac{6}{V}\right)^{2-n}.$$

Wave resistance.—Waves produced on the surface of water by the action of a body moving through it are of two distinct kinds. The first is a solitary shallow water wave generated in front of a barge moving in a narrow canal. In such a wave the particles of water are lifted up, carried forward along with the wave through a short distance, and then set down at rest, while the wave travelling onward leaves them behind. A wave of this class is described as a Wave of Translation; it possesses a certain definite amount of energy, which is transmitted with it from particle to particle as it moves, and hence when of any size it travels for great distances without external agency when once created. Such a wave is consequently only a cause of resistance to the generating body while it is being formed. The second class occur in series; and the particles of water oscillate backwards and forwards: the translation along with the wave is relatively small and for most purposes may be neglected: they are therefore described as Oscillating Waves. In a purely oscillating wave the particles describe closed curves resembling an ellipse which becomes a circle in deep water. The speed of an oscillating wave in deep water depends on its length only, the speed in knots of a wave of length λ feet being

$$V^2 = 1{\cdot}8\lambda.$$

This formula shows that the length λ of a wave travelling at the same speed in the same direction as a vessel of length L is

$$\lambda = \frac{c^2}{1{\cdot}8} \cdot L,$$

where c is the speed co-efficient.

An oscillating wave possesses both kinetic and potential energy in nearly equal amounts, but as was first pointed out by OSBORNE REYNOLDS, the kinetic energy is not transmitted along with the wave but remains behind, and therefore when such a wave travels onwards into still water its height necessarily diminishes unless it is kept up by external agency, such as a moving body which supplies it with energy. Waves then are a cause of resistance, not only when a new wave is continually being created as the vessel moves, but also when waves already existing are kept up to their full height. The energy of a complete wave is proportional to its length and the square of its height; and of this a certain definite fraction has to be supplied by the vessel as it moves through a wave length. Hence the resistance due to a wave system of a given type varies as the square of the height. If the type remained the same at different speeds the height of the waves would vary as the square of the speed, and the corresponding wave resistance as the fourth power of the speed, a law of resistance already mentioned.

A third class of waves not necessary here to consider are the "capillary" waves, so called because their motion is in great measure governed by capillary action, that is, by what is known to physicists as "surface tension." They are of very minute size, and are also known as "ripples." The wave resistance of a model vessel would be affected by surface tension if the model was small enough. No effect of this kind appears to have been noticed at present.

Interference.—If the residuary resistance of a vessel as determined by a set of speed trials upon a model be divided by the fourth power of the speed the quotients are in general neither constant nor continuously increasing or diminishing. On the contrary they show very distinctly a periodic change, being alternately greater and less than a certain mean value. The cause of this remarkable result was conclusively shown by FROUDE to be the interference of two distinct wave systems—one created at the bow, the other at the stern of the vessel. The experimental demonstration of this consisted in comparing the residuary resistances at a given speed of a set of models, the fore body and after body of which were the same in all, but which had different lengths of middle body. If for simplicity we suppose that the bow and stern generate simple waves of heights $h_1 h_2$ at points distant s from each other, measured along the side of the vessel, the result of the combination by a principle well known in physical science will be a simple wave of height h given by the formula

$$h^2 = h_1^2 + h_2^2 + 2h_1h_2 \cdot \cos 2\pi\frac{s}{\lambda},$$

λ being the wave length which is the same for all, being connected as before explained with the speed of the vessel. Hence when s is changed by varying the length of middle body the residuary resistance suffers a periodic change, and this conclusion was exactly verified by the experiment. When a given model is tried at various speeds, s remains nearly the same, but λ as well as $h_1 h_2$ varies as the square of the speed, the formula then shows that the resistance, while increasing rapidly on the whole, suffers a periodic change whereby the rate of increase is alternately excessive and moderate.

The fact that the residuary resistance is a periodic function of the speed is shown graphically by the "humps" and "hollows" which are found in curves of resistance, and is mainly accounted for by interference. But the interference is of a more complex kind than in the simple case supposed, and it would not be safe to conclude that interference is the sole cause of variation in type of the wave system especially at certain critical speeds. The form of water surface has been investigated by LORD KELVIN (Sir W. Thomson), but a rational formula for wave resistance is probably unattainable, varying as it must according to the lines of the vessel.

Approximate Formulæ.—From what has been said it appears that—omitting (1) the resistance of the air, (2) the resistance of various appendages to the vessel herself and her propelling apparatus, an item which may be considerable, and which must be separately estimated—the resistance of a vessel in Pounds per Ton is given by the general formula

$$r = av^2 + xv^4,$$

where the first term gives the surface friction in terms of a co-efficient a, which can be calculated with a fair degree of approximation, while the second term gives the wave resistance in terms of x, a periodic function of c. The character of the resistance will depend on the value of c, the second term being relatively small at low speeds.

The values of c which occur in actual vessels may be grouped as follows:

(1.) In steamers employed for the transport of merchandise c ranges from ·5 to ·7, and by the formula already given the length of waves travelling at the same speed as the vessel rarely exceeds one-fourth the length of the vessel and is usually much less. The wave resistance is in this case one-fourth or less of the whole, and the single constant formula

$$R = K \, . \, \Delta^{\frac{2}{3}} \, . \, V^2$$

may conveniently be employed. The value of K for displacements in tons and speeds in knots ranges from ·55 to ·66 in full-sized sea-going vessels, excluding any resistance due to the nature or condition of the wetted surface and to appendages.

(2.) In recent ironclads and in mail steamers the value of c ranges from ·7 to ·95, and the length of waves travelling at the same speed as the vessel increases to nearly one-half her length. The wave resistance now becomes nearly one-half the whole, and the term representing it cannot be merged into the term representing the surface friction. Already, before FROUDE's researches, this had been recognized, and formulæ had been given showing that the resistance of a ship increased faster than the square of the speed. By far the most important of these is the formula given by BOURGOIS in a treatise referred to further on.

$$R = k_1 V + k_2 V^2 + k \, . \, B \, . \, V^4.$$

The first two terms of this formula represent surface friction and eddy resistance which may be better effected in the way already explained. The physical meaning of the third term was only partially understood by BOURGOIS, but it is now, evident that it amounts to taking an average value of the periodic function, and assuming that the beam B of the vessel is the linear dimension which is principally effective in the production of wave resistance. On substituting we find the average value of x to be

$$x_0 = b \cdot \frac{L}{\beta D},$$

where b is a numerical co-efficient. The value of k employed by BOURGOIS was ·14 in French units for all cases except where the beam is as much as one-fourth the length, when it is increased to ·16. The value of b, which corresponds to $k = {·}14$, is ·23, and in some types gives good results, but it may be doubted whether it applies so universally as BOURGOIS assumed. The values of both x_0 and a necessarily depend on the lines of the vessel, so that no fixed relation can exist between the two, but the author has found that the formula

$$r = 8c^2(1 + c^2)$$

gives good results in a great variety of types, though in heavy ironclads the number 8 should be reduced to 7, or in some cases perhaps still further. The same restrictions must be understood as in the preceding case. The resistance of vessels of small draught of water is much greater, and may be approximately estimated by the formulæ already given. The resistance of the air and of appendages may be included by a suitable addition to the constant a.

(3.) In cruisers and torpedo gunboats, by the use of engine power amounting to from 2 to 5 H.P. per ton, values of c are obtained exceeding unity, sometimes even reaching 1·4, the waves are now about the same length as the vessel, and at this critical point the periodic variation of x is so great that no formula is of much value.

(4.) Beyond this speed no full-sized vessel can be propelled from the impossibility of putting sufficient engine power on board, but in torpedo boats a power of 15 H.P. to the ton can be employed, and we find values of c ranging from 1·8 to 2·3. The character of the wave resistance has now altogether changed, as might be expected since the waves are now two or three times the size of the vessel. It increases much more slowly, probably nearly as the square of the speed. The total resistance of a torpedo boat appears to be about $30c^2$ pounds per ton.

Effective Horse Power.—From the formulæ for the resistance of a vessel we may immediately deduce formulæ for the *effective* horse power required to propel her. The first of these gives the absolute power

$$E.H.P. = \frac{\Delta^{\frac{2}{3}} V^3}{C},$$

where C is a constant at low speeds, which under the restrictions already mentioned may be taken as 500 or 600. If applied to high speeds the value of C is much reduced as it diminishes rapidly with the speed. The second gives the *effective* horse power per ton

$$e.h.p. = \frac{c^2 + c^4}{C} V,$$

where the constant C will, subject to the remarks already made, usually be from 40 to 45, but may sometimes be increased to 50 or even more.

It must be distinctly understood, however, that as formulæ of this class take no account of the periodic variation of the resistance indicated by the "humps and hollows" of the resistance curve, no certain and close estimate of the E.H.P.

can be made except by the [illegible] of comparison. If a full-sized vessel of the same type exists, of which the E.H.P. is known, the principle may be applied without much error to the *total* resistance; but if the type be new, a model must be tried and the principle employed to determine the *residuary* resistance alone: the surface friction, being relatively much greater in the model, as already explained must be separately calculated.

Propellers in general.—Let us next consider briefly the means by which the vessel is propelled through the water.

Every propeller operates by driving astern some or all of the water passing through it, the reaction of which furnishes a propelling force equal and opposite to the "thrust" of the propeller. Since the resistance is directly astern, the velocity impressed on the water must be sternward as far as it is of any utility for the purpose of driving the vessel. Some forms of propeller—as, for example, the screw—give lateral motions to the water, but the energy thus employed is wasted. An ideally perfect propeller, then, impresses upon every particle of water passing through it a reaction astern which for simplicity we may suppose the same for all. This water may for the present be supposed to be initially at rest, and therefore to be passing the ship with the velocity V, which of course is the speed of the ship. After passing through the propeller this velocity is increased to v, $v - V$ being the absolute velocity of the current or "race" produced by its action. For convenience we write

$$v - V = \sigma V, \quad v - V = sv,$$

where σ, s are two fractions described as "slip-ratios," the velocity $v - V$ being the absolute "slip" of the propeller. What is called in ordinary language the slip per cent. is $100s$, but in calculations the fraction σ is often the more useful.

The quantity of water Q acted on per second may in like manner be expressed in two ways. In the first we consider the sectional area A of the race formed when the propeller is acting; in the second the area A_0 through which the same quantity of water passes by the motion of the ship before reaching the propeller. Thus we have

$$Q = Av = A_0 V.$$

The propelling reaction or "thrust" of the propeller is, reasoning as in Art. 270,

$$R = \frac{w}{g} Q(v - V) = \frac{w}{g} \cdot A_0 \sigma V^2,$$

a quantity which must be equal to the resistance of the ship. It is convenient to reckon areas in square feet and velocities in knots, then the value of the constant w/g for sea water is about 17/3.

At moderate speeds the resistance of the vessel as explained above is $K \cdot \Delta^{\frac{2}{3}} V^2$ where K is a co-efficient; hence, equating thrust and resistance, we find that for all speeds with the same notation as before

$$\sigma = \frac{K\Delta^{\frac{2}{3}}}{5 \cdot 7 A_0} = \frac{K}{5 \cdot 7} \cdot \frac{L}{\Delta^{\frac{1}{3}}} \cdot \frac{\beta BD}{35 A_0}.$$

In sea-going vessels the value of $\beta L/\Delta^{\frac{1}{3}}$ varies little and may be taken on an average as 10, though in vessels of small draught of water it may exceed 20. Adopting the value 10 and assuming $K = 5 \cdot 7$, we find

$$\sigma = \frac{BD}{35 A_0},$$

a convenient formula for obtaining a rough idea of the minimum size of propeller necessary for a given slip. It should be observed that the slip is constant only so, long as K is constant, and it therefore increases at high speeds where the resistance cannot be regarded as varying as the square of the speed.

The energy exerted by the engines per second is employed in changing the velocity relatively to the ship of the quantity of water Q, and in overcoming various useless resistances. Omitting the waste

$$\text{Energy exerted} = wQ \cdot \frac{v^2 - V^2}{2g},$$

where the useful work is RV; that is

$$\text{Useful work} = wQ \cdot \frac{V(v - V)}{g}.$$

Hence the efficiency of an ideally perfect propeller when operating on water initially at rest is

$$\text{Efficiency} = \frac{2V}{v + V} = \frac{2}{2 + \sigma}.$$

In such a propeller the only loss is in the kinetic energy of the propeller race, a loss which cannot be avoided when the water is initially at rest. The case where the water is not at rest but moves along with the ship before the propeller acts upon it will be mentioned further on.

Evidently the efficiency is greater the smaller the slip-ratio σ, but this involves an increase in the area A_0, which measures the quantity of water acted on. Hence in every propeller, in the absence of frictional losses and of any disturbance due to the passage of the vessel, the *efficiency is greater the greater the quantity of water upon which it operates*. Let us now apply these general principles to particular cases.

Jet-Propeller.—In the jet-propeller the water is drawn into the vessel through suitable orifices in her bottom and by means of a large centrifugal pump, frequently described as a "turbine" projected through two nozzles pointing astern, one on each side of the vessel. Here the theoretical conditions are exactly satisfied, and the efficiency apart from frictional losses is consequently found from the formula just given, while the joint area of the nozzles is

$$A = \frac{A_0}{1 + \sigma} = \frac{BD}{35(1 + \sigma)\sigma}.$$

For constructive and other reasons the size of the orifices must not be too large, and σ is constantly not less than unity in practical cases. We have therefore

$$\text{Efficiency} = \frac{2}{2 + \sigma} = \frac{2}{3}.$$

The losses in the pump left out of account in this calculation are necessarily large, the efficiency of centrifugal pumps in cases like the present not exceeding ·5, so that the efficiency of jet propulsion can hardly be estimated as greater than ·33 even when designed in the best way. Assuming the constructive difficulties involved in large orifices overcome it would still be undesirable to make σ much less than unity, for the hydraulic resistances would be relatively increased. To illustrate this point let us suppose that by improper arrangements at the orifices of entry the head due to the velocity of V with which the water enters the vessel to be wasted. The energy exerted per second by the engines will be increased from $wQ(v^2 - V^2)/2g$ to $wQv^2/2g$, and the efficiency becomes

$$\text{Efficiency} = \frac{2V(v - V)}{v^2} = \frac{}{(1 + \sigma)^2}.$$

This is easily seen to be greatest when σ is unity, the maximum efficiency being ·25. And if other hydraulic resistances were considered, the same conclusion would be reached, namely, that the efficiency is greatest for a certain value of σ which cannot be very small. The question is closely analogous to that of a simple reaction wheel already considered in Art. 263. It is probable, therefore, that a jet-propeller cannot compete with other forms of propeller so far as economy is concerned, though the considerable advantages it offers in other respects may render it advisable to employ it in special cases.

Paddles.—On observing the action of paddles two streams are seen proceeding from the floats, which play the part of the jets in a jet-propeller. In the most efficient kinds the floats have a "feathering" movement, being mounted on axes, upon which they turn so as to enter and leave the water without any considerable shock. The streams are simple jets of sectional area not very different from that of the floats themselves, and are driven astern with about the same velocity. If then v be the speed of the paddles calculated from their *effective* diameter and revolutions, V the speed of the ship, the propelling reaction is given by the same formula as for jets, while the energy exerted per second is greater, being Rv. Hence the efficiency is

$$\text{Efficiency} = \frac{V}{v} \cdot \frac{1}{1+\sigma}.$$

For given velocities this is less than that of a simple jet when no losses are considered except such as are necessarily involved in the action of the propeller, the reason being that the value just found includes the loss due to breaking up the water as the paddles press on it and drive it upwards in a mass of foam before it settles down to the comparatively undisturbed motion of the race. The waste of energy in this process is equal to the kinetic energy of the race, and the total waste in the paddles is therefore double that in the jet. On the other hand, the paddles act on a very much larger body of water, the value of σ being ·5 or less instead of unity, and the energy wasted in other ways is much less; the efficiency of propulsion is consequently much greater in smooth water when the paddles are properly immersed, probably exceeding ·5 in good examples.

Screw-Propellers in general. In rough water the efficiency of paddles is greatly reduced, and this is also the case when the immersion varies in consequence of the consumption of coal on a long voyage or from other causes. Even in smooth water paddles work to advantage only at the particular speed for which they have been designed in consequence of the change of immersion due to the alteration in position of the waves accompanying the vessel. When the draught of water permits, paddles are therefore almost always replaced by a screw.

In an ideally perfect screw-propeller the race would consist of the water passing through the screw disc, to which would be communicated a sternward velocity as in paddles. The diameter of the screw (supposed single) will be somewhat less than the draught of water (D), and A_0 the area occupied before reaching the screw may therefore be taken as about $\pi D^2/4$, while $Q = A_0 V$ will be the quantity of water acted on. Assuming as before σV as the change of velocity produced on passing through the screw and applying the roughly approximate formula previously given,

$$= \frac{BD}{35 \times \frac{\pi}{4} D^2} = \frac{B}{27 \cdot 5 D}$$

Assuming $D=4B$, this gives for the slip-ratio

$$\sigma=\frac{1}{11}:s=\frac{1}{12},$$

corresponding to a slip of 8½ per cent.

This calculation is of interest as giving a theoretical minimum value for the slip of a screw; the actual average must be much greater, because the whole of the water passing through the screw disc is not moved astern, and the other assumptions made are all of a nature to reduce the calculated result.

Efficiency of Screws.—Though a screw, like every other propeller, operates by impressing a sternward velocity upon the water, yet the manner in which it does this is so entirely different from the action of a paddle that it is desirable to consider the question from a different point of view. Imagine a tube of uniform transverse section to be formed into a cylindrical spiral of uniform pitch z_0, and let the axis of the spiral be a shaft projecting astern exactly parallel to the direction of motion of the vessel. The tube being fixed to the shaft rotates with it at N revolutions per second where

$$V=Nz_0.$$

Neglecting the disturbance of the water by the passage of the vessel, the effect of this arrangement is that the spiral tube screws its way through the water without disturbing it in any way in the absence of friction, the velocity U with which the water moves through the tube being $V.\operatorname{cosec}\alpha$ where α is the pitch angle. Under these circumstances the tube has no propelling effect; but now, suppose that a portion of the tube is taken and its curvature altered so that the pitch, while remaining equal to z_0 at the end where the water enters, gradually increases to z_1 at the end where the water issues, the radius of the spiral being unchanged. The effect of this is that the stream flowing through the tube, while retaining the same mean velocity and pressure, has its direction altered by the small angle ϕ, by which the pitch angle at entrance differs from that at exit. By reasoning as on pages 489, 519, it is now easy to find the resultant action upon the tube of the water inside which will be given by the formula

$$P=a\,.\,Q\,.\,U\phi=aSU^2\phi,$$

in which S is the sectional area of the tube and a a co-efficient which might be exactly calculated. The reasoning here given may be compared with that in Art. 230, p. 505, in which an equivalent result is arrived at.

It was pointed out by FROUDE that a screw blade might be considered as a body moving nearly edgewise through the water, the small angle of obliquity ϕ (depending as it does on the slip) being described as the "slip angle." This small angle is different at each point of the blade, but does not exceed 10° in practical cases. A particle of water in contact with the blade traces out upon it a spiral curve, and each of the spiral elements into which the blade may be thus divided behaves nearly as the spiral tube just described, deflecting through the small angle ϕ a stream the breadth and therefore the sectional area of which is proportional to the length of the element. Hence the propelling reaction is a force P normal to the blade given by the same formula as in the tube, S being now the area of the element. In ideal cases the co-efficient a can be calculated, but in an actual screw blade must be determined experimentally.

In addition to the normal force there will be a tangential force fSU^2 due to friction, f being a co-efficient much less than a. Calling this F and $\sqrt{f/a}$, γ,

$$\frac{F}{P}=\frac{f}{a\phi}=\frac{\gamma^2}{\phi},$$

thus for a given slip angle the ratio F/P is given just as in the case of the friction of a solid screw in its nut, discussed on page 241. Introducing a "friction angle" which we may now call ϕ' to distinguish it from ϕ and proceeding as in the article cited it will be found that

$$\text{Efficiency} = \frac{\tan\{a - \phi\}}{\tan\{a + \phi'\}}.$$

In determining the maximum value of this we must remember that θ and ϕ are independent, but that ϕ' is connected with ϕ. When ϕ' is small $\phi\phi' = \gamma^2$. Hence the maximum efficiency is when $\phi = \phi' = \gamma$ and $a = 45°$, the value being

$$\text{Maximum Efficiency} = \left(\frac{1 - \gamma}{1 + \gamma}\right)^2.$$

Thus the friction and efficiency of screw propellers as determined by this calculation, which is due to FROUDE, are governed by laws closely analogous to those which govern an ordinary screw and its nut.

The value ascribed to the co-efficient γ for a simple element by FROUDE was ·0685, corresponding to a maximum efficiency of 76 per cent. at a slip of about 13 per cent. To make a similar calculation applicable exactly to an ordinary screw blade it would be necessary to suppose that γ had the same, or at any rate some known, value for all elements of the blade, but although quantitative results are unattainable, the principle of the calculation is undoubtedly correct. There must always be a slip of maximum efficiency which cannot be very small, and at small slips the waste by friction is enormously great.

Experiment on model screws bears out this conclusion. Such experiments have been systematically made by Mr. R. E. Froude and others in great numbers, with the result of showing that in good examples the efficiency varies little at slips between 15 and 30 per cent., being then about ·66 rising to nearly 70 per cent. at a slip of about 20 per cent.

Disturbed Water.—The conclusions we have arrived at appear at first sight contrary to experience, for we know that the slip of screw propellers is commonly less than 15 per cent., and often less than the theoretical minimum of 8½ per cent. obtained above. The reason of this is that the screw works in water which is not at rest, but travels onwards along with the ship with a mean velocity u, which probably often reaches 10 per cent. of the speed of the ship. Hence the water enters the screw not with velocity V, but with velocity $V - u$, and the real slip is correspondingly increased, being probably seldom less than 20 per cent. in good examples. The effect on the efficiency of the screw is complicated. In the first place, the useful work done in propelling the ship is greater for the same real slip, and therefore for the same turning moment and speed, so that *prima facie* the efficiency is increased. But on the other hand, a screw of ordinary dimensions sucks more water through it than would naturally flow there, an action which augments the resistance of the ship unless the screw is placed further astern than is possible for constructive reasons. In a screw with many blades of considerable length there would be little if any suction, but too great blade area is a cause of great loss by friction. FROUDE stated that the augmentation was often as much as 40 or 50 per cent., but it was afterwards explained by his son that this estimate included the resistance of thick square stern posts and appendages to the propeller, the augmentation proper varying from 8 to 18 per cent. The lower value applies to twin screws and vessels with fine lines. Experiment appears to show that in models the loss by augmentation on the average about compensates for the direct gain by working in disturbed water, the efficiency of a model screw being about the same when a corresponding vessel is run ahead of it as when the vessel is removed.

The best results are doubtless obtained by an exact adaptation of the dimensions, number, and form of the screw blades to the type of vessel. At present such adaptation can only be effected by the principle of comparison from some example known to give good results. The method is fully explained by Mr. Sydney Barnaby in his work on *Marine Propellers.*

Indicated Power.—From what has been said it appears that the power required to drive a propeller will be $(1 + e_1)E.H.P.$ where e_1 is a fraction, which in the best examples of jet, paddle, or screw will seldom be much less than ·5. This addition of 50 per cent. to the effective power is due to waste of energy in giving various motions to the water acted on by the propeller, including the production of eddies, by surface friction of blades and otherwise. To obtain the indicated power we must now consider the friction of the engines and other resistances, such as air-pumps, feed-pumps, and the like. These consist (as described on page 258) of two parts, a variable part proportional to the mean effective pressure, and a constant part most conveniently expressed as a fraction of the effective pressure at full speed. Thus the formula

$$I.H.P. = (1 + e_1 + e_2) \,.\, E.H.P. + e_3 \,.\, E_1 H_1 P_1 \,.\, \frac{N}{N_1}$$

gives with sufficient accuracy for the present purpose the indicated power at the given speed of vessel and revolutions (N) of the engines in a set of speed trials, where e_2, e_3 are fractions and the suffix 1 refers to full speed. The counter-efficiency at full speed is $1 + e_1 + e_2 + e_3$, which in screw propulsion in the best examples is about 1·8, and taking e_1 as ·5 we find $e_2 + e_3 = \cdot 3$, of which at least one-half is due to the constant friction. At lower speeds the efficiency of propulsion is much less, because the effect of the constant friction is relatively great.

The ratio $\Delta^{\frac{2}{3}} V^3 / I.H.P.$ is described as the "displacement constant." It has long been known that it is not the same at different speeds in a set of progressive speed trials, but that it has a maximum value at a certain speed (about $v = \cdot 7$ in full sized vessels), diminishing considerably both at high speeds and at low speeds. The explanation of this is sufficiently clear from what has been said. It can, however, be used as a means of comparison if care be taken to compare only vessels at corresponding speeds with engines working at the same fraction of their full power.

REFERENCES.

A full account of the earlier experiments on the resistance of ships, including all that had been done on the subject before the time of FROUDE, will be found in a treatise published in 1857 entitled *Mémoire sur la Resistance de l'Eau*, by Captain (afterwards Admiral) Bourgois of the French Navy.

FROUDE'S researches are contained in two *Reports on Surface Friction* presented to the British Association in 1874, and in various papers for the most part published in the *Transactions of the Institution of Naval Architects.* It is to be hoped these papers will be published in a collected form.

Sir W. H. White's well-known work on *Naval Architecture* will also be found invaluable.

APPENDIX.

B. ORGANIZATION OF THE CLASSES IN ENGINEERING AND NAVAL ARCHITECTURE IN THE ROYAL NAVAL COLLEGE.

A SCHOOL of naval architecture was founded in Portsmouth dockyard so long ago as 1810, but, after existing for more than twenty years, was abolished in 1833. In 18[illegible] it was re-established, but only to be once more abolished in 1853. In 1860 the Institution of Naval Architects was founded, and by its influence a third school was commenced under the direction of the Science and Art Department at South Kensington. For particulars respecting the two earlier of these schools the reader is referred to a paper by SCOTT RUSSELL in the *Transactions I.N.A.* for 1863. The third was afterwards incorporated with the Royal Naval College, of which it now forms a department.

This department is divided into two classes, of which the junior serves as the final stage in the training of the engineer officers of the navy, the majority of whom spend nine months at Greenwich immediately on entering the service, after several years spent in the dockyard. (See p. 647.) The senior is an advanced class, consisting partly of a small number of engineer officers selected by competition from the preceding, and partly of students in naval architecture originally selected by competition from the dockyard apprentices to join the junior class. The full course in the advanced class lasts three years, of which one is spent in the junior, and two in the senior class. There are also private students who generally go through the full course. The programme of these classes differs in some important respects from that of most other technical colleges, and it may be useful to describe it briefly here.

The three principal branches of study are:

I. Pure and Applied Mathematics;
II. Applied Mechanics;
III. { Naval Architecture,
{ Marine Engineering:

to each of which the time allotted is about the same. In addition, there is a course in Physics and Chemistry. The mathematical course includes the theory of electricity, while the technical applications to electric lighting and torpedo work are included in the laboratory course on physics. The following remarks will be confined to the second and third of the principal subjects.

IN APPLIED MECHANICS the subjects are

A. Elementary Subjects.
B. { Stability and Oscillation of Ships.
{ Theory of the Steam Engine.
C. Wave motion. Resistance and Propulsion of Ships.

Subjects A. are pretty closely represented by the present treatise, but there are a few omissions and some additions, especially in graphical statics and elementary theory of the steam engine. The course lasts two years, each subject being commenced in the junior class and completed in the senior.

Subjects B. are commenced in the second year and completed in the third. The first is studied by students in naval architecture only, and the lectures on it are at present given by the Instructor in Naval Architecture. The second is studied by students in engineering only.

Subjects C. occupy the greater part of the third year.

In NAVAL ARCHITECTURE the course followed is very fully explained in a paper by Mr. (now Sir) W. H. White in the *Transactions I.N.A.* for 1877 (Vol. XVIII., p. 361), and it need not therefore be further considered here.

In MARINE ENGINEERING the course for the junior class occupies nine hours a week. Each of the principal parts of the marine engine, including the boiler and propeller, are taken in detail, the dimensions proper for that part determined, and the other practical questions considered which are involved in its design. An example is set, and the student is expected to work out a design from the data proposed, and to produce working drawings. About 30 of these drawings are prepared in the session, the subjects being:

Details of principal parts of Engine—
Piston, Piston Rod, Connecting Rod, Cross-head and Guides, Thrust-block, Crank-shaft, Cylinders and Fittings.

Propeller—
Shafting and Couplings, Boss, Blades.

Slide Valves—
Zeuner's Diagrams for Solid and Open Bar Links, Valve-ellipse, Construction and setting of Slide Valves, Link Motion.

Boilers—
Dimensions and Structural Details, Fittings.

Condensers and Air Pumps—
Fittings and General Arrangement.

The foregoing course is gone through by all students in engineering. Those who are selected to enter the advanced class devote eight hours a week on the average to the subject in two following sessions. In the second year detailed drawings are made of the parts, and three views of the general arrangements, of a set of marine engines of large power suitable to propel a given ship at a given speed. The drawings of the details and propeller are completed, and the general drawings pencilled. In the third year the boilers are designed, and drawings made showing the disposition of the pipes and auxiliary engines. The general drawings are completed, the whole design being represented by a set of about 20 drawings.

The practical training of students both in naval architecture and in engineering takes place in the dockyards for a period of at least 4 years before entering the College and during the three summer months in which the College is closed. This is a point of great importance, for, quite irrespectively of the absolute necessity of such training for its own sake, no theoretical course can be thoroughly understood without some preliminary knowledge of a practical kind. A College workshop is a very imperfect substitute and occupies time which is better spent elsewhere. The author, however, must not be understood to depreciate the importance of a "mechanical laboratory," provided with testing machines, hydraulic apparatus, steam engines, and the like, for the purpose of studying mechanics experimentally. Such a laboratory, when properly organized, is capable of rendering great service, but it in no way replaces training in a large workshop carried on for commercial purposes. Nor are these remarks intended to apply to the lower grades of technical education, in which the workshop to a great extent plays the part of a laboratory.

In the author's opinion, much the same may be said as to the use of models in teaching mechanics. An engineer does not use models; he employs drawings almost exclusively; and so, in the instruction of professional students, models are of little value for descriptive purposes. Nor should they be used to demonstrate the laws of motion. But in explaining a mechanical principle, a model is sometimes of service; it plays the same part as the figure in a proposition of Euclid in aiding the conception of the learner. And, as before, in the lower grades of technical education, models may properly be used for demonstrative purposes, while, in the case of non-professional students, they are often indispensable for descriptive purposes. In the "steam" department of the Royal Naval College, organized for the purpose of imparting to the executive officers of the navy a knowledge of the mechanism and working of a marine engine, models are freely used in this way. On the subject of technical education in naval architecture, the reader is referred to two valuable papers by Mr. John and Mr. W. Denny in the *Transactions of the Institution of Naval Architects*, the first in Vol. XIX., p. 120; the second in Vol. XXII., page 144.

[1892.] In the year 1877 a college for the training of engineer students was established first at Portsmouth, on board the *Marlborough*, afterwards at Keyham. In recent years a certain proportion of the engineer officers on entering the Navy have been sent afloat at once instead of passing through Greenwich, and the majority of the students of naval architecture have been selected for study at Greenwich from the Keyham students. A large part of the time at Keyham is spent in practical work in the dockyard and engine factory; but of late a part of the instruction in applied mechanics and marine engineering has been carried out there, leaving more time for the development of the subjects at Greenwich.

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